Periods and Nori Motives

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June 8, 2015

Part II

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Nori Motives

Chapter 6

Nori's diagram category

We explain Nori's construction of an abelian category attached to the representation of a diagram and establish some properties for it. The construction is completely formal. It mimicks the standard construction of the Tannakian dual of a rigid tensor category with a fibre functor. Only, we do not have a tensor product or even a category but only what we should think of as the fibre functor.

The results are due to Nori. Notes from some of his talks are available [N, N1]. There is a also a sketch in Levine's survey [L1] §5.3. In the proofs of the main results we follow closely the diploma thesis of von Wangenheim in [vW].

6.1 Main results

6.1.1 Diagrams and representations

Let R be a noetherian, commutative ring with unit.

Definition 6.1.1. A *diagram* D is a directed graph on a set of vertices $V(D)$ and edges $E(D)$. A *diagram* with *identities* is a diagram with a choice of a distinguished edge $\mathrm{id}_v : v \to v$ for every $v \in D$. A diagram is called *finite* if it has only finitely many vertices. A *finite full subdiagram* of a diagram D is a diagram containing a finite subset of vertices of D and all edges (in D) between them.

By abuse of notation we often write $v \in D$ instead of $v \in V(D)$. The set of all directed edges between $p, q \in D$ is denoted by $D(p, q)$.

Remark 6.1.2. One may view a diagram as a category where composition of morphisms is not defined. The notion of a diagram with identity edges is not standard. The notion is useful later when we consider multiplicative structures.

Example 6.1.3. Let $\mathcal{C}$ be a small category. Then we can associate a diagram $D(\mathcal{C})$ with vertices the set of objects in $\mathcal{C}$ and edges given by morphisms. It is even a diagram with identities. By abuse of notation we usually also write $\mathcal{C}$ for the diagram.

Definition 6.1.4. A *representation* T of a diagram D in a small category $\mathcal{C}$ is a map T of directed graphs from D to $D(\mathcal{C})$. A *representation* T of a diagram D with identities is a representation such that id is mapped to id.

For $p, q \in D$ and every edge m from p to q we denote their images simply by Tp, Tq and $Tm : Tp \to Tq$ (mostly without brackets).

Remark 6.1.5. Alternatively, a representation is defined as a functor from the *path category* $\mathcal{P}(D)$ to $\mathcal{C}$. Recall that the objects of the path category are the vertices of D, and the morphisms are sequences of directed edges $e_1e_2\dots e_n$ for $n \geq 0$ with the edge e_i starting in the end point of e_{i-1} for $i = 2, \dots, n$. Morphisms are composed by concatenating edges.

We are particularly interested in representations in categories of modules.

Definition 6.1.6. Let R be a noetherian commutative ring with unit. By $R-$Mod we denote the category of finitely generated R-modules. By $R-$Proj we denote the subcategory of finitely generated projective R-modules.

Note that these categories are essentially small by passing to isomorphic objects, so we will not worry about smallness from now on.

Definition 6.1.7. Let S be a commutative unital R-algebra and $T : D \to R-$Mod a representation. We denote T_S the representation

$$D \xrightarrow{T} R-\mathrm{Mod} \xrightarrow{\otimes_R S} S-\mathrm{Mod}\ .$$

Definition 6.1.8. Let T be a representation of D in $R-$Mod. We define the *ring of endomorphisms* of T by

$$\mathrm{End}(T) := \left\{ (e_p)_{p\in D} \in \prod_{p\in D} \mathrm{End}_R(Tp) | e_q \circ Tm = Tm \circ e_p \ \forall p, q \in D\ \forall m \in D(p, q) \right\}.$$

Remark 6.1.9. In other words, an element of $\mathrm{End}(T)$ consists of tuples $(e_p)_{p\in V(D)}$ of endomorphisms of Tp, such that all diagrams of the following form commute:

$$\begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ \downarrow{\scriptstyle e_p} & & \downarrow{\scriptstyle e_q} \\ Tp & \xrightarrow{Tm} & Tq \end{array}$$

Note that the ring of endomorphisms does not change when we replace D by the path category $\mathcal{P}(D)$.

6.1.2 Explicit construction of the diagram category

The diagram category can be characterized by a universal property, but it also has a simple explicit description that we give first.

Definition 6.1.10 (Nori)**.** Let R be a noetherian commutative ring with unit. Let T be a representation of D in $R-$Mod.

1. Assume D is finite. Then we put

$$\mathcal{C}(D,T) = \mathrm{End}(T)-\mathrm{Mod}$$

   the category of finitely generated R-modules equipped with an R-linear operation of the algebra $\mathrm{End}(T)$.

2. In general let

$$\mathcal{C}(D,T) = 2-\mathrm{colim}_F \mathcal{C}(F,T|_F)$$

   where F runs through the system of finite subdiagrams of D.

   More explicitly: the objects of $\mathcal{C}(D,T)$ are the objects of $\mathcal{C}(F,T|_F)$ for some finite subdiagram F. For $X \in \mathcal{C}(F,T|_F)$ and $F \subset F'$ we write $X_{F'}$ for the image of X in $\mathcal{C}(F',T|_{F'})$. For objects $X,Y \in \mathcal{C}(D,T)$, we put

$$\mathrm{Mor}_{\mathcal{C}(D,T)}(X,Y) = \varinjlim_F \mathrm{Mor}_{\mathcal{C}(F,T|_F)}(X_F,Y_F)\ .$$

The category $\mathcal{C}(D,T)$ is called the *diagram category*. With

$$f_T : \mathcal{C}(D,T) \longrightarrow R-\mathrm{Mod}$$

we denote the forgetful functor.

Remark 6.1.11. The representation $T : D \longrightarrow \mathcal{C}(D,T)$ extends to a functor on the path category $\mathcal{P}(D)$. By construction the diagram categories $\mathcal{C}(D,T)$ and $\mathcal{C}(\mathcal{P}(D),T)$ agree. The point of view of the path category will be useful Chapter 7, in particular in Definition 7.2.1.

In section 6.5 we will prove that under additional conditions for R, satisfied in the cases of most interest, there is the following even more direct description of $\mathcal{C}(D,T)$ as comodules over a coalgebra.

Theorem 6.1.12. *If the representation T takes values in free modules over a field or Dedekind domain R, the diagram category is equivalent to the category of finitely generated comodules (see Definition 6.5.4) over the coalgebra $A(D,T)$ where*

$$A(D,T) = \mathrm{colim}_F A(F,T) = \mathrm{colim}_F \mathrm{End}(T|_F)^\vee$$

with F running through the system of all finite subdiagrams of D and $^\vee$ the R-dual.

The proof of this theorem is given in Section 6.5.

6.1.3 Universal property: Statement

Theorem 6.1.13 (Nori). *Let D be a diagram and*

$$T : D \longrightarrow R\text{--Mod}$$

a representation of D.

Then there exists an R-linear abelian category $\mathcal{C}(D,T)$, together with a representation

$$\tilde{T} : D \longrightarrow \mathcal{C}(D,T),$$

and a faithful, exact, R-linear functor f_T, such that:

1. *T factorizes over $D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R\text{--Mod}$.*
2. *$\tilde{T}$ satisfies the following universal property: Given*
   (a) *another R-linear, abelian category $\mathcal{A}$,*
   (b) *an R-linear, faithful, exact functor, $f : \mathcal{A} \to R\text{--Mod}$,*
   (c) *another representation $F : D \to \mathcal{A}$,*

   such that $f \circ F = T$, then there exists a functor $L(F)$ - unique up to unique isomorphism of functors - such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{C}(D,T) \arrow[dr, "f_T"] \arrow[dd, dashed, "L(F)"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] \arrow[dr, "F"'] & & R\text{--Mod} . \\
 & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}$$

The category $\mathcal{C}(D,T)$ together with $\tilde{T}$ and f_T is uniquely determined by this property up to unique equivalence of categories. It is explicitly described by the diagram category of Definition 6.1.10. It is functorial in D in the obvious sense.

The proof will be given in Section 6.4. We are going to view f_T as an extension of T from D to $\mathcal{C}(D,T)$ and sometimes write simply T instead of f_T.

The universal property generalizes easily.

Corollary 6.1.14. *Let D, R, T be as in Theorem 6.1.19. Let $\mathcal{A}$ and f, F be as in loc.cit. 2. (a)-(c). Moreover, let S be a faithfully flat commutative unitary R-algebra S and*

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S-\mathrm{Mod}$. Then there exists a functor $L(F) : \mathcal{C}(D,T) \to \mathcal{A}$ and an isomorphism of functors

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

such that

$$\begin{tikzcd}
 & \mathcal{C}(D,T) \arrow[dr, "(f_T)_S"] \arrow[dd, dashed, "L(F)"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T_S"] \arrow[dr, "F"'] & & S-\mathrm{Mod} \\
 & \mathcal{A} \arrow[ur, "f_S"'] &
\end{tikzcd}$$

commutes up to ϕ and $\tilde{\phi}$. The pair $(L(F), \tilde{\phi})$ is unique up to unique isomorphism of functors.

The proof will also be given in Section 6.4.

The following properties provide a better understanding of the nature of the category $\mathcal{C}(D,T)$.

Proposition 6.1.15.

1. *As an abelian category $\mathcal{C}(D,T)$ is generated by the $\tilde{T}v$ where v runs through the set of vertices of D, i.e., it agrees with its smallest full subcategory such that the inclusion is exact containing all such $\tilde{T}v$.*
2. *Each object of $\mathcal{C}(D,T)$ is a subquotient of a finite direct sum of objects of the form $\tilde{T}v$.*
3. *If $\alpha : v \to v'$ is an edge in D such that $T\alpha$ is an isomorphism, then $\tilde{T}\alpha$ is also an isomorphism.*

Proof. Let $\mathcal{C}' \subset \mathcal{C}(D,T)$ be the subcategory generated by all $\tilde{T}v$. By definition, the representation $\tilde{T}$ factors through $\mathcal{C}'$. By the universal property of $\mathcal{C}(D,T)$, we obtain a functor $\mathcal{C}(D,T) \to \mathcal{C}'$, hence an equivalence of subcategories of $R-\mathrm{Mod}$.

The second statement follows from the first criterion since the full subcategory in $\mathcal{C}(D,T)$ of subquotients of finite direct sums is abelian, hence agrees with $\mathcal{C}(D,T)$. The assertion on morphisms follows since the functor $f_T : \mathcal{C}(D,T) \to R{-}\mathrm{Mod}$ is faithful and exact between abelian categories. Kernel and cokernel of $\tilde{T}\alpha$ vanish if kernel and cokernel of $T\alpha$ vanish. $\square$

Remark 6.1.16. We will later give a direct proof, see Proposition 6.3.20. It will be used in the proof of the universal property.

The diagram category only weakly depends on T.

Corollary 6.1.17. *Let D be a diagram and $T, T' : D \to R{-}\mathrm{Mod}$ two representations. Let S be a faithfully flat R-algebra and $\phi : T_S \to T'_S$ be an isomorphism of representations in $S{-}\mathrm{Mod}$. Then it induces an equivalence of categories*

$$\Phi : \mathcal{C}(D,T) \to \mathcal{C}(D,T').$$

Proof. We apply the universal property of Corollary 6.1.14 to the representation T and the abelian category $\mathcal{A} = \mathcal{C}(D,T')$. This yields a functor $\Phi : \mathcal{C}(D,T) \to \mathcal{C}(D,T')$. By interchanging the role of T and T' we also get a functor Φ' in the opposite direction. We claim that they are inverse to each other. The composition $\Phi' \circ \Phi$ can be seen as the universal functor for the representation of D in the abelian category $\mathcal{C}(D,T)$ via T. By the uniqueness part of the universal property, it is the identity. $\square$

Corollary 6.1.18. *Let D_2 be a diagram. Let $T_2 : D_2 \to R{-}\mathrm{Mod}$ be a representation. Let*

$$D_2 \xrightarrow{\tilde{T}_2} \mathcal{C}(D_2, T_2) \xrightarrow{f_{T_2}} R{-}\mathrm{Mod}$$

be the factorization via the diagram category.

Let $D_1 \subset D_2$ be a full subdiagram. It has the representation $T_1 = T_2|_{D_1}$ obtained by restricting T_2. Let

$$D_1 \xrightarrow{T_1} \mathcal{C}(D_1, T_1) \xrightarrow{f_{T_1}} R{-}\mathrm{Mod}$$

be the factorization via the diagram category. Let $\iota : \mathcal{C}(D_1,T_1) \to \mathcal{C}(D_2,T_2)$ be the functor induced from the inclusion of diagrams. Moreover, we assume that there is a representation $F : D_2 \to \mathcal{C}(D_1,T_1)$ compatible with T_2, i.e., such that there is an isomorphism of functors

$$T_2 \to f_{T_2} \circ \iota \circ F = f_{T_1} \circ F\ .$$

Then ι is an equivalence of categories.

Proof. Let $T'_2 = f_{T_1} \circ F : D_2 \to R{-}\mathrm{Mod}$ and denote $T'_1 = T'_2|_{D_1} : D_1 \to R{-}\mathrm{Mod}$. Note that T_2 and T'_2 and T_1 and T'_1 are isomorphic by assumption.

By the universal property of the diagram category, the representation F induces a functor

$$\pi' : \mathcal{C}(D_2, T_2') \to \mathcal{C}(D_1, T_1) \ .$$

It induces $\pi : \mathcal{C}(D_2, T_2)$ by precomposition with the equivalence Φ from Corollary 6.1.17. We claim that $\iota \circ \pi$ and $\pi \circ \iota$ are isomorphic to the identity functor.

By the uniqueness part of the universal property, the composition $\iota \circ \pi' : \mathcal{C}(D_2, T_2') \to \mathcal{C}(D_2, T_2)$ is induced from the representation $\iota \circ F$ of D_2 in the abelian category $\mathcal{C}(D_2, T_2)$. By the proof of Corollary 6.1.17 this is the equivalence Φ^{-1}. In particular, $\iota \circ \pi$ is the identity.

The argument for $\pi \circ \iota$ on $\mathcal{C}(D_1, T_1)$ is analogous. □

The most important ingredient for the proof of the universal property is the following special case.

Theorem 6.1.19. *Let R be a noetherian ring and $\mathcal{A}$ an abelian, R-linear category. Let*

$$T : \mathcal{A} \longrightarrow R\text{−Mod}$$

be a faithful, exact, R-linear functor and

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A}, T) \xrightarrow{f_T} R\text{−Mod}$$

the factorization via its diagram category (see Definition 6.1.10). Then $\tilde{T}$ is an equivalence of categories.

The proof of this theorem will be given in Section 6.3.

6.1.4 Discussion of the Tannakian case

The above may be viewed as a generalization of Tannaka duality. We explain this in more detail. We are not going to use the considerations in the sequel.

Let k be a field, $\mathcal{C}$ a k-linear abelian tensor category, and

$$T : \mathcal{C} \longrightarrow k\text{−Vect}$$

a k-linear faithful tensor functor, all in the sense of [DM]. By standard Tannakian formalism (cf [Sa] and [DM]), there is a k-bialgebra A such that the category is equivalent to the category of A-comodules on finite dimensional k-vector spaces.

On the other hand, if we regard $\mathcal{C}$ as a diagram (with identities) and T as a representation into finite dimensional vector spaces, we can view the diagram category of $\mathcal{C}$ as the category $A(\mathcal{C}, T)$−Comod by Theorem 6.1.12. By Theorem 6.1.19 the category $\mathcal{C}$ is equivalent to its diagram category $A(\mathcal{C}, T)$−Comod. The construction of the two coalgebras A and $A(\mathcal{C}, T)$ coincides. Thus Nori implicitely shows that we can recover the coalgebra structure of A just by looking at the representations of $\mathcal{C}$.

The *algebra* structure on $A(\mathcal{C}, T)$ is induced from the tensor product on $\mathcal{C}$ (see also Section 7.1). This defines a pro-algebraic scheme $\mathrm{Spec}A(\mathcal{C}, T)$. The *coalgebra* structure turns $\mathrm{Spec}A(\mathcal{C}, T)$ into a monoid scheme. We may interpret $A(\mathcal{C}, T)-$Comod as the category of finite-dimensional representations of this monoid scheme.

If the tensor structure is rigid in addition, $\mathcal{C}(D, T)$ becomes what Deligne and Milne call a *neutral Tannakian category* [DM]. The rigidity structure induces an antipodal map, making $A(\mathcal{C}, T)$ into a Hopf algebra. This yields the structure of a *group scheme* on $\mathrm{Spec}A(\mathcal{C}, T)$, rather than only a monoid scheme.

We record the outcome of the discussion:

Theorem 6.1.20. *Let R be a field and $\mathcal{C}$ be a neutral R-linear Tannakian category with faithful exact fibre functor $T : \mathcal{C} \to R-\mathrm{Mod}$. Then $A(\mathcal{C}, T)$ is equal to the Hopf algebra of the Tannakian dual.*

Proof. By construction, see [DM] Theorem 2.11 and its proof. □

A similar result holds in the case that R is a Dedekind domain and

$$T : D \longrightarrow R-\mathrm{Proj}$$

a representation into finitely generated projective R-modules. Again by Theorem 6.1.12, the diagram category $\mathcal{C}(D, T)$ equals $A(\mathcal{C}, T)-$Comod, where $A(\mathcal{C}, T)$ is projective over R. Wedhorn shows in [Wed] that if $\mathrm{Spec}A(\mathcal{C}, T)$ is a group scheme it is the same to have a representation of $\mathrm{Spec}A(\mathcal{C}, T)$ on a finitely generated R-module M and to endow M with an $A(\mathcal{C}, T)$-comodule structure.

6.2 First properties of the diagram category

Let R be a unitary commutative noetherian ring, D a diagram and $T : D \to R-\mathrm{Mod}$ a representation. We investigate the category $\mathcal{C}(D, T)$ introduced in Definition 6.1.10.

Lemma 6.2.1. *If D is a finite diagram, then* $\mathrm{End}(T)$ *is an R-algebra which is finitely generated as an R-module.*

Proof. For any $p \in D$ the module Tp is finitely generated. Since R is noetherian, the algebra $\mathrm{End}_R(Tp)$ then is finitely generated as R-module. Thus $\mathrm{End}(T)$ becomes a unitary subalgebra of $\prod_{p \in Ob(D)} \mathrm{End}_R(Tp)$. Since $V(D)$ is finite and R is noetherian,

$$\mathrm{End}(T) \subset \prod_{p \in Ob(D)} \mathrm{End}_R(Tp)$$

is finitely generated as R-module. □

Lemma 6.2.2. *Let D be a finite diagram and $T : D \to R-\mathrm{Mod}$ a representation. Then:*

1. *Let S be a flat R-algebra. Then:*
$$\mathrm{End}_S(T_S) = \mathrm{End}_R(T) \otimes S$$
2. *Let $F : D' \to D$ be morphism of diagrams and $T' = T \circ F$ the induced representation. Then F induces a canonical R-algebra homomorphism*
$$F^* : \mathrm{End}(T) \to \mathrm{End}(T') \ .$$

Proof. The algebra $\mathrm{End}(T)$ is defined as a limit, i.e., a kernel

$$0 \to \mathrm{End}(T) \to \prod_{p\in V(D)} \mathrm{End}_R(Tp) \xrightarrow{\phi} \prod_{m\in D(p,q)} \mathrm{Hom}_R(Tp, Tq)$$

with $\phi(p)(m) = e_q \circ Tm - Tm \circ e_p$. As S is flat over R, this remains exact after tensoring with S. As the R-module Tp is finitely presented and S flat, we have

$$\mathrm{End}_R(Tp) \otimes S = \mathrm{End}_S(T_S p) \ .$$

Hence we get

$$0 \to \mathrm{End}(T|_F) \otimes S \to \prod_{p\in V(D)} \mathrm{End}_S(T_S(p)) \xrightarrow{\phi} \prod_{m\in D(p,q)} \mathrm{Hom}_S(T_S(p), T_S(q)) \ .$$

This is the defining sequence for $\mathrm{End}(T_S)$.

The morphism of diagrams $F : D' \to D$ induces a homomorphism

$$\prod_{p\in V(D)} \mathrm{End}_R(Tp) \to \prod_{p'\in V(D')} \mathrm{End}_R(T'p'),$$

by mapping $e = (e_p)_p$ to $F^*(e)$ with $(F^*(e))_{p'} = e_{f(p')}$ in $\mathrm{End}_R(T'p') = \mathrm{End}_R(Tf(p'))$. It is compatible with the induced homomorphism

$$\prod_{m\in D(p,q)} \mathrm{Hom}_R(Tp, Tq) \to \prod_{m'\in D'(p',q')} \mathrm{Hom}_R(T'p', T'q').$$

Hence it induces a homomorphism on the kernels. □

Proposition 6.2.3. *Let R be unitary commutative noetherian ring, D a finite diagram and $T : D \longrightarrow R-\mathrm{Mod}$ be a representation. For any $p \in D$ the object Tp is a natural left $\mathrm{End}(T)$-module. This induces a representation*

$$\tilde{T} : D \longrightarrow \mathrm{End}(T)-\mathrm{Mod},$$

such that T factorises via

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R-\mathrm{Mod}.$$

Proof. For all $p \in D$ the projection

$$pr : \mathrm{End}(T) \to \mathrm{End}_R(Tp)$$

induces a well-defined action of $\mathrm{End}(T)$ on Tp making Tp into a left $\mathrm{End}(T)$-module. To check that $\tilde{T}$ is a representation of left $\mathrm{End}(T)$-modules, we need $Tm \in \mathrm{Hom}_R(Tp, Tq)$ to be $\mathrm{End}(T)$-linear for all $p, q \in D, m \in D(p,q)$. This corresponds directly to the commutativity of the diagram in Remark 6.1.9. □

Now let D be general. We study the system of finite subdiagrams $F \subset D$. Recall that subdiagrams are full, i.e., they have the same edges.

Corollary 6.2.4. *The finite subdiagrams of D induce a directed system of abelian categories $\big(\mathcal{C}(D, T|_F)\big)_{F\subset D \mathit{finite}}$ with exact, faithful R-linear functors as transition maps.*

Proof. The transition functors are induced from the inclusion via Lemma 6.2.2. □

Recall that we have defined $\mathcal{C}(D,T)$ as 2-colimit of this system, see Definition 6.1.10.

Proposition 6.2.5. *The 2-colimit $\mathcal{C}(D,T)$ exists. It provides an R-linear abelian category such that the composition of the induced representation with the forgetful functor*

$$\begin{array}{ccccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) & \xrightarrow{f_T} & R\text{−Mod} \\ p & \mapsto & Tp & \mapsto & Tp. \end{array}$$

yields a factorization of T. The functor f_T is R-linear, faithful and exact.

Proof. It is a straightforward calculation that the limit category inherits all structures of an R-linear abelian category. It has well-defined (co)products and (co)kernels because the transition functors are exact. It has a well-defined R-linear structure as all transition functors are R-linear. Finally, one shows that every kernel resp. cokernel is a monomorphism resp. epimorphism using the fact that all transition functors are faithful and exact.

So for every $p \in D$ the R-module Tp becomes an $\mathrm{End}(T|_F)$-module for all finite $F \subset D$ with $p \in F$. Thus it represents an object in $\mathcal{C}(D,T)$. This induces a representation

$$\begin{array}{ccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) \\ p & \mapsto & Tp. \end{array}$$

The forgetful functor is exact, faithful and R-linear. Composition with the forgetful functor f_T obviously yields the initial diagram T. □

We now consider functoriality in D.

Lemma 6.2.6. *Let D_1, D_2 be diagrams and $G : D_1 \to D_2$ a map of diagrams. Let further $T : D_2 \to R{-}\mathrm{Mod}$ be a representation and*

$$D_2 \xrightarrow{\tilde{T}} \mathcal{C}(D_2, T) \xrightarrow{f_T} R{-}\mathrm{Mod}$$

the factorization of T through the diagram category $\mathcal{C}(D_2, T)$ as constructed in Proposition 6.2.5. Let

$$D_1 \xrightarrow{\widetilde{T \circ G}} \mathcal{C}(D_1, T \circ G) \xrightarrow{f_{T\circ G}} R{-}\mathrm{Mod}$$

be the factorization of $T \circ G$.

Then there exists a faithful R-linear, exact functor $\mathcal{G}$, such that the following diagram commutes.

$$\begin{tikzcd}
D_1 \arrow[rr, "G"] \arrow[d, "\widetilde{T \circ G}"] & & D_2 \arrow[d, "\tilde{T}"] \\
\mathcal{C}(D_1, T \circ G) \arrow[rr, dashed, "\mathcal{G}"] \arrow[dr, "f_{T\circ G}"] & & \mathcal{C}(D_2, T) \arrow[dl, "f_T"] \\
& R{-}\mathrm{Mod} &
\end{tikzcd}$$

Proof. Let D_1, D_2 be finite diagrams first. Let $T_1 = T \circ G|_{D_1}$ and $T_2 = T|_{D_2}$. The homomorphism

$$G^* : \mathrm{End}(T_2) \to \mathrm{End}(T_1)$$

of Lemma 6.2.2 induces by restriction of scalars a functor on diagram categories as required.

Let now D_1 be finite and D_2 arbitrary. Let E_2 be finite full subdiagram of D_2 containing $G(D_1)$. We apply the finite case to $G : D_1 \to E_2$ and obtain a functor

$$\mathcal{C}(D_1, T_1) \to \mathcal{C}(E_2, T_2)$$

which we compose with the canonical functor $\mathcal{C}(E_2, T_2) \to \mathcal{C}(D_2, T_2)$. By functoriality, it is independent of the choice of E_2.

Let now D_1 and D_2 be arbitrary. For every finite subdiagram $E_1 \subset E_1$ we have constructed

$$\mathcal{C}(E_1, T_1) \to \mathcal{C}(D_2, T_2)\ .$$

They are compatible and hence define a functor on the limit. $\square$

Isomorphic representations have equivalent diagram categories. More precisely:

Lemma 6.2.7. *Let $T_1, T_2 : D \to R-\mathrm{Mod}$ be representations and $\phi : T_1 \to T_2$ an isomorphism of representations. Then ϕ induces an equivalence of categories $\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2)$ together with an isomorphism of representations*

$$\tilde{\phi} : \Phi \circ \tilde{T}_1 \to \tilde{T}_2$$

such that $f_{T_2} \circ \tilde{\phi} = \phi$.

Proof. We only sketch the argument which is analogous to the proof of Lemma 6.2.6. It suffices to consider the case $D = F$ finite. The functor

$$\Phi : \mathrm{End}(T_1)-\mathrm{Mod} \to \mathrm{End}(T_2)-\mathrm{Mod}$$

is the extension of scalars for the R-algebra isomorphism $\mathrm{End}(T_1) \to \mathrm{End}(T_2)$ induced by conjugating by ϕ. $\square$

6.3 The diagram category of an abelian category

In this section we give the proof of Theorem 6.1.19: the diagram category of the diagram category of an abelian category with respect to a representation given by an exact faithful functor is the abelian category itself.

We fix a commutative noetherian ring R with unit and an R-linear abelian category $\mathcal{A}$. By R-algebra we mean a unital R-algebra, not necessarily commutative.

6.3.1 A calculus of tensors

We start with some general constructions of functors. We fix a unital R-algebra E, finitely generated as R-module, not necessarily commutative. The most important case is $E = R$, but this is not enough for our application.

In the simpler case where R is a field, the constructions in this sections can already be found in [DMOS].

Definition 6.3.1. Let E be an R-algebra which is finitely generated as R-module. We denote $E-\mathrm{Mod}$ the category of finitely generated left E-modules.

The algebra E and the objects of $E-\mathrm{Mod}$ are noetherian because R is.

Definition 6.3.2. Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

be a morphism of R-algebras. We say that p is a *right E-module in $\mathcal{A}$*.

Example 6.3.3. Let $\mathcal{A}$ be the category of left R'-modules for some R-algebra R'. Then a right E-module in $\mathcal{A}$ is the same thing as an (R', E)-bimodule, i.e., a left R'-module with the structure of a right E-module.

Lemma 6.3.4. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R-algebra and p a right E-module in $\mathcal{A}$. Then*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \to R-\mathrm{Mod}$$

can naturally be viewed as a functor to $E-\mathrm{Mod}$.

Proof. For every $q \in \mathcal{A}$, the algebra E operates on $\mathrm{Hom}_{\mathcal{A}}(p, q)$ via functoriality. □

Proposition 6.3.5. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and p a right E-module in $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \longrightarrow E-\mathrm{Mod}$$

has an R-linear left adjoint

$$p \otimes_E \_ : E-\mathrm{Mod} \longrightarrow \mathcal{A}.$$

It is right exact. It satisfies

$$p \otimes_E E = p,$$

and on endomorphisms of E we have (using $\mathrm{End}_E(E) \cong E^{op}$)

$$\begin{array}{cccc} p \otimes_E \_ : & \mathrm{End}_E(E) & \longrightarrow & \mathrm{End}_{\mathcal{A}}(p) \\ & a & \longmapsto & f(a). \end{array}$$

Proof. Right exactness of $p \otimes_E \_$ follows from the universal property. For every $M \in E-\mathrm{Mod}$, the value of $p \otimes_E M$ is uniquely determined by the universal property. In the case of $M = E$, it is satisfied by p itself because we have for all $q \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}}(p, q) = \mathrm{Hom}_E(E, \mathrm{Hom}_{\mathcal{A}}(p, q)).$$

This identification also implies the formula on endomorphisms of $M = E$.

By compatibility with direct sums, this implies that $p \otimes_E E^n = \bigoplus_{i=1}^n p$ for free E-modules. For the same reason, morphisms $E^m \xrightarrow{(a_{ij})_{ij}} E^n$ between free E-modules must be mapped to $\bigoplus_{i=1}^m p \xrightarrow{f(a_{ij})_{ij}} \bigoplus_{i=1}^n p$.

Let M be a finitely presented left E-module. We fix a finite presentation

$$E^{m_1} \xrightarrow{(a_{ij})_{ij}} E^{m_0} \overset{\pi_a}{\twoheadrightarrow} M \to 0.$$

Since $p \otimes_E \_$ preserves cokernels (if it exists), we need to define

$$p \otimes_E M := \mathrm{Coker}(p^{m_1} \xrightarrow{\tilde{A} := f(a_{ij})_{ij}} p^{m_0}).$$

We check that it satisfies the universal property. Indeed, for all $q \in \mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_1}, q) \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_0}, q) \arrow[l] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{A}}(p \otimes M, q) \arrow[l] \arrow[d, dashed] & 0 \arrow[l] \\
\mathrm{Hom}_E(E^{m_1}, \mathrm{Hom}_{\mathcal{A}}(p, q)) & \mathrm{Hom}_E(E^{m_0}, \mathrm{Hom}_{\mathcal{A}}(p, q)) \arrow[l] & \mathrm{Hom}_E(M, \mathrm{Hom}_{\mathcal{A}}(p, q)) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Hence the dashed arrow exists and is an isomorphism.

The universal property implies that $p \otimes_E M$ is independent of the choice of presentation and functorial. We can also make this explicit. For a morphism between arbitrary modules $\varphi : M \to N$ we choose lifts

$$\begin{tikzcd}
E^{m_1} \arrow[r, "A"] \arrow[d, "\varphi^1"] & E^{m_0} \arrow[r, "\pi_A"] \arrow[d, "\varphi^0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
E^{n_1} \arrow[r, "B"] & E^{n_0} \arrow[r, "\pi_B"] & N \arrow[r] & 0.
\end{tikzcd}$$

The respective diagram in $\mathcal{A}$,

$$\begin{tikzcd}
p^{m_1} \arrow[r, "\tilde{A}"] \arrow[d, "\tilde{\varphi^1}"] & p^{m_0} \arrow[r, "\pi_{\tilde{A}}"] \arrow[d, "\tilde{\varphi^0}"] & \mathrm{Coker}(\tilde{A}) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
p^{n_1} \arrow[r, "\tilde{B}"] & p^{n_0} \arrow[r, "\pi_{\tilde{B}}"] & \mathrm{Coker}(\tilde{B}) \arrow[r] & 0.
\end{tikzcd}$$

induces a unique morphism $p \otimes_E \_(\varphi) : p \otimes_E M \to p \otimes_E N$ that keeps the diagram commutative. It is independent of the choice of lifts as different lifts of projective resolutions are homotopic. This finishes the construction. $\square$

Corollary 6.3.6. *Let E be an R-algebra finitely generated as R-module and $\mathcal{A}$ an R-linear abelian category. Let*

$$T : \mathcal{A} \longrightarrow E\text{−Mod}$$

be an exact, R-linear functor into the category of finitely generated E-modules. Further, let p be a right E-module $\mathcal{A}$ with structure given by

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

a morphism of R-algebras. Then the composition

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p) \xrightarrow{T} \mathrm{End}_E(Tp).$$

induces a right action on Tp, making it into an E-bimodule. The composition

$$\begin{array}{ccccc} E-\mathrm{Mod} & \xrightarrow{p\otimes_E \text{-}} & \mathcal{A} & \xrightarrow{T} & E-\mathrm{Mod} \\ M & \mapsto & p\otimes_E M & \mapsto & Tp\otimes_E M \end{array}$$

becomes the usual tensor functor of E-modules.

Proof. It is obvious that the composition

$$\begin{array}{ccccc} E-\mathrm{Mod} & \xrightarrow{p\otimes_E \text{-}} & \mathcal{A} & \xrightarrow{T} & E-\mathrm{Mod} \\ E^n & \mapsto & p\otimes_E E^n & \mapsto & Tp\otimes_E E^n \end{array}$$

induces the usual tensor functor

$$Tp\otimes_E \_ : E-\mathrm{Mod} \longrightarrow E-\mathrm{Mod}$$

on free E-modules. For arbitrary finitely generated E-modules this follows from the fact that $Tp\otimes_E \_$ is right exact and T is exact. □

Remark 6.3.7. Let E be an R-algebra, let M be a right E-module and N be a left E-module. We obtain the tensor product $M\otimes_E N$ by dividing out the equivalence relation $m\cdot e\otimes n \sim m\otimes e\cdot n$ for all $m\in M, n\in N, e\in E$ of the tensor product $M\otimes_R N$ of R-modules. We will now see that a similar approach holds for the abstract tensor products $p\otimes_R M$ and $p\otimes_E M$ in $\mathcal{A}$ as defined in Proposition 6.3.5. For the easier case that R is a field, this approach has been used in [DM].

Lemma 6.3.8. *Let $\mathcal{A}$ be an R-linear, abelian category, E a not necessarily commutative R-algebra which is finitely generated as R-module and $p\in\mathcal{A}$ a right E-module in $\mathcal{A}$. Let $M\in E-\mathrm{Mod}$ and $E'\in E-\mathrm{Mod}$ be in addition a right E-module. Then $p\otimes_E E'$ is a right E-module in $\mathcal{A}$ and we have*

$$p\otimes_E(E'\otimes_E M) = (p\otimes_E E')\otimes_E M.$$

Moreover,

$$(p\otimes_E E)\otimes_R M = p\otimes_R M.$$

Proof. The right E-module structure on $p\otimes_E E'$ is defined by functoriality. The equalities are immediate from the universal property. □

Proposition 6.3.9. *Let $\mathcal{A}$ be an R-linear, abelian category. Let further E be a unital R-algebra with finite generating family $e_1,\dots,e_m$. Let p a right E-module in $\mathcal{A}$ with structure given by*

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p).$$

Let M be a left E-module.

Then $p \otimes_E M$ is isomorphic to the cokernel of the map

$$\Sigma : \bigoplus_{i=1}^{m} (p \otimes_R M) \longrightarrow p \otimes_R M$$

given by

$$\sum_{i=1}^{m} \left(f(e_i) \otimes \mathrm{id}_M - \mathrm{id}_p \otimes e_i \mathrm{id}_M\right) \pi_i$$

with π_i the projection to the i-summand.

More suggestively (even if not quite correct), we write

$$\Sigma : (x_i \otimes v_i)_{i=1}^m \mapsto \sum_{i=1}^{m} (f(e_i)(x_i) \otimes v_i - x_i \otimes (e_i \cdot v_i))$$

for $x_i \in p$ and $v_i \in M$.

Proof. Consider the sequence

$$\bigoplus_{i=1}^{m} E \otimes_R E \longrightarrow E \otimes E \longrightarrow E \longrightarrow 0$$

where the first map is given by

$$(x_i \otimes y_i)_{i=1}^m \mapsto \sum_{i=1}^{m} x_i e_i \otimes y_i - x_i \otimes e_i y_i$$

and the second is multiplication. We claim that it is exact. The sequence is exact in E because E is unital. The composition of the two maps is zero, hence the cokernel maps to E. The elements in the cokernel satisfy the relation $\bar{x}e_i \otimes \bar{y} = \bar{x} \otimes e_i\bar{y}$ for all $\bar{x}, \bar{y}$ and $i = 1, \ldots, m$. The e_i generate E, hence $\bar{x}e \otimes \bar{y} = \bar{x} \otimes e\bar{y}$ for all $\bar{x}, \bar{y}$ and all $e \in E$. Hence the cokernel equals $E \otimes_E E$ which is E via the multiplication map.

Now we tensor the sequence from the left by p and from the right by M and obtain an exact sequence

$$\bigoplus_{i=1}^{m} p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E E \otimes_E M \to 0.$$

Applying the computation rules of Lemma 6.3.8, we get the sequence in the proposition. ☐

Similarly to Proposition 6.3.5 and Corollary 6.3.6, but less general, we construct a contravariant functor $\mathrm{Hom}_R(p, \_)$:

Proposition 6.3.10. *Let $\mathcal{A}$ be an R-linear abelian category, and p be an object of $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(\_, p) : \mathcal{A}^\circ \longrightarrow R\mathrm{-Mod}$$

has a left adjoint

$$\mathrm{Hom}_R(\_, p) : R\mathrm{-Mod} \longrightarrow \mathcal{A}^\circ.$$

This means that for all $M \in R\mathrm{-Mod}$ and $q \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(q, \mathrm{Hom}_R(M, p)) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(q, p)).$$

It is left exact. It satisfies

$$\mathrm{Hom}_R(R, p) = p.$$

If

$$T : \mathcal{A} \longrightarrow R\mathrm{-Mod}$$

is an exact, R-linear functor into the category of finitely generated R-modules then the composition

$$\begin{array}{ccccc} R\mathrm{-Mod} & \overset{\mathrm{Hom}(\_,p)}{\longrightarrow} & \mathcal{A} & \overset{T}{\longrightarrow} & R\mathrm{-Mod} \\ M & \mapsto & \mathrm{Hom}_R(M, p) & \mapsto & \mathrm{Hom}_R(M, Tp) \end{array}$$

is the usual $\mathrm{Hom}(\_, Tp)$-functor in $R\mathrm{-Mod}$.

Proof. The arguments are the same as in the proof of Proposition 6.3.5 and Corollary 6.3.6. □

Remark 6.3.11. Let $\mathcal{A}$ be an R-linear, abelian category. The functors $\mathrm{Hom}_R(\_, p)$ as defined in Proposition 6.3.10 and $p \otimes_R \_$ as defined in Proposition 6.3.6 are also functorial in p, i.e., we have even functors

$$\mathrm{Hom}_R(\_, \_) : (R\mathrm{-Mod})^\circ \times \mathcal{A} \longrightarrow \mathcal{A}$$

and

$$\_ \otimes_R \_ : \mathcal{A} \times R\mathrm{-Mod} \longrightarrow \mathcal{A}.$$

We will denote the image of a morphism $p \xrightarrow{\alpha} q$ under the functor $\mathrm{Hom}_R(M, \_)$ by

$$\mathrm{Hom}_R(M, p) \xrightarrow{\alpha\circ} \mathrm{Hom}_R(M, q)$$

This notation $\alpha\circ$ is natural since by composition

$$\begin{array}{ccccc} \mathcal{A} & \overset{\mathrm{Hom}(M,\_)}{\longrightarrow} & \mathcal{A} & \overset{T}{\longrightarrow} & R\mathrm{-Mod} \\ p & \mapsto & \mathrm{Hom}_R(M, p) & \mapsto & \mathrm{Hom}_R(M, Tp) \end{array}$$

$T(\alpha\circ)$ becomes the usual left action of $T\alpha$ on $\mathrm{Hom}_R(M, Tp)$.

Proof. This follows from the universal property. $\square$

We will now check that the above functors have very similar properties to usual tensor and Hom-functors in $R-$Mod.

Lemma 6.3.12. *Let $\mathcal{A}$ be an R-linear, abelian category and M a finitely generated R-module. Then the functor $\mathrm{Hom}_R(M, \_)$ is right-adjoint to the functor $\_ \otimes_R M$.*

If

$$T : \mathcal{A} \longrightarrow R-\mathrm{Mod}$$

is an R-linear, exact functor into finitely generated R-modules, the composed functors $T \circ \mathrm{Hom}_R(M, \_)$ and $T \circ (\_ \otimes_R M)$ yield the usual hom-tensor adjunction in $R-$Mod.

Proof. The assertion follows from the universal property and the identification $T \circ \mathrm{Hom}_R(M, \_) = \mathrm{Hom}_R(M, T\_)$ in Proposition 6.3.10 and $T \circ \_ \otimes_R M = (T\_) \otimes_R M$ in Proposition 6.3.6. $\square$

6.3.2 Construction of the equivalence

Definition 6.3.13. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a not necessarily abelian subcategory. With $\langle \mathcal{S} \rangle$ we denote the smallest full abelian subcategory of $\mathcal{A}$ containing $\mathcal{S}$, i.e., the intersection of all full subcategories of $\mathcal{A}$ that are abelian, contain $\mathcal{S}$, and for which the inclusion functor is exact.

Lemma 6.3.14. *Let $\mathcal{A} = \langle F \rangle$ for a finite set of objects. Let $T : \langle F \rangle \to R-\mathrm{Mod}$ be a faithful exact functor. Then the inclusion $F \to \langle F \rangle$ induces an equivalence*

$$\mathrm{End}(T|_F)-\mathrm{Mod} \longrightarrow \mathcal{C}(\langle F \rangle, T).$$

Proof. Let $E = \mathrm{End}(T|_F)$. Its elements are tuples of endomorphisms of Tp for $p \in F$ commuting with all morphisms $p \to q$ in F.

We have to show that $E = \mathrm{End}(T)$. In other words, that any element of E defines a unique endomorphism of Tq for all objects q of $\langle F \rangle$ and commutes with all morphisms in $\langle F \rangle$.

Any object q is a subquotient of a finite direct sum of copies of objects $p \in F$. The operation of E on Tp with $p \in F$ extends uniquely to an operation on direct sums, kernels and cokernels of morphisms. It is also easy to see that the action commutes with Tf for all morphisms f between these objects. This means that it extends to all objects $\langle F \rangle$, compatible with all morphisms. $\square$

We first concentrate on the case $\mathcal{A} = \langle p \rangle$. From now on, we abbreviate $\mathrm{End}(T|_{\{p\}})$ by $E(p)$.

Lemma 6.3.15. *Let $\langle p\rangle = \mathcal{A}$ be an abelian category. Let $\langle p\rangle \xrightarrow{T} R\text{−Mod}$ a faithful exact R-linear functor into the category of finitely generated R-modules. Let*

$$\langle p\rangle \xrightarrow{\tilde{T}} E(p)\text{−Mod} \xrightarrow{f_T} R\text{−Mod}$$

be the factorization via the diagram category of T constructed in Proposition 6.2.5. Then:

1. *There exists an object $X(p) \in Ob(\langle p\rangle)$ such that*

$$\tilde{T}(X(p)) = E(p).$$

2. *The object $X(p)$ has a right $E(p)$-module structure in $\mathcal{A}$*

$$E(p)^{op} \to \mathrm{End}_{\mathcal{A}}(X(p))$$

such that the induced $E(p)$-module structure on $E(p)$ is the product.

3. *There is an isomorphism*

$$\tau : X(p) \otimes_{E(p)} \tilde{T}p \to p$$

which is natural in $f \in \mathrm{End}_{\mathcal{A}}(p)$, i.e.,

$$\begin{tikzcd} p \arrow[r, "f"] \arrow[d, "\tau"] & p \arrow[d, "\tau"] \\ X(p) \otimes_{E(p)} \tilde{T}p \arrow[r, "id \otimes \tilde{T}f"] & X(p) \otimes_{E(p)} \tilde{T}p \end{tikzcd}$$

An easier construction of $X(p)$ in the field case can be found in [DM], the construction for R being a noetherian ring is due to Nori [N].

Proof. We consider the object $\mathrm{Hom}_R(Tp, p) \in \mathcal{A}$. Via the contravariant functor

$$\begin{array}{ccc} R\text{−Mod} & \xrightarrow{\mathrm{Hom}(\_,p)} & \mathcal{A} \\ Tp & \mapsto & \mathrm{Hom}_R(Tp, p) \end{array}$$

of Proposition 6.3.10 it is a right $\mathrm{End}_R(Tp)$-module in $\mathcal{A}$ which, after applying T just becomes the usual right $\mathrm{End}(Tp)$-module $\mathrm{Hom}_R(Tp, Tp)$. For each $\varphi \in \mathrm{End}(Tp)$, k we will write $\circ\varphi$ for the action on $\mathrm{Hom}(Tp, p)$ as well. By Lemma 6.3.12 the functors $\mathrm{Hom}_R(Tp, \_)$ and $\_ \otimes_R Tp$ are adjoint, so we obtain an evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp, p) \otimes_R Tp \longrightarrow p$$

that becomes the usual evaluation in $R-$Mod after applying T. Our aim is now to define $X(p)$ as a suitable subobject of $\mathrm{Hom}_R(Tp,p) \in \mathcal{A}$. The structures on $X(p)$ will be induced from the structures on $\mathrm{Hom}_R(Tp,p)$.

Let $M \in R-$Mod. We consider the functor

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Hom}_R(M,\_)} & \mathcal{A} \\ p & \mapsto & \mathrm{Hom}_R(M,p) \end{array}$$

of Remark 6.3.11. The endomorphism ring $\mathrm{End}_{\mathcal{A}}(p)) \subset \mathrm{End}_R(Tp)$ is finitely generated as R-module, since T is faithful and R is noetherian. Let $\alpha_1, ..., \alpha_n$ be a generating family. Since

$$E(p) = \{\varphi \in \mathrm{End}(Tp) | T\alpha \circ \varphi = \varphi \circ T\alpha \ \forall \alpha : p \to p\},$$

we can write $E(p)$ as the kernel of

$$\begin{array}{ccc} \mathrm{Hom}(Tp,Tp) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp,Tp) \\ u & \mapsto & u \circ T\alpha_i - T\alpha_i \circ u \end{array}$$

By the exactness of T, the kernel $X(p)$ of

$$\begin{array}{ccc} \mathrm{Hom}(Tp,p) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp,p) \\ u & \mapsto & u \circ T\alpha_i - \alpha_i \circ u \end{array}$$

is a preimage of $E(p)$ under T in $\mathcal{A}$.

By construction, the right $\mathrm{End}_R(Tp)$-module structure on $\mathrm{Hom}_R(Tp,p)$ restricts to a right $E(p)$-module structure on $X(p)$ whose image under $\tilde{T}$ yields the natural $E(p)$ right-module structure on $E(p)$.

Now consider the evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp,p) \otimes_R Tp \longrightarrow p$$

mentioned at the beginning of the proof. By Proposition 6.3.9 we know that the cokernel of the map Σ defined there is isomorphic to $X(p) \otimes_{E(p)} \tilde{T}p$. The diagram

$$\begin{tikzcd} \bigoplus_{i=1}^k (X(p) \otimes_R Tp) \arrow[r, "\Sigma"] & X(p) \otimes_R Tp \arrow[r, "\mathrm{incl} \otimes id"] \arrow[dr, "\mathrm{Coker}(\Sigma)"] & \mathrm{Hom}_R(Tp,p) \otimes_R Tp \arrow[r, "\tilde{\mathrm{ev}}"] & p \\ & & X(p) \otimes_{E(p)} \tilde{T}p & \end{tikzcd}$$

in $\mathcal{A}$ maps via T to the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{k}(E(p)\otimes_R Tp) \arrow[r, "\Sigma"] & E(p)\otimes_R Tp \arrow[r, "\mathrm{incl}\otimes id"] \arrow[dr, "\mathrm{Coker}(\Sigma)"] & \mathrm{Hom}_R(Tp,Tp)\otimes_R Tp \arrow[r, "\mathrm{ev}"] & Tp \\
 & & E(p)\otimes_{E(p)}\tilde{T}p &
\end{tikzcd}$$

in $R-$Mod, where the composition of the horizontal maps becomes zero. Since T is faithful, the respective horizontal maps in $\mathcal{A}$ are zero as well and induce a map

$$\tau : X(p)\otimes_{E(p)} Tp \longrightarrow p$$

that keeps the diagram commutative. By definition of Σ in Proposition 6.3.9, the respective map

$$\tilde{T}\tau : E(p)\otimes_{E(p)}\tilde{T}p \longrightarrow \tilde{T}p$$

becomes the natural evaluation isomorphism of E-modules. Since $\tilde{T}$ is faithful, τ is an isomorphism as well.

Naturality in f holds since $\tilde{T}$ is faithful and

$$\begin{tikzcd}
\tilde{T}p \arrow[r, "\tilde{T}f"] \arrow[d, "\tilde{T}\tau"] & \tilde{T}p \arrow[d, "\tilde{T}\tau"] \\
E(p)\otimes_{E(p)}\tilde{T}p \arrow[r, "id\otimes\tilde{T}f"] & E(p)\otimes_{E(p)}\tilde{T}p
\end{tikzcd}$$

commutes in $E(p)-$Mod. □

Proposition 6.3.16. *Let $\langle p\rangle = \mathcal{A}$ be an R-linear, abelian category and*

$$\mathcal{A} \xrightarrow{T} R-\mathrm{Mod}$$

be as in Theorem 6.1.19. Let

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A},T) \xrightarrow{f_T} R-\mathrm{Mod}$$

be the factorization of T via its diagram category. Then $\tilde{T}$ is an equivalence of categories with inverse given by $X(p)\otimes_{E(p)}$ _ with $X(p)$ the object constructed in Lemma 6.3.15.

Proof. We have $\mathcal{A} = \langle p\rangle$, thus $\mathcal{C}(\mathcal{A},T) = E(p)-$Mod. Consider the object $X(p)$ of Lemma 6.3.15. It is a right $E(p)$-module in $\mathcal{A}$, in other words

$$\begin{array}{cccc} f: & \big(E(p)\big)^{op} & \longrightarrow & \mathrm{End}_{\mathcal{A}}(X(p)) \\ & \varphi & \longmapsto & \circ\varphi \end{array}$$

We apply Corollary 6.3.6 to $E = E(p)$, the object $X(p)$, the above f and the functor

$$\tilde{T} : \langle p\rangle \longrightarrow E(p)\text{−Mod} .$$

It yields the functor

$$X(p) \otimes_{E(p)} \_ : E(p)\text{−Mod} \longrightarrow \langle p\rangle$$

such that the composition

$$\begin{array}{ccccc} E(p)\text{−Mod} & \stackrel{X(p)\otimes_{E(p)}\_}{\longrightarrow} & \langle p\rangle & \stackrel{\tilde{T}}{\longrightarrow} & E(p)\text{−Mod} \\ M & \longmapsto & X(p)\otimes_{E(p)} M & \mapsto & \tilde{T}(X(p))\otimes_{E(p)} M = E(p)\otimes_{E(p)} M \end{array}$$

becomes the usual tensor product of $E(p)$-modules, and therefore yields the identity functor.

We want to check that $X(p) \otimes_{E(p)} \_$ is a left-inverse functor of $\tilde{T}$ as well. Thus we need to find a natural isomorphism τ, i.e., for all objects $p_1, p_2 \in \mathcal{A}$ we need isomorphisms τ_{p_1}, τ_{p_2} such that for morphisms $f : p_1 \to p_2$ the following diagram commutes:

$$\begin{tikzcd} X(p)\otimes_{E(p)} \tilde{T}p_1 \arrow[r, "id\otimes\tilde{T}f"] \arrow[d, "\tau_{p_1}"] & X(p)\otimes_{E(p)} \tilde{T}p_2 \arrow[d, "\tau_{p_2}"] \\ p_1 \arrow[r, "f"] & p_2 \end{tikzcd}$$

Since the functors T and f_T are faithful and exact, and we have $T = f_t \circ \tilde{T}$, we know that $\tilde{T}$ is faithful and exact as well. We have already shown that $\tilde{T} \circ X(p) \otimes_{E(p)} \_$ is the identity functor. So $X(p) \otimes_{E(p)} \_$ is faithful exact as well. Since $\mathcal{A}$ is generated by p, it suffices to find a natural isomorphism for p and its endomorphisms. This is provided by the isomorphism τ of Lemma 6.3.15. $\square$

Proof of Theorem 6.1.19. If $\mathcal{A}$ is generated by one object p, then the functor $\tilde{T}$ is an equivalence of categories by Proposition 6.3.16. It remains to reduce to this case.

The diagram category $\mathcal{C}(\mathcal{A}, T)$ arises as a direct limit, hence we have

$$2\text{−colim}_{F\subset Ob(\mathcal{A})}\mathrm{End}(T|_F)\text{−Mod}$$

and in the same way we have

$$\mathcal{A} = 2\text{−colim}_{F\subset Ob(\mathcal{A})}\langle F\rangle$$

with F ranging over the system of full subcategories of $\mathcal{A}$ that contain only a finite number of objects. Moreover, by Lemma 6.3.14, we have $\mathrm{End}(T|_F) = \mathrm{End}(T|_{\langle F\rangle})$. Hence it suffices to check equivalence of categories

$$\langle F\rangle \xrightarrow{\tilde{T}|_{\langle F\rangle}} \mathrm{End}(T|_F)\text{−Mod}$$

for all abelian categories that are generated by a finite number of objects.

We now claim that $\langle F\rangle \cong \langle\bigoplus_{p\in F} p\rangle$ are equivalent: indeed, since any endomorphism of $\bigoplus_{p\in F} p$ is of the form $(a_{pq})_{p,q\in F}$ with $a_{pq} : p \to q$, and since F has all finite direct sums, we know that $\langle\bigoplus_{p\in F} p\rangle$ is a full subcategory of $\langle F\rangle$. On the other hand, for any $q, q' \in F$ the inclusion $q \hookrightarrow \bigoplus_{p\in F} p$ is a kernel and the projection $\bigoplus_{p\in F} p \twoheadrightarrow q'$ is a cokernel, so for any set of morphisms $(a_{qq'})_{q,q'\in F}$, the morphism $a_{qq'} : q \to q'$ by composition

$$q \hookrightarrow \bigoplus_{p\in F} \xrightarrow{(a_{pp'})_{p,p'\in F}} \bigoplus_{p'\in F} p' \twoheadrightarrow q'$$

is contained in $\langle\bigoplus_{p\in F} p\rangle$. Thus $\langle F\rangle$ is a full subcategory of $\langle\bigoplus_{p\in F} p\rangle$.

Similarly one sees that $\mathrm{End}(T|_{\{p\}})$−Mod is equivalent to $\mathrm{End}(T|_F)$−Mod. So we can even assume that our abelian category is generated by just one object $q = \bigoplus_{p\in F} p$. □

6.3.3 Examples and applications

We work out a couple of explicit examples in order to demonstrate the strength of Theorem 6.1.19. We also use the arguments of the proof to deduce an additional property of the diagram property as a first step towards its universal property.

Throughout let R be a noetherian unital ring.

Example 6.3.17. Let $T : R\text{−Mod} \to R\text{−Mod}$ be the identity functor viewed as a representation. Note that R−Mod is generated by the object R^n. By Theorem 6.1.19 and Lemma 6.3.14, we have equivalences of categories

$$\mathrm{End}(T|_{\{R^n\}})\text{−Mod} \longrightarrow \mathcal{C}(R\text{−Mod}, T) \longrightarrow R\text{−Mod}.$$

By definition, $E = \mathrm{End}(T|_{\{R^n\}})$ consists of those elements of $\mathrm{End}_R(R^n)$ which commute with all elements of $\mathrm{End}_{\mathcal{A}}(R^n)$, i.e., the center of the matrix algebra, which is R.

This can be made more interesting by playing with the representation.

Example 6.3.18 (Morita equivalence)**.** Let R be a noetherian commutative unital ring, $\mathcal{A} = R$−Mod. Let P be a flat finitely generated R-module and

$$T : R\text{−Mod} \longrightarrow R\text{−Mod}, \quad M \mapsto M \otimes_R P.$$

It is faithful and exact, hence the assumptions of Theorem 6.1.19 are satisfied and we get an equivalence

$$\mathcal{C}(R\text{−Mod}, T) \longrightarrow R\text{−Mod} .$$

Note that $\mathcal{A} = \langle R\rangle$ and hence by Lemma 6.3.14, $\mathcal{C}(R\text{−Mod}, T) = E\text{−Mod}$ with $E = \mathrm{End}_R(T|_{\{R\}}) = \mathrm{End}_R(P)$. Hence we have shown that

$$\mathrm{End}_R(P)\text{−Mod} \to R\text{−Mod}$$

is an equivalence of categories. This is a case of Morita equivalence of categories of modules.

Example 6.3.19. Let R be a noetherian commutative unital ring and E an R-algebra finitely generated as an R-module. Let

$$T : E\text{−Mod} \to R\text{−Mod}$$

be the forgetful functor. The category E−Mod is generated by the module E. Hence by Theorem 6.1.19 and Lemma 6.3.14, we have again equivalences of categories

$$E'\text{−Mod} \longrightarrow \mathcal{C}(E\text{−Mod}, T) \longrightarrow E\text{−Mod},$$

where $E' = \mathrm{End}(T|_{\{E\}})$ is the subalgebra of $\mathrm{End}_R(E)$ of endomorphisms compatible with all E-morphisms $E \to E$. Note that $\mathrm{End}_E(E) = E^{op}$ and hence E' is the centralizer of E^{op} in $\mathrm{End}_R(E)$

$$E' = C_{\mathrm{End}_R(E)}(E^{op}) = E .$$

Hence in this case the functor $\mathcal{A} \to \mathcal{C}(\mathcal{A}, T)$ is the identity.

We deduce another consequence of the explicit description of $\mathcal{C}(D, T)$.

Proposition 6.3.20. *Let D be a diagram and $T : D \to R$−Mod a representation. Let*

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R\text{−Mod}$$

its factorization. Then the category $\mathcal{C}(D, T)$ is generated by the image of $\tilde{T}$:

$$\mathcal{C}(D, T) = \langle \tilde{T}(D)\rangle .$$

Proof. It suffices to consider the case when D is finite. Let $X = \bigoplus_{p\in D} Tp$ and $\mathbb{E} = \mathrm{End}_R(X)$. Let $S \subset \mathbb{E}$ be the R-subalgebra generated by Te for $e \in E(D)$ and the projectors $p_p : X \to T(p)$. Then

$$E = \mathrm{End}(T) = C_{\mathbb{E}}(S)$$

is the commutator of S in $\mathbb{E}$. (The endomorphisms commuting with the projectors are those respecting the decomposition. By definition, $\mathrm{End}(T)$ consists of those endomorphisms of the summands commuting with all Te.)

By construction $\mathcal{C}(D,T) = E-\mathrm{Mod}$. We claim that it is equal to

$$\mathcal{A} := \langle\{\tilde{T}p | p \in D\}\rangle = \langle \tilde{X} \rangle$$

with $\tilde{X} = \bigoplus_{p\in D} \tilde{T}p$. The category has a faithful exact representation by $f_T|_{\mathcal{A}}$. Note that $f_T(\tilde{X}) = X$. By Theorem 6.1.19, the category $\mathcal{A}$ is equivalent to its diagram category $\mathcal{C}(\langle\tilde{X}\rangle, f_T) = E'-\mathrm{Mod}$ with $E' = \mathrm{End}(f_T|_{\mathcal{A}})$. By Lemma 6.3.14, E' consists of elements of $\mathbb{E}$ commuting with all elements of $\mathrm{End}_{\mathcal{A}}(\tilde{X})$. Note that

$$\mathrm{End}_{\mathcal{A}}(\tilde{X}) = \mathrm{End}_E(X) = C_{\mathbb{E}}(E)$$

and hence

$$E' = C_{\mathbb{E}}(C_{\mathbb{E}}(E)) = C_{\mathbb{E}}(C_{\mathbb{E}}(C_{\mathbb{E}}(S)) = C_{\mathbb{E}}(S)$$

because a triple commutator equals the simple commutator. We have shown $E = E'$ and the two categories are equivalent.

□

Remark 6.3.21. This is a direct proof of Proposition 6.1.15.

6.4 Universal property of the diagram category

At the end of this section we will be able to establish the universal property of the diagram category.

Let $T : D \longrightarrow R-\mathrm{Mod}$ be a diagram and

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R-\mathrm{Mod}$$

the factorization of T via its diagram category. Let $\mathcal{A}$ be another R-linear abelian category, $F : D \to \mathcal{A}$ a representation, and $f : \mathcal{A} \to R-\mathrm{Mod}$ a faithful, exact, R-linear functor into the categories of finitely generated R-modules such that $f \circ F = T$.

Our aim is to deduce that there exists - uniquely up to isomorphism - an R-linear exact faithful functor

$$L(F) : \mathcal{C}(D,T) \to \mathcal{A},$$

making the following diagram commute:

$$\begin{tikzcd}
 & D \arrow[dl, "\tilde{T}"'] \arrow[dr, "F"] & \\
\mathcal{C}(D,T) \arrow[rr, dashed, "\exists! L(F)"] \arrow[dr, "f_T"'] & & \mathcal{A} \arrow[dl, "T_{\mathcal{A}}"] \\
 & R-\mathrm{Mod} &
\end{tikzcd}$$

Proposition 6.4.1. *There is a functor $L(F)$ making the diagram commute.*

Proof. We can regard $\mathcal{A}$ as a diagram and obtain a representation

$$\mathcal{A} \xrightarrow{T_{\mathcal{A}}} R\text{−Mod},$$

that factorizes via its diagram category

$$\mathcal{A} \xrightarrow{\tilde{T}_{\mathcal{A}}} \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \xrightarrow{f_{T_{\mathcal{A}}}} R\text{−Mod}.$$

We obtain the following commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}_D"] \arrow[ddr, "T"] & & \mathcal{A} \arrow[d, "\tilde{T}_{\mathcal{A}}"] \arrow[ddl, "T_{\mathcal{A}}"'] \\
\mathcal{C}(D,T) \arrow[dr, "f_T"] & & \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \arrow[dl, "f_{T_{\mathcal{A}}}"'] \\
& R\text{−Mod} &
\end{tikzcd}$$

By functoriality of the diagram category (see Proposition 6.2.6) there exists an R-linear faithful exact functor $\mathcal{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}_D"] & & \mathcal{A} \arrow[d, "\tilde{T}_{\mathcal{A}}"] \\
\mathcal{C}(D,T) \arrow[rr, dashed, "\mathcal{F}"] \arrow[dr, "f_T"] & & \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \arrow[dl, "f_{T_{\mathcal{A}}}"'] \\
& R\text{−Mod} &
\end{tikzcd}$$

Since $\mathcal{A}$ is R-linear, abelian, and T is faithful, exact, R-linear, we know by Proposition 6.1.19, that $\tilde{T}_{\mathcal{A}}$ is an equivalence of categories. The functor

$$L(F) : \mathcal{C}(D, T) \to \mathcal{A},$$

is given by the composition of $\mathcal{F}$ with the inverse of $\tilde{T}_{\mathcal{A}}$. Since an equivalence of R-linear categories is exact, faithful and R-linear, $L(F)$ is so as well, as it is the composition of such functors. □

Proposition 6.4.2. *The functor $L(F)$ is unique up to unique isomorphism.*

Proof. Let L' be another functor satisfying the condition in the diagram. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}(D,T)$ on which $L' = L(F)$. We claim that the inclusion is an equivalence of categories. Without loss of generality, we may assume D is finite.

Note that the subcategory is full because $f : \mathcal{A} \to R\text{−Mod}$ is faithful. It contains all objects of the form $\tilde{T}p$ for $p \in D$. As the functors are additive, this implies that they also have to agree (up to canonical isomorphism) on finite direct sums of objects. As the functors are exact, they also have to agree on and all kernels and cokernels. Hence $\mathcal{C}'$ is the full abelian subcategory of $\mathcal{C}(D,T)$ generated by $\tilde{T}(D)$. By Proposition 6.3.20 this is all of $\mathcal{C}(D,T)$. □

Proof of Theorem 6.1.13. Let $T : D \to R\text{−Mod}$ be a representation and $f : \mathcal{A} \to R\text{−Mod}$, $F : D \to \mathcal{A}$ as in the statement. By Proposition 6.4.1 the functor $L(F)$ exists. It is unique by Proposition 6.4.2. Hence $\mathcal{C}(D,T)$ satisfies the universal property of Theorem 6.1.13.

Let $\mathcal{C}$ be another category satisfying the universal property. By the universal property for $\mathcal{C}(D,T)$ and the representation of D in $\mathcal{C}$, we get a functor $\Psi : \mathcal{C}(D,T) \to \mathcal{C}$. By interchanging their roles, we obtain a functor Ψ' in the opposite direction. Their composition $\Psi' \circ \Psi$ satisfies the universal property for $\mathcal{C}(D,T)$ and the representation $\tilde{T}$. By the uniqueness part, it is isomorphic to the identity functor. The same argument also applies to $\Psi \circ \Psi'$. Hence Ψ is an equivalence of categories.

Functoriality of $\mathcal{C}(D,T)$ in D is Lemma 6.2.6. □

The generalized universal property follows by a trick.

Proof of Corollary 6.1.14. Let $T : D \to R\text{−Mod}$, $f : \mathcal{A} \to R\text{−Mod}$ und $F : D \to \mathcal{A}$ be as in the corollary. Let S be a faithfully flat R-algebra and

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S\text{−Mod}$. We first show the existence of $L(F)$.

Let $\mathcal{A}'$ be the category with objects of the form (V_1, V_2, ψ) where $V_1 \in R\text{−Mod}$, $V_2 \in \mathcal{A}$ and $\psi : V_1 \otimes_R S \to f(V_2) \otimes_R S$ an isomorphism. Morphisms are defined as pairs of morphisms in $R\text{−Mod}$ and $\mathcal{A}$ such the obvious diagram commutes. This category is abelian because S is flat over R. Kernels and cokernels are taken componentwise. Let $f' : \mathcal{A}' \to R\text{−Mod}$ be the projection to the first component. It is faithful and exact because S is faithfully flat over R.

The data T, F and ϕ define a representation $F' : D \to \mathcal{A}'$ compatible with T. By the universal property of Theorem 6.1.13, we obtain a factorization

$$F' : D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{L(F')} \mathcal{A}' \ .$$

We define $L(F)$ as the composition of $L(F')$ with the projection to the second component. The transformation

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

is defined on $X \in \mathcal{C}(D,T)$ using the isomorphism ψ part of the object $L(F')(X) \in \mathcal{A}'$.

Conversely, the triple $(f, L(F), \tilde{\phi})$ satisfies the universal property of $L(F')$. By the uniqueness part of the universal property, this means that it agrees with $L(F')$. This makes $L(F)$ unique. □

6.5 The diagram category as a category of comodules

Under more restrictive assumptions on R and T, we can give a description of the diagram category of comodules, see Theorem 6.1.12.

6.5.1 Preliminary discussion

In [DM] Deligne and Milne note that if R is a field, E a finite-dimensional R-algebra, and V an E-module that is finite-dimensional as R-vector space then V has a natural structure as comodule over the coalgebra $E^\vee := \mathrm{Hom}_R(E,R)$. For an algebra E finitely generated as an R-module over an arbitrary noetherian ring R, the R-dual $E^\vee$ does not even necessarily carry a natural structure of an R-coalgebra. The problem is that the dual map to the algebra multiplication

$$E^\vee \xrightarrow{\mu^*} (E \otimes_R E)^\vee$$

does not generally define a comultiplication because the canonical map

$$\rho : E^\vee \otimes_R E^\vee \to \mathrm{Hom}(E, E^\vee) \cong (E \otimes_R E)^\vee$$

fails to be an isomorphism in general. In this chapter we will see that this isomorphism holds true for the R-algebras $\mathrm{End}(T_F)$ if we assume that R is a Dedekind domain or field. We will then show that by

$$\begin{aligned}\mathcal{C}(D,T) &= 2\text{−}\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)\text{−}\mathrm{Mod})\\ &= 2\text{−}\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)^\vee\text{−}\mathrm{Comod}) = (2\text{−}\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee)\text{−}\mathrm{Comod}\end{aligned}$$

we can view the diagram category $\mathcal{C}(D,T)$ as the category of finitely generated comodules over the coalgebra $2\text{−}\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee$.

Remark 6.5.1. Note that the category of comodules over an arbitrary coalgebra C is not abelian in general, since the tensor product $X \otimes_R -$ is right exact, but in general not left exact. If C is flat as R-algebra (e.g. free), then the category of C-comodules is abelian [MM, pg. 219].

6.5.2 Coalgebras and comodules

Let R be a noetherian ring with unit.

Proposition 6.5.2. *Let E be an R-algebra which is finitely generated as R-module. Then the canonical map*

$$\begin{array}{rccc} \rho: & E^\vee \otimes_R M & \to & \mathrm{Hom}(E, M) \\ & \varphi \otimes m & \mapsto & (n \mapsto \varphi(n) \cdot m) \end{array}$$

becomes an isomorphism for all R-modules M if and only if E is projective.

Proof. [Str, Proposition 5.2] □

Lemma 6.5.3. *Let E be an R-algebra which is finitely generated and projective as an R-module.*

1. *The R-dual module $E^\vee$ carries a natural structure of a counital coalgebra.*
2. *Any left E-module that is finitely generated as R-module carries a natural structure as left $E^\vee$-comodule.*
3. *We obtain an equivalence of categories between the category of finitely generated left E-modules and the category of finitely generated left $E^\vee$-comodules.*

Proof. By the repeated application of Proposition 6.5.2, this becomes a straightforward calculation. We will sketch the main steps of the proof.

1. If we dualize the associativity constraint of E we obtain a commutative diagram of the form

$$\begin{tikzcd} (E \otimes_R E \otimes_R E)^\vee & (E \otimes_R E)^\vee \arrow[l, "(\mu \otimes id)^*"'] \\ (E \otimes_R E)^\vee \arrow[u, "(id \otimes \mu)^*"] & E^\vee . \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \end{tikzcd}$$

   By the use of the isomorphism in Propostion 6.5.2 and Hom-Tensor adjunction we obtain the commutative diagram

$$\begin{tikzcd} E^\vee \otimes_R E^\vee \otimes_R E^\vee & E^\vee \otimes_R E^\vee \arrow[l, "\mu^* \otimes id^*"'] \\ E^\vee \otimes_R E^\vee \arrow[u, "id^* \otimes \mu^*"] & E^\vee , \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \end{tikzcd}$$

which induces a cocommutative comultiplication on $E^\vee$. Similarly we obtain the counit diagram, so $E^\vee$ naturally gets a coalgebra structure.

2. For an E-module M we analogously dualize the respective diagram

$$\begin{tikzcd} M & E \otimes_R M \arrow[l, "m"'] \\ E \otimes_R M \arrow[u, "m"] & E \otimes_R E \otimes_R M \arrow[l, "\mu \otimes id"] \arrow[u, "id \otimes m"'] \end{tikzcd}$$

and use Proposition 6.5.2 and Hom-Tensor adjunction to see that the E-multiplication induces a well-defined $E^\vee$-comultiplication

$$\begin{tikzcd} M \arrow[r, "\hat{m}"] \arrow[d, "\hat{m}"] & E^\vee \otimes_R M \arrow[d, "id \otimes \hat{m}"] \\ E^\vee \otimes_R M \arrow[r, "\mu^* \otimes id"] & E^\vee \otimes_R E^\vee \otimes_R M \end{tikzcd}$$

on M.

3. For any homomorphism $f : M \longrightarrow N$ of left E-modules, the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ E \otimes_R M \arrow[u, "\mu"] \arrow[r, "id \otimes f"] & E \otimes_R N \arrow[u, "\mu"'] \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} E^\vee \otimes_R M \arrow[r, "id \otimes f"] & E^\vee \otimes_R N, \\ M \arrow[u, "\hat{\mu}"] \arrow[r, "f"] & N \arrow[u, "\hat{\mu}"'] \end{tikzcd}$$

thus f is a homomorphism of left $E^\vee$-comodules.

4. Conversely, we can dualize the $E^\vee$-comodule structure to obtain a $(E^\vee)^\vee = E$-module structure. The two constructions are inverse to each other.

□

Definition 6.5.4. Let A be a coalgebra over R. Then we denote by $A-$Comod the category of comodules over A that are finitely generated as a R-modules.

Recall that $R-$Proj denotes the category of finitely generated projective R-modules.

Corollary 6.5.5. *Let R be a field or Dedekind domain, D a diagram and*

$$T : D \longrightarrow R-\mathrm{Proj}$$

a representation. Set $A(D,T) := \varinjlim_{F\subset D\ finite} \mathrm{End}(T_F)^\vee$. Then $A(D,T)$ has the structure of a coalgebra and the diagram category of T is the abelian category $A(D,T)-$Comod.

Proof. For any finite subset $F \subset D$ the algebra $\mathrm{End}(T_F)$ is a submodule of the finitely generated projective R-module $\prod_{p\in F}\mathrm{End}(T_p)$. Since R is a field or Dedekind domain, for a finitely generated module to be projective is equivalent to being torsion free. Hence the submodule $\mathrm{End}(T_F)$ is also finitely generated and torsion-free, or equivalently, projective. By the previous lemma, $\mathrm{End}(T_F)^\vee$ is an R-coalgebra and $\mathrm{End}(T_F)-\mathrm{Mod} \cong \mathrm{End}(T_F)^\vee-\mathrm{Comod}$. From now on, we denote $\mathrm{End}(T_F)^\vee$ with $A(F,T)$. Taking limits over the direct system of finite subdiagrams as in Definition 6.1.10, we obtain

$$\begin{aligned}\mathcal{C}(D,T) &:= 2-\mathrm{colim}_{F\subset D\ \mathrm{finite}}\mathrm{End}(T_F)-\mathrm{Mod}\\ &= 2-\mathrm{colim}_{F\subset D\ \mathrm{finite}}A(F,T)-\mathrm{Comod}.\end{aligned}$$

Since the category of coalgebras is cocomplete, $A(D,T) = \varinjlim_{F\subset D} A(F,T)$ is a coalgebra as well.

We now need to show that the categories $2-\mathrm{colim}_{F\subset D\ \mathrm{finite}}(A(F,T)-\mathrm{Comod})$ and $A(D,T)-$Comod are equivalent. For any finite F the canonical map $A(F,T) \longrightarrow A(D,T)$ via restriction of scalars induces a functor

$$\phi_F : A(F,T)-\mathrm{Comod} \longrightarrow A(D,T)-\mathrm{Comod}$$

and therefore by the universal property a unique functor

$$u : \varinjlim A(F,T)-\mathrm{Comod} \longrightarrow A(D,T)-\mathrm{Comod}.$$

such that for all finite $F', F'' \subset D$ with $F' \subset F''$ and the canonical functors

$$\psi_F : A(F',T)-\mathrm{Comod} \longrightarrow \varinjlim_{F\subset D} A(F,T)-\mathrm{Comod}$$

the following diagram commutes:

$$\begin{tikzcd}
A(F',T)-\mathrm{Comod} \arrow[rr, "\phi_{F'F''}"] \arrow[dr, "\psi_{F'}"] \arrow[ddr, "\phi_{F'}"'] & & A(F'',T)-\mathrm{Comod} \arrow[dl, "\psi_{F''}"'] \arrow[ddl, "\phi_{F''}"] \\
& \varinjlim_{F\subset D}(A(F,T)-\mathrm{Comod}) \arrow[d, "\exists! u"] & \\
& A(D,T)-\mathrm{Comod} &
\end{tikzcd}$$

We construct an inverse map to u: Let M be an $A(D,T)$-comodule and

$$m : M \to M \otimes_R A(D,T)$$

be the comultiplication. Let $M = \langle x_1, .., x_n \rangle_R$. Then $m(x_i) = \sum_{k=1}^{n} a_{ki} \otimes x_k$ for certain $a_{ki} \in A(D,T)$. Every a_{ki} is already contained in an $A(F,T)$ for sufficiently large F. By taking the union of these finitely many F, we can assume that all a_{ki} are contained in one coalgebra $A(F,T)$. Since $x_1, .., x_n$ generate M as R-module, m defines a comultiplication

$$\tilde{m} : M \to M \otimes_R A(F,T).$$

So M is an $A(F,T)$-comodule in a natural way, thus via ψ_F an object of $2-\mathrm{colim}_I(A_i-\mathrm{Comod})$. □

We also need to understand the behavior of $A(D,T)$ under base-change.

Lemma 6.5.6 (Base change)**.** *Let R be a field or a Dedekind domain and $T : D \to R-\mathrm{Proj}$ a representation. Let $R \to S$ be flat. Then*

$$A(D, T_S) = A(D,T) \otimes_R S \ .$$

Proof. Let $F \subset D$ be a finite subdiagram. Recall that

$$A(F,T) = \mathrm{Hom}_R(\mathrm{End}(T|_F), R) \ .$$

Both R and $\mathrm{End}_R(T|_F)$ are projective because R is a field or a Dedekind domain. Hence by Lemma 6.2.2

$$\mathrm{Hom}_R(\mathrm{End}_R(T|_F), R) \otimes S \cong \mathrm{Hom}_S(\mathrm{End}_R(T|_F) \otimes S, S) \cong \mathrm{Hom}_S(\mathrm{End}_S((T_S)|_F), S).$$

This is nothing but $A(F, T_S)$. Tensor products commute with direct limits, hence the statement for $A(D,T)$ follows immediately. □

Chapter 7

More on diagrams

We study additional structures on a diagram and a representation that lead to the construction of a tensor product on the diagram category. The aim is then to turn it into a rigid tensor category with a faithful exact functor to a category of R-modules. The chapter is formal, but the assumptions are tailored to the application to Nori motives.

A particularly puzzling and subtle question is how the question of graded commutativity of the Künneth formula is dealt with.

We continue to work in the setting of Chapter 6.

7.1 Multiplicative structure

Let R a fixed noetherian unital commutative ring.

Recall that $R-$Proj is the category of projective R-modules of finite type over R. We only consider representations $T : D \longrightarrow R-$Proj where D is a diagram with identities, see Definition 6.1.1.

Definition 7.1.1. Let D_1, D_2 be diagrams with identities. Then $D_1 \times D_2$ is defined as the diagram with vertices of the form (v, w) for v a vertex of D_1, w a vertex of D_2, and with edges of the form (α, id) and (id, β) for α an edge of D_1 and β an edge of D_2 and with $\mathrm{id} = (\mathrm{id}, \mathrm{id})$.

Remark 7.1.2. Levine in [L1] p.466 seems to define $D_1 \times D_2$ by taking the product of the graphs in the ordinary sense. He claims (in the notation of loc. cit.) a map of diagrams

$$H_*\mathrm{Sch}' \times H_*\mathrm{Sch}' \to H_*\mathrm{Sch}'.$$

It is not clear to us how this is defined on general pairs of edges. If α, β are edges corresponding to boundary maps and hence lower the degree by 1, then we would expect $\alpha \times \beta$ to lower the degree by 2. However, there are no such edges in $H_*\mathrm{Sch}'$.

Our restricted version of products of diagrams is enough to get the implications we want.

In order to control signs in the Künneth formula, we need to work in a graded commutative setting.

Definition 7.1.3. A *graded diagram* is a diagram D with identities together with a map

$$|\cdot| : \{\text{vertices of } D\ \} \to \mathbb{Z}/2\mathbb{Z}\ .$$

For an edge $\gamma : v \to v'$ we put $|\gamma| = |v| - |v'|$. If D is a graded diagram, $D \times D$ is equipped with the grading $|(v,w)| = |v| + |w|$.

A *commutative product structure* on a graded D is a map of graded diagrams

$$\times : D \times D \to D$$

together with choices of edges

$$\begin{aligned} \alpha_{v,w} &: v \times w \to w \times v \\ \beta_{v,w,u} &: v \times (w \times u) \to (v \times w) \times u \\ \beta'_{v,w,u} &: (v \times w) \times u \to v \times (w \times u) \end{aligned}$$

for all vertices v, w, h of D.

A *graded multiplicative representation* T of a graded diagram with commutative product structure is a representation of T in $R-$Proj together with a choice of isomorphism

$$\tau_{(v,w)} : T(v \times w) \to T(v) \otimes T(w)$$

such that:

1. The composition

$$T(v) \otimes T(w) \xrightarrow{\tau_{(v,w)}^{-1}} T(v \times w) \xrightarrow{T(\alpha_{v,w})} T(w \times v) \xrightarrow{\tau_{(w,v)}} T(w) \otimes T(v)$$

   is $(-1)^{|v||w|}$ times the natural map of R-modules.

2. If $\gamma : v \to v'$ is an edge, then the diagram

$$\begin{tikzcd} T(v \times w) \arrow[r, "T(\gamma\times\mathrm{id})"] \arrow[d, "\tau"'] & T(v' \times w) \arrow[d, "\tau"] \\ T(v) \otimes T(w) \arrow[r, "(-1)^{|\gamma||w|}T(\gamma)\otimes\mathrm{id}"] & T(v') \otimes T(w) \end{tikzcd}$$

   commutes.

3. If $\gamma : v \to v'$ is an edge, then the diagram

$$\begin{array}{ccc} T(w \times v) & \xrightarrow{T(\mathrm{id}\times\gamma)} & T(w \times v') \\ \tau\downarrow & & \downarrow\tau \\ T(w) \otimes T(v) & \xrightarrow{\mathrm{id}\otimes T(\gamma)} & T(w) \otimes T(v') \end{array}$$

   commutes.

4. The diagram

$$\begin{array}{ccc} T(v \times (w \times u)) & \xrightarrow{T(\beta_{v,w,u})} & T((v \times w) \times u) \\ \downarrow & & \downarrow \\ T(v) \otimes T(w \times u) & & T(v \times w) \otimes T(u) \\ \downarrow & & \downarrow \\ T(v) \otimes (T(w) \otimes T(u)) & \longrightarrow & (T(v) \otimes T(w)) \otimes T(u) \end{array}$$

   commutes under the standard identification

$$T(v) \otimes (T(w) \otimes T(u)) \cong (T(v) \otimes T(w)) \otimes T(u).$$

   The maps $T(\beta_{v,w,u})$ and $T(\beta'_{v,w,u})$ are inverse to each other.

A *unit* for a graded diagram with commutative product structure D is a vertex $\mathbf{1}$ of degree 0 together with a choice of edges

$$u_v : v \to \mathbf{1} \times v$$

for all vertices of v. A graded multiplicative representation is *unital* if $T(u_v)$ is an isomorphism for all vertices v.

Remark 7.1.4. 1. In particular, $T(\alpha_{v,w})$ and $T(\beta_{v,w,u})$ are isomorphisms. If $v = w$ then $T(\alpha_{v,v}) = (-1)^{|v|}$. If $\mathbf{1}$ is a unit, then $T(\mathbf{1})$ satisfies $T(\mathbf{1}) \cong T(\mathbf{1}) \otimes T(\mathbf{1})$. Hence it is a free R-module of rank 1.

2. Note that the first and the second factor are *not* treated symmetrically. There is a choice of sign convention involved. The convention above is chosen to be conform with the one of Section 1.3. Eventually, we want to view relative singular cohomology as graded multiplicative representation in the above sense.

3. For the purposes immediately at hand, the choice of $\beta'_{v,w,u}$ is not needed. However, it is needed later on in the definition of the product structure on the localized diagram, see Remark 7.2.2.

Let $T : D \longrightarrow R-\mathrm{Proj}$ be a representation of a diagram with identities. Recall that we defined its diagram category $\mathcal{C}(D,T)$ (see Definition 6.1.10). It R is a field or a Dedekind domain, then $\mathcal{C}(D,T)$ can be described as the category of $A(D,T)$-comodules of finite type over R for the coalgebra $A(D,T)$ defined in Theorem 6.1.12.

Proposition 7.1.5. *Let D be a graded diagram with commutative product structure with unit and T a unital graded multiplicative representation of D in $R-\mathrm{Proj}$*

$$T : D \longrightarrow R-\mathrm{Proj}.$$

1. *Then $\mathcal{C}(D,T)$ carries the structure of a commutative and associative tensor category with unit and $T : \mathcal{C}(D,T) \to R-\mathrm{Mod}$ is a tensor functor. On the generators $\tilde{T}(v)$ of $\mathcal{C}(D,T)$ the associativity constraint is induced by the edges $\beta_{v,w,u}$, the commutativity constraint is induced by the edges $\alpha_{v,w}$, the unit object is $\tilde{\mathbf{1}}$ with unital maps induced from the edges u_v.*

2. *If, in addition, R is a field or a Dedekind domain, the coalgebra $A(D,T)$ carries a natural structure of commutative bialgebra (with unit and counit).*

The unit object is going to be denoted $\mathbf{1}$.

Proof. We consider finite diagrams F and F' such that

$$\{v \times w | v, w \in F\} \subset F' \ .$$

We are going to define natural maps

$$\mu_F^* : \mathrm{End}(T|_{F'}) \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F).$$

Assume this for a moment. Let $X, Y \in \mathcal{C}(D,T)$. We want to define $X \otimes Y$ in $\mathcal{C}(D,T) = 2-\mathrm{colim}_F \mathcal{C}(F,T)$. Let F such that $X, Y \in \mathcal{C}(F,T)$. This means that X and Y are finitely generated R-modules with an action of $\mathrm{End}(T|_F)$. We equip the R-module $X \otimes Y$ with a structure of $\mathrm{End}(T|_{F'})$-module. It is given by

$$\mathrm{End}(T|_{F'}) \otimes X \otimes Y \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F) \otimes X \otimes Y \to X \otimes Y$$

where we have used the comultiplication map μ_F^* and the module structures of X and Y. This will be independent of the choice of F and F'. Properties of $\otimes$ on $\mathcal{C}(D,T)$ follow from properties of μ_F^*.

If R is a field or a Dedekind domain, let

$$\mu_F : A(F,T) \otimes A(F,T) \to A(F',T)$$

be dual to μ_F^*. Passing to the direct limit defines a multiplication μ on $A(D,T)$.

We now turn to the construction of μ_F^*. Let $a \in \mathrm{End}(T|_{F'})$, i.e., a compatible system of endomorphisms $a_v \in \mathrm{End}(T(v))$ for $v \in F'$. We describe its image $\mu_F^*(a)$. Let $(v,w) \in F \times F$. The isomorphism

$$\tau : T(v \times w) \to T(v) \otimes T(w)$$

induces an isomorphism

$$\mathrm{End}(T(v \times w)) \cong \mathrm{End}(T(v)) \otimes \mathrm{End}(T(w)).$$

We define the (v,w)-component of $\mu^*(a)$ by the image of $a_{v\times w}$ under this isomorphism.

In order to show that this is a well-defined element of $\mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F)$, we need to check that diagrams of the form

$$\begin{tikzcd}
T(v) \otimes T(w) \arrow[r, "\mu^*(a)_{(v,w)}"] \arrow[d, "T(\alpha)\otimes T(\beta)"'] & T(v) \otimes T(w) \arrow[d, "T(\alpha)\otimes T(\beta)"] \\
T(v') \otimes T(w') \arrow[r, "\mu^*(a)_{(v',w')}"'] & T(v') \otimes T(w')
\end{tikzcd}$$

commute for all edges $\alpha : v \to v'$, $\beta : w \to w'$ in F. We factor

$$T(\alpha) \otimes T(\beta) = (T(\mathrm{id}) \otimes T(\beta)) \circ (T(\alpha) \circ T(\mathrm{id}))$$

and check the factors separately.

Consider the diagram

$$\begin{tikzcd}
T(v \times w) \arrow[rrr, "a_{v\times w}"'] \arrow[dr, "\tau"] \arrow[ddd, "T(\alpha\times\mathrm{id})"'] & & & T(v \times w) \arrow[dl, "\tau"'] \arrow[ddd, "T(\alpha\times\mathrm{id})"] \\
& T(v) \otimes T(w) \arrow[r, "\mu^*(a)_{(v,w)}"] \arrow[d, "T(\alpha)\otimes T(\mathrm{id})"'] & T(v) \otimes T(w) \arrow[d, "T(\alpha)\otimes T(\mathrm{id})"] & \\
& T(v') \otimes T(w) \arrow[r, "\mu^*(a)_{(v',w)}"'] & T(v') \otimes T(w) & \\
T(v' \times w) \arrow[rrr, "a_{v'\times w}"] \arrow[ur, "\tau"] & & & T(v' \times w) \arrow[ul, "\tau"']
\end{tikzcd}$$

The outer square commutes because a is a diagram endomorphism. Top and bottom commute by definition of $\mu^*(a)$. Left and right commute by property (3) up to the same sign $(-1)^{|w||\alpha|}$. Hence the middle square commutes without signs. The analogous diagram for $\mathrm{id} \times \beta$ commutes on the nose. Hence $\mu^*(a)$ is well-defined.

We now want to compare the (v,w)-component to the (w,v)-component. Recall that there is a distinguished edge $\alpha_{v,w} : v \times w \to w \times v$. Consider the diagram

$$\begin{tikzcd}
 & T(v)\otimes T(w) \arrow[r, "\mu^*(a)_{(v,w)}"] \arrow[ddd] & T(v)\otimes T(w) \arrow[ddd] & \\
T(v\times w) \arrow[ur, "\tau"] \arrow[rrr, "a_{v\times w}"] \arrow[d, "T(\alpha_{v,w})"'] & & & T(v\times w) \arrow[ul, "\tau"'] \arrow[d, "T(\alpha_{v,w})"'] \\
T(w\times v) \arrow[dr, "\tau"'] \arrow[rrr, "a_{w\times v}"] & & & T(w\times v) \arrow[dl, "\tau"] \\
 & T(w)\otimes T(v) \arrow[r, "\mu^*(a)_{(w,v)}"'] & T(w)\otimes T(v) &
\end{tikzcd}$$

By the construction of $\mu^*(a)_{(v,w)}$ (resp. $\mu^*(a)_{(w,v)}$), the upper (resp. lower) tilted square commutes. By naturality, the middle rectangle with $\alpha_{v,w}$ commutes. By property (1) of a representation of a graded diagram with commutative product, the left and right faces commute where the vertical maps are $(-1)^{|v||w|}$ times the natural commutativity of tensor products of T-modules. Hence the inner square also commutes without the sign factors. This is cocommutativity of μ^*.

The associativity assumption (3) for representations of diagrams with product structure implies the coassociativity of μ^*.

The compatibility of multiplication and comultiplication is built into the definition.

In order to define a unit object in $\mathcal{C}(D,T)$ it suffices to define a counit for $\mathrm{End}(T|_F)$. Assume $\mathbf{1} \in F$. The counit

$$u^* : \mathrm{End}(T|_F) \subset \prod_{v\in F} \mathrm{End}(T(v)) \to \mathrm{End}(T(\mathbf{1})) = R$$

is the natural projection. The assumption on unitality of T allows to check that the required diagrams commute. $\square$

Remark 7.1.6. The proof of Proposition 7.1.5 works without any changes in the arguments when we weaken the assumptions as follows: in Definition 7.1.3 replace $\times$ by a map of diagrams

$$\times : D \times D \to \mathcal{P}(D)$$

where $\mathcal{P}(D)$ is the path category of D: objects are the vertices of D and morphisms the paths. A representation T of D extends canonically to a functor on $\mathcal{P}(D)$.

7.2 Localization

The purpose of this section is to give a diagram version of the localization of a tensor category with respect to one object, i.e., a distinguished object X becomes invertible with respect to tensor product. This is the standard construction used to pass e.g. from effective motives to all motives.

We restrict to the case when R is a field or a Dedekind domain and all representations of diagrams take values in $R-$Proj.

Definition 7.2.1 (Localization of diagrams)**.** Let D^{eff} be a graded diagram with a commutative product structure with unit $\mathbf{1}$. Let $v_0 \in D^{\mathrm{eff}}$ be a vertex. The *localized diagram* D has vertices and edges as follows:

1. for every v a vertex of D^{eff} and $n \in \mathbb{Z}$ a vertex denoted $v(n)$;
2. for every edge $\alpha : v \to w$ in D^{eff} and every $n \in \mathbb{Z}$, an edge denoted $\alpha(n) : v(n) \to w(n)$ in D;
3. for every vertex v in D^{eff} and every $n \in \mathbb{Z}$ an edge denoted $(v \times v_0)(n) \to v(n+1)$.

Put $|v(n)| = |v|$.

We equip D with a weak commutative product structure in the sense of Remark 7.1.6

$$\times : D \times D \to \mathcal{P}(D) \qquad v(n) \times w(m) \mapsto (v \times w)(n+m)$$

together with

$$\begin{aligned} \alpha_{v(n),w(m)} &= \alpha_{v,w}(n+m) \\ \beta_{v(n),w(m),u(r)} &= \beta_{v,w,u}(n+m+r) \\ \beta'_{v(n),w(m),u(r)} &= \beta'_{v,w,u}(n+m+r) \end{aligned}$$

Let $\mathbf{1}(0)$ together with

$$u_{v(n)} = u_v(n)$$

be the unit.

Note that there is a natural inclusion of multiplicative diagrams $D^{\mathrm{eff}} \to D$ which maps a vertex v to $v(0)$.

Remark 7.2.2. The above definition does not spell out $\times$ on edges. It is induced from the product structure on D^{eff} for edges of type (2). For edges of type (3) there is an obvious sequence of edges. We take their composition in $\mathcal{P}(D)$. E.g. for $\gamma_{v,n} : (v \times v_0)(n) \to v(n+1)$ and $\mathrm{id}_{w(m)} = \mathrm{id}_w(m) : w(m) \to w(m)$ we have

$$\gamma_{v,n} \times \mathrm{id}(m) : (v \times v_0)(n) \times w(m) \to v(n+1) \times w(m)$$

via

$$\begin{aligned}(v \times v_0)(n) \times w(m) = ((v \times v_0) \times w)(n+m) \\ \xrightarrow{\beta'_{v,v_0,w}(n+m)} (v \times (v_0 \times w))(n+m) \\ \xrightarrow{\mathrm{id}\times\alpha_{v_0,w}(n+m)} (v \times (w \times v_0))(n+m) \\ \xrightarrow{\beta_{v,w,v_0}(n+m)} ((v \times w) \times v_0)(n+m) \\ \xrightarrow{\gamma_{v\times w,n+m}} (v \times w)(n+m+1) = v(n+1) \times w(m)\ .\end{aligned}$$

Assumption 7.2.3. Let R be a field or a Dedekind domain. Let T be a multiplicative unital representation of D^{eff} with values in $R-\mathrm{Proj}$ such that $T(v_0)$ is locally free of rank 1 as R-module.

Lemma 7.2.4. *Under Assumption 7.2.3, the representation T extends uniquely to a graded multiplicative representation of D such that $T(v(n)) = T(v) \otimes T(v_0)^{\otimes n}$ for all vertices and $T(\alpha(n)) = T(\alpha) \otimes T(\mathrm{id})^{\otimes n}$ for all edges. It is multiplicative and unital with the choice*

$$\begin{tikzcd} T(v(n) \times w(m)) \arrow[r, "\tau_{v(n),w(m)}"] \arrow[d, "\tau_{v,w}(n+m)"'] & T(v(n)) \otimes T(w(m)) \arrow[d, "="] \\ T(v) \otimes T(w) \otimes T(v_0)^{\otimes n+m} \arrow[r, "\cong"] & T(v) \otimes T(v_0)^{\otimes n} \otimes T(w) \otimes T(v_0)^{\otimes m} \end{tikzcd}$$

where the last line is the natural isomorphism.

Proof. Define T on the vertices and edges of D via the formula. It is tedious but straightforward to check the conditions. ☐

Proposition 7.2.5. *Let D^{eff}, D and T be as above. Assume Assumption 7.2.3. Let $A(D,T)$ and $A(D^{\mathrm{eff}},T)$ be the corresponding bialgebras. Then:*

1. *$\mathcal{C}(D,T)$ is the localization of the category $\mathcal{C}(D^{\mathrm{eff}},T)$ with respect to the object $\tilde{T}(v_0)$.*
2. *Let $\chi \in \mathrm{End}(T(v_0))^\vee = A(\{v_0\},T)$ be the dual of* $\mathrm{id} \in \mathrm{End}(T(v_0))$*. We view it in $A(D^{\mathrm{eff}},T)$. Then $A(D,T) = A(D^{\mathrm{eff}},T)_\chi$ (localization of algebras).*

Proof. Let $D^{\geq n} \subset D$ be the subdiagram with vertices of the form $v(n')$ with $n' \geq n$. Clearly, $D = \mathrm{colim}_n D^{\geq n}$, and hence

$$\mathcal{C}(D,T) \cong 2{-}\mathrm{colim}_n \mathcal{C}(D^{\geq n},T)\ .$$

Consider the morphism of diagrams

$$D^{\geq n} \to D^{\geq n+1},\ \ v(m) \mapsto v(m+1).$$

It is clearly an isomorphism. We equip $\mathcal{C}(D^{\geq n+1}, T)$ with a new fibre functor $f_T \otimes T(v_0)^\vee$. It is faithful exact. The map $v(m) \mapsto \tilde{T}(v(m+1))$ is a representation of $D^{\geq n}$ in the abelian category $\mathcal{C}(D^{\geq n+1}, T)$ with fibre functor $f_T \otimes T(v_0)^\vee$. By the universal property, this induces a functor

$$\mathcal{C}(D^{\geq n}, T) \to \mathcal{C}(D^{\geq n+1}, T) \ .$$

The converse functor is constructed in the same way. Hence

$$\mathcal{C}(D^{\geq n}, T) \cong \mathcal{C}(D^{\geq n+1}, T), \qquad A(D^{\geq n}, T) \cong A(D^{\geq n+1}, T).$$

The map of graded diagrams with commutative product and unit

$$D^{\mathrm{eff}} \to D^{\geq 0}$$

induces an equivalence on tensor categories. Indeed, we represent $D^{\geq 0}$ in $\mathcal{C}(D^{\mathrm{eff}}, T)$ by mapping $v(m)$ to $\tilde{T}(v) \otimes \tilde{T}(v_0)^m$. By the universal property (see Corollary 6.1.18), this implies that there is a faithful exact functor

$$\mathcal{C}(D^{\geq 0}, T) \to \mathcal{C}(D^{\mathrm{eff}}, T)$$

inverse to the obvious inclusion. Hence we also have $A(D^{\mathrm{eff}}, T) \cong A(D^{\geq 0}, T)$ as unital bialgebras.

On the level of coalgebras, this implies

$$A(D, T) = \mathrm{colim}_n A(D^{\geq n}, T) = \mathrm{colim}_n A(D^{\mathrm{eff}}, T)$$

because $A(D^{\geq n}, T)$ isomorphic to $A(D^{\mathrm{eff}}, T)$ as coalgebras. $A(D^{\mathrm{eff}}, T)$ also has a multiplication, but the $A(D^{\geq n}, T)$ for general $n \in \mathbb{Z}$ do not. However, they carry a weak $A(D^{\mathrm{eff}}, T)$-module structure analogous to Remark 7.1.6 corresponding to the map of graded diagrams

$$D^{\mathrm{eff}} \times D^{\geq n} \to \mathcal{P}(D^{\geq n}).$$

We want to describe the transition maps of the direct limit. From the point of view of $D^{\mathrm{eff}} \to D^{\mathrm{eff}}$, it is given by $v \mapsto v \times v_0$.

In order to describe the transition maps $A(D^{\mathrm{eff}}, T) \to A(D^{\mathrm{eff}}, T)$, it suffices to describe $\mathrm{End}(T|_F) \to \mathrm{End}(T|_{F'})$ where F, F' are finite subdiagrams of D^{eff} such that $v \times v_0 \in V(F')$ for all vertices $v \in V(F)$. It is induced by

$$\mathrm{End}(T(v)) \to \mathrm{End}(T(v \times v_0)) \xrightarrow{\tau} \mathrm{End}(T(v)) \otimes \mathrm{End}(T(v_0)) \quad a \mapsto a \otimes \mathrm{id}.$$

On the level of coalgebras, this corresponds to the map

$$A(D^{\mathrm{eff}}, T) \to A(D^{\mathrm{eff}}, T), \quad x \mapsto x\chi.$$

Note finally, that the direct limit $\mathrm{colim} A(D^{\mathrm{eff}}, T)$ with transition maps given by multiplication by χ agrees with the localization $A(D^{\mathrm{eff}}, T)_\chi$. $\square$

7.3 Nori's Rigidity Criterion

Implicit in Nori's construction of motives is a rigidity criterion, which we are now going to formulate and prove explicitly.

Let R be a Dedekind domain or a field and $\mathcal{C}$ an R-linear tensor category. Recall that $R-$Mod is the category of finitely generated R-modules and $R-$Proj the category of finitely generated projective R-modules.

We assume that the tensor product on $\mathcal{C}$ is associative, commutative and unital. Let $\mathbf{1}$ be the unit object. Let $T : \mathcal{C} \to R-$Mod be a faithful tensor functor with values in $R-$Mod. In particular, $T(\mathbf{1}) \cong R$.

We introduce an ad-hoc notion.

Definition 7.3.1. Let V be an object of $\mathcal{C}$. We say that V *admits a perfect duality* if there is morphism

$$q : V \otimes V \to \mathbf{1}$$

or

$$\mathbf{1} \to V \otimes V$$

such that $T(V)$ is projective and $T(q)$ (respectively its dual) is a non-degenerate bilinear form.

Definition 7.3.2. Let V be an object of $\mathcal{C}$. By $\langle V \rangle_\otimes$ we denote the smallest full abelian unital tensor subcategory of $\mathcal{C}$ containing V.

We start with the simplest case of the criterion.

Lemma 7.3.3. *Let V be an object such that $\mathcal{C} = \langle V \rangle_\otimes$ and such that V admits a perfect duality. Then $\mathcal{C}$ is rigid.*

Proof. By standard Tannakian formalism, $\mathcal{C}$ is the category of comodules for a bialgebra A, which is commutative and of finite type as an R-algebra. Indeed: The construction of A as a coalgebra was explained in Proposition 6.1.12. We may view $\mathcal{C}$ as graded diagram (with trivial grading) with a unital commutative product structure in the sense of Definition 7.1.3. The fibre functor T is a unital graded multiplicative representation. The algebra structure on A is the one of Proposition 7.1.5. It is easy to see that A is generated by $A(\{V\}, T,)$ as an algebra. The argument is given in more detail below.

We want to show that A is a Hopf algebra, or equivalently, that the algebraic monoid $M = \mathrm{Spec}A$ is an algebraic group.

By Lemma 7.3.6 it suffices to show that there is a closed immersion $M \to G$ of monoids into an algebraic group G. We are going to construct this group or rather its ring of regular functions. We have

$$A = \lim A_n$$

with $A_n = A(\mathcal{C}_n, T)$ for $\mathcal{C}_n = \langle \mathbf{1}, V, V^{\otimes 2}, \dots, V^{\otimes n}\rangle$, the smallest full abelian subcategory containing $\mathbf{1}, V, \dots, V^{\otimes n}$. By construction, there is a surjective map

$$\bigoplus_{i=0}^{n} \mathrm{End}_R((T(V)^{\otimes i})^\vee \to A_n$$

or, dually, an injective map

$$A_n^\vee \to \bigoplus_{i=0}^{n} \mathrm{End}_R(T(V)^{\otimes i})$$

where $A_n^\vee$ consists of those endomorphisms compatible with all morphisms in $\mathcal{C}_n$. In the limit, there is a surjection of bialgebras

$$\bigoplus_{i=0}^{\infty} \mathrm{End}_R((T(V)^{\otimes i})^\vee) \to A$$

and the kernel is generated by the relation defined by compatibility with morphisms in $\mathcal{C}$. One such relation is the commutativity constraint, hence the map factors via the symmetric algebra

$$S^*(\mathrm{End}(T(V)^\vee) \to A\ .$$

Note that $S^*(\mathrm{End}(T(V)^\vee)$ is canonically the ring of regular functions on the algebraic monoid $\mathrm{End}(T(V))$. Another morphism in $\mathcal{C}$ is the pairing $q : V \otimes V \to \mathbf{1}$. We want to work out the explicit equation induced by q.

We choose a basis $e_1, \dots, e_r$ of $T(V)$. Let

$$a_{i,j} = T(q)(e_i, e_j) \in R$$

By assumption, the matrix is invertible. Let X_{st} be the matrix coefficients on $\mathrm{End}(T(V))$ corresponding to the basis e_i. Compatibility with q gives for every pair (i, j) the equation

$$\begin{aligned} a_{ij} &= q(e_i, e_j) \\ &= q((X_{rs})e_i, (X_{r's'})e_j) \\ &= q\left(\sum_r X_{ri}e_r, \sum_{r'} X_{r'j}e_{r'}\right) \\ &= \sum_{r,r'} X_{ri}X_{r'j}q(e_r, e_{r'}) \\ &= \sum_{r,r'} X_{ri}X_{r'j}a_{rr'} \end{aligned}$$

Note that the latter is the (i, j)-term in the product of matrices

$$(X_{ir})^t(a_{rr'})(X_{r'j})\ .$$

Let $(b_{ij}) = (a_{ij})^{-1}$. With

$$(Y_{ij}) = (b_{ij})(X_{i'r})^t(a_{rr'})$$

we have the coordinates of the inverse matrix. In other words, our set of equations defines the isometry group $G(q) \subset \mathrm{End}(T(V))$. We now have expressed A as quotient of the ring of regular functions of $G(q)$.

The argument works in the same way, if we are given

$$q : \mathbf{1} \to V \otimes V$$

instead. $\square$

Proposition 7.3.4 (Nori)**.** *Let $\mathcal{C}$ and $T : \mathcal{C} \to R$−Mod be as defined at the beginning of the section. Let $\{V_i | i \in I\}$ be a set of objects of $\mathcal{C}$ with the properties:*

1. *It generates $\mathcal{C}$ as an abelian tensor category, i.e., the smallest full abelian tensor subcategory of $\mathcal{C}$ containing all V_i is equal to $\mathcal{C}$.*
2. *For every V_i there is an object W_i and a morphism*

   $$q_i : V_i \otimes W_i \to \mathbf{1}$$

   such that $T(q_i) : T(V_i) \otimes T(W_i) \to T(\mathbf{1}) = R$ is a perfect pairing of free R-modules.

Then $\mathcal{C}$ is rigid, i.e., for every object V there is a dual object $V^\vee$ such that

$$\mathrm{Hom}(V \otimes A, B) = \mathrm{Hom}(A, V^\vee \otimes B)\ , \qquad \mathrm{Hom}(V^\vee \otimes A, B) = \mathrm{Hom}(A, V \otimes B)\ .$$

This means that the Tannakian dual of $\mathcal{C}$ is not only a monoid but a group.

Remark 7.3.5. The Proposition also holds with the dual assumption, existence of morphisms

$$q_i : \mathbf{1} \to V_i \otimes W_i$$

such that $T(q_i)^\vee : T(V)^\vee \otimes T(W_i)^\vee \to R$ is a perfect pairing.

Proof. Consider $V_i' = V_i \oplus W_i$. The pairing q_i extends to a symmetric map q_i' on $V_i' \otimes V_i'$ such that $T(q_i')$ is non-degenerate. We now replace V_i by V_i'. Without loss of generality, we can assume $V_i = W_i$.

For any finite subset $J \subset I$, let $V_J = \bigoplus_{j \in J} V_j$. Let q_J be the orthogonal sum of the q_j for $j \in J$. It is again a symmetric perfect pairing.

For every object V of $\mathcal{C}$, we write $\langle V \rangle_\otimes$ for the smallest full abelian tensor subcategory of $\mathcal{C}$ containing V. By assumption we have

$$\mathcal{C} = \bigcup_J \langle V_J \rangle_\otimes$$

We apply the standard Tannakian machinery. It attaches to every $\langle V_J \rangle_\otimes$ an R-bialgebra A_J such that $\langle V_J \rangle_\otimes$ is equivalent to the category of A_J-comodules. If we put

$$A = \lim A_J$$

then $\mathcal{C}$ will be equivalent to the category of A-comodules. It suffices to show that A_J is a Hopf-algebra. This is the case by Lemma 7.3.3. □

Finally, the missing lemma on monoids.

Lemma 7.3.6. *Let R be noetherian ring, G be an algebraic group scheme of finite type over R and $M \subset G$ a closed immersion of a submonoid with $1 \in M(R)$. Then M is an algebraic group scheme over R.*

Proof. This seems to be well-known. It is appears as an exercise in [Re] 3.5.1 2. We give the argument:

Let S be any finitely generated R-algebra. We have to show that the value $S \mapsto M(S)$ is a group. We take base change of the situation to S. Hence without loss of generality, it suffices to consider $R = S$. If $g \in G(R)$, we denote the isomorphism $G \to G$ induced by left multiplication with g also by $g : G \to G$. Take any $g \in G(R)$ such that $gM \subset M$ (for example $g \in M(R)$). Then one has

$$M \supseteq gM \supseteq g^2M \supseteq \cdots$$

As G is Noetherian, this sequence stabilizes, say at $s \in \mathbb{N}$:

$$g^sM = g^{s+1}M$$

as closed subschemes of G. Since every g^s is an isomorphism, we obtain that

$$M = g^{-s}g^sM = g^{-s}g^{s+1}M = gM$$

as closed subschemes of G. So for every $g \in M(R)$ we showed that $gM = M$. Since $1 \in M(R)$, this implies that $M(R)$ is a subgroup. □

Chapter 8

Nori motives

We explain Nori's construction of an abelian category of motives. It is defined as the diagram category (see Chapters 6 and 7) of a certain diagram. It is universal for all cohomology theories that can be compared with singular cohomology.

8.1 Essentials of Nori Motives

As before, we denote $\mathbb{Z}-\mathrm{Mod}$ the category of finitely generated $\mathbb{Z}$-modules and $\mathbb{Z}-\mathrm{Proj}$ the category of finitely generated free $\mathbb{Z}$-modules.

8.1.1 Definition

Let k be a subfield of $\mathbb{C}$. For a variety X over k, we define singular cohomology as singular cohomology of $X(\mathbb{C}) = X \times_k \mathbb{C}$. As in Chapter 2.1, we denote it simply by $H^i(X, \mathbb{Z})$.

Definition 8.1.1. Let k be a subfield of $\mathbb{C}$. The diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of *effective pairs* consists of triples (X, Y, i) with X a k-variety, $Y \subset X$ a closed subvariety and an integer i. There are two types of edges between effective pairs:

1. (functoriality) For every morphism $f : X \to X'$ with $f(Y) \subset Y'$ an edge
$$f^* : (X', Y', i) \to (X, Y, i)\ .$$
2. (coboundary) For every chain $X \supset Y \supset Z$ of closed k-subschemes of X an edge
$$\partial : (Y, Z, i) \to (X, Y, i + 1)\ .$$

The diagram has identities (see Definition 6.1.1) given by the identity morphism. The diagram is graded (see Definition 7.1.3) by $|(X, Y, i)| = i$.

Proposition 8.1.2. *The assignment*

$$H^* : \mathrm{Pairs}^{\mathrm{eff}} \to \mathbb{Z}\text{−Mod}$$

which maps to (X, Y, i) *relative singular cohomology* $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z})$ *is a representation in the sense of Definition 6.1.4. It maps* $(\mathbb{G}_m, \{1\}, 1)$ *to* $\mathbb{Z}$*.*

Proof. Relative singular cohomology was defined in 2.1.1. By definition, it is contraviantly functorial. This defines H^* on edges of type 1. The connecting morphism for triples, see Corollary 2.1.4, defines the representation on edges of type 2. We compute $H^1(\mathbb{G}_m, \{1\}, \mathbb{Z})$ via the sequence for relative cohomology

$$H^0(\mathbb{C}^*, \mathbb{Z}) \to H^0(\{1\}, \mathbb{Z}) \to H^1(\mathbb{C}^*, \{1\}, \mathbb{Z}) \to H^1(\mathbb{C}^*, \mathbb{Z}) \to H^1(\{1\}, \mathbb{Z})$$

The first map is an isomorphism. The last group vanishes for dimension reasons. Finally, $H^1(\mathbb{C}^*, \mathbb{Z}) \cong \mathbb{Z}$ because $\mathbb{C}^*$ is homotopy equivalent to the unit circle. □

Definition 8.1.3.

1. The category of effective *mixed Nori motives* $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} = \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ is defined as the diagram category $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$ from Theorem 6.1.13.

2. For an effective pair (X, Y, i), we write $H^i_{\mathrm{Nori}}(X, Y)$ for the corresponding object in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$. We put

$$\mathbf{1}(-1) = H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\}) \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \ ,$$

   the *Lefschetz motive*.

3. The category $\mathcal{MM}_{\mathrm{Nori}} = \mathcal{MM}_{\mathrm{Nori}}(k)$ of *Nori motives* is defined as the localization of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ with respect to $\mathbb{Z}(-1)$.

4. We also write H^* for the extension of H^* to $\mathcal{MM}_{\mathrm{Nori}}$.

Remark 8.1.4. This is equivalent to Nori's orginal definition by Theorem 8.3.4.

8.1.2 Main results

Theorem 8.1.5 (Nori)**.**

1. *$\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ has a natural structure of commutative tensor category with unit such that H^* is a tensor functor.*

2. *$\mathcal{MM}_{\mathrm{Nori}}$ is a rigid tensor category.*

3. *$\mathcal{MM}_{\mathrm{Nori}}$ is equivalent to the category of representations of a pro-algebraic group scheme $G_{\mathrm{mot}}(k, \mathbb{Z})$ over $\mathbb{Z}$.*

For the proof see Section 8.3.1.

Definition 8.1.6. The group scheme $G_{\mathrm{mot}}(k, \mathbb{Z})$ is called the *motivic Galois group* in the sense of Nori.

Remark 8.1.7. The first statement also holds with the coefficient ring $\mathbb{Z}$ replaced by any noetherian ring R. The other two hold if R is a Dedekind ring R or field. Of particular interest is the case $R = \mathbb{Q}$.

The proof of this theorem will take the rest of the chapter. We now explain the key ideas. In order to define the tensor structure, we would like to apply the abstract machine developed in Section 7.1. However, the shape of the Künneth formula

$$H^n(X \times Y, \mathbb{Q}) \cong \bigoplus_{i+j=n} H^i(X, \mathbb{Q}) \otimes H^i(Y, \mathbb{Q})$$

is not of the required kind. Nori introduces a subdiagram of *good pairs* where relative cohomology is concentrated in a single degree and free, so that the Künneth formula simplifies. The key insight now becomes that it is possible to recover *all* pairs from good pairs. This is done via an algebraic skeletal filtration constructed from the Basic Lemma as discussed in Section 2.5. As a byproduct, we will also know that $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ and $\mathcal{MM}_{\mathrm{Nori}}$ are given as representations of an algebra monoid. In the next step, we have to verify rigidity, i.e., we have to show that the monoid is an algebraic group. We do this by verifying the abstract criterion of Section 7.3.

On the way, we need to establish a general "motivic" property of Nori motives.

Theorem 8.1.8. *There is a natural contravariant triangulated functor*

$$R : K_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}})$$

on the homotopy category of bounded homological complexes in $\mathbb{Z}[\mathrm{Var}]$ *such that for every effective pair* (X, Y, i) *we have*

$$H^i(R(\mathrm{Cone}(Y \to X)) = H^i_{\mathrm{Nori}}(X, Y).$$

For the proof see Section 8.3.1. The theorem allows, for example, to define motives of simplicial varieties or motives with support.

The category of motives is supposed to be the universal abelian category such that all cohomology theories with suitable properties factor via the category of motives. We do not yet have such a theory, even though it is reasonable to conjecture that $\mathcal{MM}_{\mathrm{Nori}}$ is the correct description. In any case, it does have a universal property which is good enough for many applications.

Theorem 8.1.9 (Universal property)**.** *Let* $\mathcal{A}$ *be an abelian category with a faithful exact functor* $f : \mathcal{A} \to R\mathrm{-Mod}$ *for a noetherian ring* R*. Let*

$$H'^* : \mathrm{Pairs}^{\mathrm{eff}} \to \mathcal{A}$$

be a representation. Assume that there is an extension $R \to S$ *such that* S *is faithfully flat over* R *and* $\mathbb{Z}$ *and an isomorphism of representations*

$$\Phi : H'^*_S \to (f \circ H'^*)_S.$$

Then H'^* extends to $\mathcal{MM}_{\text{Nori}}$:

$$\text{Pairs}^{\text{eff}} \to \mathcal{MM}_{\text{Nori}} \to \mathcal{A}[H'^*(\mathbf{1}(-1))]^{-1}.$$

More precisely, there exists a functor $L(H'^*) : \mathcal{MM}_{\text{Nori}} \to \mathcal{A}[\mathbf{1}(-1)]^{-1}$ and an isomorphism of functors

$$\tilde{\Phi} : (f_{H^*})_S \to f_S \circ L(H'^*)$$

such that

$$\begin{tikzcd}
& \mathcal{MM}_{\text{Nori}} \arrow[dr, "(f_{H^*})_S"] \arrow[dd, dashed, "L(H'^*)"] & \\
\text{Pairs}^{\text{eff}} \arrow[ur, "\tilde{H}^*"] \arrow[rr, "H^*_S"] \arrow[dr, "H'^*"'] & & S\text{-Mod} \\
& \mathcal{A}[H'^*(\mathbf{1}(-1))]^{-1} \arrow[ur, "f_S"'] &
\end{tikzcd}$$

commutes up to ϕ and $\tilde{\phi}$. The pair $(L(H'^*), \tilde{\phi})$ is unique up to unique isomorphism of functors.

If, moreover, $\mathcal{A}$ is a tensor category, f a tensor functor and H'^* a graded multiplicative representation on Good$^{\text{eff}}$*, then $L(H'^*)$ is a tensor functor and $\tilde{\phi}$ is an isomorphism of tensor functors.*

For the proof see Section 8.3.1. This means that $\mathcal{MM}_{\text{Nori}}$ is universal for all cohomology theories with a comparison isomorphism to singular cohomology. Actually, it suffice to have a representation of Good$^{\text{eff}}$ or VGood$^{\text{eff}}$, see Definition 8.2.1.

Example 8.1.10. Let $R = k$, $\mathcal{A} = k-$Mod, H'^* algebraic de Rham cohomology see Chapter 3. Let $S = \mathbb{C}$, and let the comparison isomorphism Φ be the period isomorphism of Chapter 5. By the universal property, de Rham cohomology extends to $\mathcal{MM}_{\text{Nori}}$. We will study this example in a lot more detail in Part III in order to understand the period algebra.

Example 8.1.11. Let $R = \mathbb{Z}$, $\mathcal{A}$ the category of mixed $\mathbb{Z}$-Hodge structures, H'^* the functor assigning a mixed Hodge structure to a variety or a pair. Then $S = \mathbb{Z}$ and Φ is the functor mapping a Hodge structure to the underlying $\mathbb{Z}$-module. By the universal property, H'^* factors canonically via $\mathcal{MM}_{\text{Nori}}$. In other words, motives define mixed Hodge structures.

Example 8.1.12. Let ℓ be a prime, $R = \mathbb{Z}_\ell$, and $\mathcal{A}$ the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous operation of $\mathrm{Gal}(\bar{k}/k)$. Let H'^* be ℓ-adic cohomology over $\bar{k}$. For X a variety and $Y \subset X$ a closed subvariety with open complement $j : U \to X$, we have

$$(X, Y, i) \mapsto H^i_{et}(X_{\bar{k}}, j_!\mathbb{Z}_\ell).$$

In this case, we let $S = \mathbb{Z}_l$ and use the comparison isomorphism between ℓ-adic and singular cohomology.

Corollary 8.1.13. *The category $\mathcal{MM}_{\mathrm{Nori}}$ is independent of the choice of embedding $\sigma : k \to \mathbb{C}$. More precisely, $\sigma' : k \to \mathbb{C}$ be another embedding. Let H'^* be singular cohomology with respect to this embedding. Then there is an equivalence of categories*

$$\mathcal{MM}_{\mathrm{Nori}}(\sigma) \to \mathcal{MM}_{\mathrm{Nori}}(\sigma').$$

Proof. Use $S = \mathbb{Z}_\ell$ and the comparison isomorphism given by comparing both singular cohomology functors with ℓ-adic cohomology. This induces the functor. □

Remark 8.1.14. Note that the equivalence is *not* canonical. In the argument above it depends on the choice of embeddings of $\bar{k}$ into $\mathbb{C}$ extending σ and σ', respectively. If we are willing to work with rational coefficients instead, we can compare both singular cohomologies with algebraic de Rham cohomology (with $S = k$). This gives a compatible system of comparison equivalences.

8.2 Yoga of good pairs

We now turn to alternative descriptions of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ better suited to the tensor structure.

8.2.1 Good pairs and good filtrations

Definition 8.2.1. Let k be a subfield of $\mathbb{C}$.

1. The diagram $\mathrm{Good}^{\mathrm{eff}}$ of *effective good pairs* is the full subdiagram of $\mathrm{Pairs}^{\mathrm{eff}}$ with vertices the triples (X, Y, i) such that singular cohomology satisfies
$$H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) = 0, \text{ unless } j = i.$$
and is free for $j = i$.

2. The diagram $\mathrm{VGood}^{\mathrm{eff}}$ of *effective very good pairs* is the full subdiagram of those effective good pairs (X, Y, i) with X affine, $X \smallsetminus Y$ smooth and either X of dimension i and Y of dimension $i - 1$, or $X = Y$ of dimension less than i.

We will later (see Definition 8.3.2) also introduce the diagrams Pairs of *pairs*, Good of *good pairs* and VGood of *very good pairs* as localization (see Definition 7.2.1) with respect to $(\mathbb{G}_m, \{1\}, 1)$.

Good pairs exist in abundance by the basic lemma, see Theorem 2.5.2.

Our first aim is to show that the diagram categories attached to $\text{Pairs}^{\text{eff}}$, Good^{eff} and $\text{VGood}^{\text{eff}}$ are equivalent. By the general principles of diagram categories this means that we have to represent the diagram $\text{Pairs}^{\text{eff}}$ in $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$. We do this in two steps: a general variety is replaced by the Čech complex attached to an affine cover; affine varieties are replaced by complexes of very good pairs using the key idea of Nori. The construction proceeds in a complicated way because both steps involve choices which have to be made in a compatible way. We handle this problem in the same way as in [Hu3].

We start in the affine case. Using induction, one gets from the Basic Lemma 2.5.2:

Proposition 8.2.2. *Every affine variety X has a filtration*

$$\emptyset = F_{-1}X \subset F_0X \subset \cdots \subset F_{n-1}X \subset F_nX = X,$$

such that $(F_jX, F_{j-1}X, j)$ is very good.

Filtrations of the above type are called *very good filtrations.*

Proof. Let $\dim X = n$. Put $F_nX = X$. Choose a subvariety of dimension $n-1$ which contains all singular points of X. By the Basic Lemma 2.5.2, there is a subvariety $F_{n-1}X$ of dimension $n-1$ such that $(F_nX, F_{n-1}X, n)$ is good. By construction $F_{n-1}X \smallsetminus F_{n-1}X$ is smooth and hence the pair is very good. We continue by induction. □

Corollary 8.2.3. *Let X be an affine variety. The inductive system of all very good filtrations of X is filtered and functorial.*

Proof. Let F_*X and F'_*X be two very good filtrations of X. $F_{n-1}X \cup F'_{n-1}X$ has dimension $n-1$. By the Basic Lemma 2.5.2, there is subvariety $G_{n-1}X \subset X$ of dimension $n-1$ such that $(X, G_{n-1}X, n)$ is a good pair. It is automatically very good. We continue by induction.

Consider a morphism $f : X \to X'$. Let F_*X be a very good filtration. Then $f(F_iX)$ has dimension at most i. As in the proof of Corollary 8.2.2, we construct a very good filtration F_*X' with the additional property $f(F_iX) \subset F_iX'$. □

Remark 8.2.4. This allows to construct a functor from the category of affine varieties to the diagram category $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$ as follows: Given an affine variety X, let F_*X be a very good filtration. The boundary maps of the triples $F_{i-1}X \subset F_iX \subset F_{i+1}X$ define a complex in $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$

$$\cdots \to H^i_{\text{Nori}}(F_iX, F_{i-1}X) \to H^{i+1}_{\text{Nori}}(F_{i+1}X, F_iX) \to \dots$$

Taking i-th cohomology of this complex defines an object in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ whose underlying $\mathbb{Z}$-module is nothing but singular cohomology $H^i(X, \mathbb{Z})$. Up to isomorphism it is independent of the choice of filtration. In particular, it is functorial.

We are going to refine the above construction such that is also applies to complexes of varieties.

8.2.2 Čech complexes

The next step is to replace arbitrary varieties by affine ones. The idea for the following construction is from the case of étale coverings, see [F] Definition 4.2.

Definition 8.2.5. Let X a variety. A *rigidified* affine cover is a finite open affine covering $\{U_i\}_{i\in I}$ together with a choice of an index i_x for every closed point $x \in X$ such that $x \in U_{i_x}$. We also assume that in the covering every index $i \in I$ occurs as i_x for some $x \in X$.

Let $f : X \to Y$ be a morphism of varieties, $\{U_i\}_{i\in I}$ a rigidified open cover of X and $\{V_j\}_{j\in J}$ a rigidified open cover of Y. A *morphism* of rigidified covers (over f)

$$\phi : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$$

is a map of sets $\phi : I \to J$ such that $f(U_i) \subset V_{\phi(i)}$ and for all $x \in X$ we have $\phi(i_x) = j_{f(x)}$.

Remark 8.2.6. The rigidification makes ϕ unique if it exists.

Lemma 8.2.7. *The projective system of rigidified affine covers is filtered and strictly functorial, i.e., if $f : X \to Y$ is a morphism of varieties, pull-back defines a map of projective systems.*

Proof. Any two covers have their intersection as common refinement with index set the product of the index sets. The rigidification extends in the obvious way. Preimages of rigidified covers are rigidified open covers. □

We need to generalize this to complexes of varieties. Recall from Definition 1.1.1 the additive categories $\mathbb{Z}[\mathrm{Aff}]$ and $\mathbb{Z}[\mathrm{Var}]$ with objects (affine) varieties and morphisms roughly $\mathbb{Z}$-linear combinations of morphisms of varieties. The support of a morphism in $\mathbb{Z}[\mathrm{Var}]$ is the set of morphisms occuring in the linear combination.

Definition 8.2.8. Let X_* be a homological complex of varieties, i.e., an object in $C_b(\mathbb{Z}[\mathrm{Var}])$. An *affine cover* of X_* is a complex of rigidified affine covers, i.e., for every X_n the choice of a rigidified open cover $\tilde{U}_{X_n}$ and for every $g : X_n \to X_{n-1}$ in the support of the differential $X_n \to X_{n-1}$ in the complex X_* a morphism of rigidified covers $\tilde{g} : \tilde{U}_{X_n} \to \tilde{U}_{X_{n-1}}$ over g.

Let $F_* : X_* \to Y_*$ be a morphism in $C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$, $\tilde{U}_{Y_*}$ affine covers of X_* and Y_*. A morphism of affine covers over F_* is a morphism of rigidified affine covers $f_n : \tilde{U}_{X_n} \to \tilde{U}_{Y_n}$ over every morphism in the support of F_n.

Lemma 8.2.9. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Then the projective system of rigidified affine covers of X_* is non-empty, filtered and functorial, i.e., if $f_* : X_* \to Y_*$ is a morphism of complexes and $\tilde{U}_{X_*}$ an affine cover of X_*, then there is an affine cover $\tilde{U}_{Y_*}$ and a morphism of complexes of rigidified affine covers. Any two choices are compatible in the projective system of covers.*

Proof. Let n be minimal with $X_n \neq \emptyset$. Choose a rigidified cover of X_n. The support of $X_{n+1} \to X_n$ has only finitely many elements. Choose a rigidified cover of X_{n+1} compatible with all of them. Continue inductively.

Similar constructions show the rest of the assertion. □

Definition 8.2.10. Let X be a variety and $\tilde{U}_X = \{U_i\}_{i\in I}$ a rigidified affine cover of X. We put

$$C_\star(\tilde{U}_X) \in C_-(\mathbb{Z}[\mathrm{Aff}]),$$

the Čech complex associated to the cover, i.e.,

$$C_n(\tilde{U}_X) = \coprod_{\underline{i}\in I_n} \bigcap_{i\in\underline{i}} U_i,$$

where I_n is the set of tuples $(i_0, \dots, i_n)$. The boundary maps are the ones obtained by taking the alternating sum of the boundary maps of the simplicial scheme.

If $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ is a complex, and $\tilde{U}_{X_*}$ a rigidified affine cover, let

$$C_\star(\tilde{U}_{X_*}) \in C_{-,b}(\mathbb{Z}[\mathrm{Aff}])$$

be the double complex $C_i(\tilde{U}_{X_j})$.

Note that all components of $C_\star(\tilde{U}_{X_*})$ are affine. The projective system of these complexes is filtered and functorial.

Definition 8.2.11. Let X be a variety, $\{U_i\}_{i\in I}$ a rigidified affine cover of X. A *very good filtration* on $\tilde{U}_X$ is the choice of very good filtrations for

$$\bigcap_{i\in J} U_i$$

for all $J \subset I$ compatible with all inclusions between these.

Let $f : X \to Y$ be a morphism of varieties, $\phi : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ a morphism of rigidified affine covers above f . Fix very good filtrations on both covers. The morphism ϕ is called *filtered*, if for all $J \subset I$ the induced map

$$\bigcap_{i\in I'} U_i \to \bigcap_{i\in I'} V_{\phi(i)}$$

is compatible with the filtrations.

Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ be a bounded complex of varieties, $\tilde{U}_{X_*}$ an affine cover of X_*. A *very good filtration* on $\tilde{U}_{X_*}$ is a very good filtration on all $\tilde{U}_{X_n}$ compatible with all morphisms in the support of the boundary maps.

Note that the Čech complex associated to a rigidified affine cover with very good filtration is also filtered in the sense that there is a very good filtration on all $C_n(\tilde{U}_X)$ and all morphisms in the support of the differential are compatible with the filtrations.

Lemma 8.2.12. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover. Then the inductive system of very good filtrations on $\tilde{U}_X$ is non-empty, filtered and functorial.*

The same statement also holds for a complex of varieties $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$.

Proof. Let $\tilde{U}_X = \{U_i\}_{i\in I}$ be the affine cover. We choose recursively very good filtrations on $\bigcap_{i\in J} U_i$ with decreasing order of J, compatible with the inclusions.

We extend the construction inductively to complexes, starting with the highest term of the complex. □

Definition 8.2.13. Let $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. A *very good filtration* of X_* is given by a very good filtration $F_\cdot X_n$ for all n which is compatible with all morphisms in the support of the differentials of X_*.

Lemma 8.2.14. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$ an affine cover of X_* with a very good filtration. Then the total complex of $C_\star(\tilde{U}_{X_*})$ carries a very good filtration.*

Proof. Clear by construction. □

8.2.3 Putting things together

Let $\mathcal{A}$ be an abelian category with a faithful forgetful functor $f : \mathcal{A} \to R\mathrm{-Mod}$ with R noetherian. Let $T : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{A}$ be a representation of the diagram of very good pairs.

Definition 8.2.15. Let $F_\bullet X$ be an affine variety X together with a very good filtration $F_\bullet$. We put $\tilde{R}(F_\bullet X) \in C^b(\mathcal{A})$

$$\cdots \to T(F_j X_*, F_{j-1}X_*) \to T(F_{j+1}X_*, F_j X_*) \to \dots$$

Let $F_\bullet X_*$ be a very good filtration of a complex $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. We put $\tilde{R}(F_\bullet X_*) \in C^+(\mathcal{A})$ the total complex of the double complex $\tilde{R}(F_\cdot X_n)_{n\in\mathbb{Z}}$.

Proposition 8.2.16. *Let $\mathcal{A}$ be an R-linear abelian category with a faithful forgetful functor f to R–Mod. Let $T : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{A}$ be a representation such* *that $f\circ T$ is singular cohomology with R-coefficients. Then there is a natural contravariant triangulated functor*

$$R : C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$$

on the category of bounded homological complexes in $\mathbb{Z}[\mathrm{Var}]$ *such that for every good pair* (X,Y,i) *we have*

$$H^j(R(\mathrm{Cone}(Y\to X)) = \begin{cases} 0 & j\neq i, \\ T(X,Y,i) & j=i. \end{cases}$$

Moreover, the image of $R(X)$ *in* $D^b(R\mathrm{-Mod})$ *computes singular cohomology of* $X(\mathbb{C})$.

Proof. We first define $R : C^b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$ on objects. Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Choose a rigidified affine cover $\tilde{U}_{X_*}$ of X_*. This is possible by Lemma 8.2.9. Choose a very good filtration on the cover. This is possible by 8.2.12. It induces a very good filtration on $\mathrm{Tot}C_\star(\tilde{U}_{X_*})$. Put

$$R(X_*) = \tilde{R}(\mathrm{Tot}C_\star(\tilde{U}_{X_*})).$$

Note that any other choice yields a complex isomorphic to this one in $D^+(\mathcal{A})$ because f is faithful and exact and the image of $R(X_*)$ in $D^+(R\mathrm{-Mod})$ computes singular cohomology with R-coefficients.

Let $f : X_* \to Y_*$ be a morphism. Choose a refinement $\tilde{U}'_{X_*}$ of $\tilde{U}_{X_*}$ which maps to $\tilde{U}_{Y_*}$ and a very good filtration on $\tilde{U}'_{X_*}$. Choose a refinement of the filtrations on $\tilde{U}_{X_*}$ and $\tilde{U}_{Y_*}$ compatible with the filtration on $\tilde{U}'_{X_*}$. This gives a little diagram of morphisms of complexes $\tilde{R}$ which defines $R(f)$ in $D^+(\mathcal{A})$. □

Remark 8.2.17. Nori suggests working with Ind-objects (or rather pro-object in our dual setting) in order to get functorial complexes attached to affine varieties. However, the mixing between inductive and projective systems in our construction does not make it obvious if this works out for the result we needed. In order to avoid this situation, it might, however, be possible to do the construction in two steps. This approach is used in Harrer's generalization to complexes of smooth correspondences, [Ha], which completely avoids discussing Čech complexes.

As a corollary of the construction in the proof, we also get:

Corollary 8.2.18. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover with Čech complex $C_\star(\tilde{U}_X)$. Then*

$$R(X) \to R(C_\star(\tilde{U}_X))$$

is an isomorphism in $D^+(\mathcal{A})$.

We are mostly interested in two explicit examples of complexes.

Definition 8.2.19. Consider the situation of Proposition 8.2.16. Let $Y \subset X$ be a closed subvariety with open complement U, $i \in \mathbb{Z}$. Then we put

$$R(X,Y) = R(\mathrm{Cone}(Y \to X)), \quad R_Y(X) = R(\mathrm{Cone}(U \to X)) \in D^b(\mathcal{A})$$
$$H(X,Y,i) = H^i(R(X,Y)), \quad H_Y(X,i) = H^i(R_Y(X)) \in \mathcal{A}$$

$H(X,Y,i)$ is called *relative cohomology.* $H_Y(X,i)$ is called *cohomology with support.*

8.2.4 Comparing diagram categories

We are now ready to proof the first key theorems.

Theorem 8.2.20. *The diagram categories* $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$*,* $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ *and* $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ *are equivalent.*

Proof. The inclusion of diagrams induces faithful functors

$$i : \mathcal{C}(\mathrm{VGoodeff}, H^*) \to \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*).$$

We want to apply Corollary 6.1.18. Hence it suffices to represent the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ such that the restriction of the representation to $\mathrm{VGood}^{\mathrm{eff}}$ gives back H^* (up to natural isomorphism).

We turn to the construction of the representation of $\mathrm{Pairs}^{\mathrm{eff}}$ in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$. We apply Proposition 8.2.16 to

$$H^* : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$$

and get a functor

$$R : C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)).$$

Consider an effective pair (X,Y,i) in D. It is represented by

$$H(X,Y,i) = H^i(R(X,Y)) \in \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$$

where

$$R(X,Y) = R(\mathrm{Cone}(Y \to X)) \ .$$

The construction is functorial for morphisms of pairs. This allows to represent edges of type f^*.

Finally, we need to consider edges corresponding to coboundary maps for triples $X \supset Y \supset Z$. In this case, it follows from the construction of R that there is a natural exact triangle

$$R(X,Y) \to R(X,Z) \to R(Y,Z).$$

We use the connecting morphism in cohomology to represent the edge $(Y,Z,i) \to (X,Y,i+1)$. $\square$

For further use, we record a number of corollaries.

Corollary 8.2.21. *Every object of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of a direct sum of objects of the form $H^i_{\mathrm{Nori}}(X,Y)$ for a good pair (X,Y,i) where $X = W \smallsetminus W_\infty$ and $Y = W_0 \smallsetminus (W_0 \cap W_\infty)$ with W smooth projective, $W_\infty \cup W_0$ a divisor with normal crossings.*

Proof. By Proposition 6.1.15, every object in the diagram category of $\mathrm{VGood}^{\mathrm{eff}}$ (and hence $\mathcal{MM}_{\mathrm{Nori}}$) is a subquotient of a direct sum of some $H^i_{\mathrm{Nori}}(X,Y)$ with (X,Y,i) very good. In particular, $X \smallsetminus Y$ can be assumed smooth.

We follow Nori: By resolution of singularities, there is a smooth projective variety W and a normal crossing divisor $W_0 \cup W_\infty \subset W$ together with a proper, surjective morphism $\pi : W \smallsetminus W_\infty \to X$ such that one has $\pi^{-1}(Y) = W_0 \smallsetminus W_\infty$ and $\pi : W \smallsetminus \pi^{-1}(Y) \to X \smallsetminus Y$ is an isomorphism. This implies that

$$H^*_{\mathrm{Nori}}(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)) \to H^*_{\mathrm{Nori}}(X,Y)$$

is also an isomorphism by proper base change, i.e., excision. $\square$

Remark 8.2.22. Note that the pair $(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)$ is good, but not very good in general. Replacing Y by a larger closed subset Z, one may, however, assume that $W_0 \smallsetminus (W_0 \cap W_\infty)$ is affine. Therefore, by Lemma 8.3.7, the dual of each generator can be assumed to be very good.

It is not clear to us if it suffices to construct Nori's category using the diagram of (X,Y,i) with X smooth, Y a divisor with normal crossings. The corollary says that the diagram category has the right "generators", but there might be too few "relations".

Corollary 8.2.23. *Let $Z \subset X$ be a closed immersion. Then there is a natural object $H^i_Z(X)$ in $\mathcal{MM}_{\mathrm{Nori}}$ representing cohomology with supports. There is a natural long exact sequence*

$$\cdots \to H^i_Z(X) \to H^i_{\mathrm{Nori}}(X) \to H^i_{\mathrm{Nori}}(X \smallsetminus Z) \to H^{i+1}_Z(X) \to \cdots$$

Proof. Let $U = X \smallsetminus Z$. Put

$$R_Z(X) = R(\mathrm{Cone}(U \to X)), \quad H^i_Z(X) = H^i(R_Z(X)) \ .$$

$\square$

8.3 Tensor structure

We now introduce the tensor structure using the formal set-up developed in Section 7.1. Recall that $\mathrm{Pairs}^{\mathrm{eff}}$, $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$ are graded diagrams with $|(X,Y,i)| = i$.

Proposition 8.3.1. *The graded diagrams* Good *and* $\mathrm{VGood}^{\mathrm{eff}}$ *carry a weak commutative product structure (see Definition 7.1.3) defined as follows: for all vertices* $(X,Y,i),(X',Y',i')$

$$(X,Y,i)\times(X',Y',i')=(X\times X',X\times Y'\cup Y\times X',i+i').$$

with the obvious definition on edges. Let also

$$\alpha:(X,Y,i)\times(X',Y',i')\to(X',Y',i')\times(X,Y,i)$$
$$\beta:(X,Y,i)\times((X',Y',i')\times(X'',Y'',i''))\to((X,Y,i)\times(X',Y',i'))\times(X'',Y'',i'')$$
$$\beta':((X,Y,i)\times(X',Y',i'))\times(X'',Y'',i'')\to(X,Y,i)\times((X',Y',i')\times(X'',Y'',i''))$$

be the edges given by the natural isomorphisms of varieties.

There is a unit given by $(\mathrm{Spec}\ k,\emptyset,0)$ *and*

$$u:(X,Y,i)\to(\mathrm{Spec}\ k,\emptyset,0)\times(X,Y,i)=(\mathrm{Spec}\ k\times X,\mathrm{Spec}\ k\times Y,i)$$

be given by the natural isomorphism of varieties.

Moreover, H^* *is a graded multiplicative representation (see Definition 7.1.3) with*

$$\tau:H^{i+i'}(X\times X',X\times Y'\cup X'\times Y,\mathbb{Z})\to H^i(X,Y,\mathbb{Z})\otimes H^{i'}(X',Y',\mathbb{Z})$$

the Künneth isomorphism (see Theorem 2.4.1).

Proof. If (X,Y,i) and (X',Y',i') are good pairs, then by the Küennth formula so is $(X\times X',X\times Y'\cup Y\times X',i+i')$. If they are even very good, then so is their product. Hence $\times$ is well-defined on vertices. Recall that edges id $\mathrm{Good}^{\mathrm{eff}}\times\mathrm{Good}^{\mathrm{eff}}$ are of the form $\gamma\times\mathrm{id}$ or $\mathrm{id}\times\gamma$ for an edge γ of $\mathrm{Good}^{\mathrm{eff}}$. The definition of $\times$ on these edges is the natural one.

We need to check that H^* satisfies the conditions of Definition 7.1.3. This is tedious, but straightforward from the properties of the Künneth formula, see in particular Proposition 2.4.3 for compatibility with edges of type ∂ changing the degree. Associativity and graded commutativity are stated in Proposition 2.4.2. □

Definition 8.3.2. Let Good and VGood be the localizations (see Definition 7.2.1) of $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$, respectively, with respect to the vertex $\mathbf{1}(-1)=(\mathbb{G}_m,\{1\},1)$.

Proposition 8.3.3. Good *and* VGood *are graded diagrams with a weak commutative product structure (see Remark 7.1.6). Moreover,* H^* *is a graded multiplicative representation of* Good *and* VGood.

Proof. This follows formally from the effective case and Lemma 7.2.4. The Assumption 7.2.3 that $H^*(\mathbf{1}(-1))=\mathbb{Z}$ is satisfied by Proposition 8.1.2. □

Theorem 8.3.4. *1. This definition of $\mathcal{MM}_{\mathrm{Nori}}$ is equivalent to Nori's original definition.*

2. $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \subset \mathcal{MM}_{\mathrm{Nori}}$ are commutative tensor categories with a faithful fiber functor H^*.

3. $\mathcal{MM}_{\mathrm{Nori}}$ is equivalent to the digram categories $\mathcal{C}(\mathrm{Good}, H^*)$ and $\mathcal{C}(\mathrm{VGood}, H^*)$.

Proof. We already know by Theorem 8.2.20 that

$$\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*) = \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$$

are equivalent. Moreover, this agrees with Nori's definition using either $\mathrm{Good}^{\mathrm{eff}}$ or $\mathrm{Pairs}^{\mathrm{eff}}$.

By Proposition 8.3.1, the diagrams $\mathrm{VGood}^{\mathrm{eff}}$ and $\mathrm{Good}^{\mathrm{eff}}$ carry a mulitplicative structure. Hence by Proposition 7.1.5, the category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ carries a tensor structure.

By Proposition 7.2.5, the diagram categories of the localized diagrams Good and VGood also have tensor structure and can be equivalently defined as the localization with respect to he Lefschetz object $\mathbf{1}(-1)$.

In [L1], the category of Nori motives is defined as the category of comodules of finite type over $\mathbb{Z}$ for the localization of the ring A^{eff} with respect to the element $\chi \in A(\mathbf{1}(-1))$ considered in Proposition 7.2.5. By this same Proposition, the category of A^{eff}_{χ}-comodules agrees with $\mathcal{MM}_{\mathrm{Nori}}$. $\square$

Our next aim is to establish rigidity using the criterion of Section 7.3. Hence we need to check that Poincaré duality is motivic, at least in a weak sense.

Definition 8.3.5. Let $\mathbf{1}(-1) = H^1_{\mathrm{Nori}}(\mathbb{G}_m)$ and $\mathbf{1}(-n) = \mathbf{1}(-1)^{\otimes n}$.

Lemma 8.3.6. *1. $H^{2n}_{\mathrm{Nori}}(\mathbb{P}^N) = \mathbf{1}(-n)$ for $N \geq n \geq 0$.*

2. Let Z be a projective variety of dimension n. Then $H^{2n}_{\mathrm{Nori}}(Z) \cong \mathbf{1}(-n)$.

3. Let X be a smooth variety, $Z \subset X$ a smooth, irreducible, closed subvariety of pure codimension n. Then the motive with support of Corollary 8.2.23 satisfies

$$H^{2n}_Z(X) \cong \mathbf{1}(-n).$$

Proof. 1. Embedding projective spaces linearly into higher dimensional projective spaces induces isomorphisms on cohomology and hence motives. Hence it suffices to check the top cohomology of $\mathbb{P}^N$.

We start with $\mathbb{P}^1$. Consider the standard cover of $\mathbb{P}^1$ by $U_1 = \mathbb{A}^1$ and $U_2 = \mathbb{P}^1 \smallsetminus \{0\}$. We have $U_1 \cap U_2 = \mathbb{G}_m$. By Corollary 8.2.18,

$$R(\mathbb{P}^1) \to \mathrm{Cone}\Big(R(U_1) \oplus R(U_2) \to R(\mathbb{G}_m)\Big)[-1]$$

is an isomorphism in the derived category. This induces the isomorphism $H^2_{\mathrm{Nori}}(\mathbb{P}^1) \to H^1_{\mathrm{Nori}}(\mathbb{G}_m)$. Similarly, the Čech complex (see Definition 8.2.10) for the standard affine cover of $\mathbb{P}^N$ relates $H^{2N}_{\mathrm{Nori}}(\mathbb{P}^N)$ with $H^N_{\mathrm{Nori}}(\mathbb{G}_m^N)$.

2. Let $Z \subset \mathbb{P}^N$ be a closed immersion with N large enough. Then $H^{2n}_{\mathrm{Nori}}(Z) \to H^{2n}_{\mathrm{Nori}}(\mathbb{P}^N)$ is an isomorphism in $\mathcal{MM}_{\mathrm{Nori}}$ because it is in singular cohomology.

3. We note first that under our assumptions 3. holds in singular cohomology by the Gysin isomorphism 2.1.8

$$H^0(Z) \xrightarrow{\cong} H^{2n}_Z(X).$$

For the embedding $Z \subset X$ one has the deformation to the normal cone [Fu, Sec. 5.1], i.e., a smooth scheme $D(X,Z)$ together with a morphism to $\mathbb{A}^1$ such that the fiber over 0 is given by the normal bundle $N_Z X$ of Z in X, and the other fibers by X. The product $Z \times \mathbb{A}^1$ can be embedded into $D(X,Z)$ as a closed subvariety of codimension n, inducing the embeddings of $Z \subset X$ as well as the embedding of the zero section $Z \subset N_Z X$ over 0. Hence, using the three Gysin isomorphisms and homotopy invariance, it follows that there are isomorphisms

$$H^{2n}_Z(X) \leftarrow H^{2n}_{Z\times\mathbb{A}^1}(D(X,Z)) \to H^{2n}_Z(N_Z X)$$

in singular cohomology and hence in our category. Thus, we have reduced the problem to the embedding of the zero section $Z \hookrightarrow N_Z X$. However, the normal bundle $\pi : N_Z X \to Z$ trivializes on some dense open subset $U \subset Z$. This induces an isomorphism

$$H^{2n}_Z(N_Z X) \to H^{2n}_U(\pi^{-1}(U)),$$

and we may assume that the normal bundle $N_Z X$ is trivial. In this case, we have

$$N_Z(X) = N_{Z\times\{0\}}(Z \times \mathbb{A}^n) = N_{\{0\}}(\mathbb{A}^n),$$

so that we have reached the case of $Z = \{0\} \subset \mathbb{A}^n$. Using the Künneth formula with supports and induction on n, it suffices to consider $H^2_{\{0\}}(\mathbb{A}^1)$ which is isomorphic to $H^1(\mathbb{G}_m) = \mathbf{1}(-1)$ by Corollary 8.2.23. □

The following lemma (more precisely, its dual) is formulated implicitly in [N] in order to establish rigidity of $\mathcal{MM}_{\mathrm{Nori}}$.

Lemma 8.3.7. *Let W be a smooth projective variety of dimension i, $W_0, W_\infty \subset W$ divisors such that $W_0 \cup W_\infty$ is a normal crossing divisor. Let*

$$\begin{aligned} X &= W \smallsetminus W_\infty \\ Y &= W_0 \smallsetminus W_0 \cap W_\infty \\ X' &= W \smallsetminus W_0 \\ Y' &= W_\infty \smallsetminus W_0 \cap W_\infty \end{aligned}$$

We assume that (X, Y) is a very good pair.

Then there is a morphism in $\mathcal{MM}_{\mathrm{Nori}}$

$$q : \mathbf{1} \to H^i_{\mathrm{Nori}}(X,Y) \otimes H^i_{\mathrm{Nori}}(X',Y')(i)$$

such that the dual of $H^*(q)$ is a perfect pairing.

Proof. We follow Nori's construction. The two pairs (X,Y) and (X',Y') are Poincaré dual to each other in singular cohomology, see Proposition 2.4.5 for the proof. This implies that they are both good pairs. Hence

$$H^i_{\mathrm{Nori}}(X,Y) \otimes H^i_{\mathrm{Nori}}(X',Y') \to H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X')$$

is an isomorphism. Let $\Delta = \Delta(W \smallsetminus (W_0 \cup W_\infty))$ via the diagonal map. Note that

$$X \times Y' \cup X' \times Y \subset X \times X' \smallsetminus \Delta.$$

Hence, by functoriality and the definition of cohomology with support, there is a map

$$H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X') \leftarrow H^{2i}_{\Delta}(X \times X').$$

Again, by functoriality, there is a map

$$H^{2i}_{\Delta}(X \times X') \leftarrow H^{2i}_{\bar{\Delta}}(W \times W)$$

with $\bar{\Delta} = \Delta(W)$. By Lemma 8.3.6, it is isomorphic to $\mathbf{1}(-i)$. The map q is defined by twisting the composition by (i). The dual of this map realizes Poincaré duality, hence it is a perfect pairing. □

Theorem 8.3.8 (Nori)**.** *$\mathcal{MM}_{\mathrm{Nori}}$ is rigid, hence a neutral Tannakian category. Its Tannaka dual is given by $G_{\mathrm{mot}} = \mathrm{Spec}(A(\mathrm{Good}, H^*))$.*

Proof. By Corollary 8.2.21, every object of $\mathcal{MM}_{\mathrm{Nori}}$ is a subquotient of $M = H^i_{\mathrm{Nori}}(X,Y)(j)$ for a good pair (X,Y,i) of the particular form occurring in Lemma 8.3.7. By this Lemma, they all admit a perfect pairing.

By Proposition 7.3.4, the category $\mathcal{MM}_{\mathrm{Nori}}$ is neutral Tannakian. The Hopf algebra of its Tannaka dual agrees with Nori's algebra by Theorem 6.1.20. □

8.3.1 Collection of proofs

We go through the list of theorems of Section 8.1 and give the missing proofs.

Proof of Theorem 8.1.5. By Theorem 8.3.4, the categories $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ and $\mathcal{MM}_{\mathrm{Nori}}$ are tensor categories. By construction, H^* is a tensor functor. The category $\mathcal{MM}_{\mathrm{Nori}}$ is rigid by Theorem 8.3.8. By loc. cit., we have a description of its Tannaka dual. □

Proof of Theorem 8.1.8. We apply Proposition 8.2.16 with $\mathcal{A} = \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ and $T = H^*$, $R = \mathbb{Z}$. □

Proof of Theorem 8.1.9. We apply the universal property of the diagram category (see Corollary 6.1.14) to the diagram $\mathrm{Good}^{\mathrm{eff}}$, $T = H^*$ and $F = H'^*$. This gives the universal property for $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$.

Note that $H'^*(\mathbf{1}(-1)) \cong R$ by comparison with singular cohomology. Hence everything extends to $\mathcal{MM}_{\mathrm{Nori}}$ by localizing the categories.

If $\mathcal{A}$ is a tensor category and $H'*$ a graded multiplicative representation, then all functors are tensor functors by construction. $\square$