Periods and Nori Motives

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Part III

Part III

Periods

Chapter 9

Periods of varieties

A period or more precisely a period number may be thought of as the value of an integral that occurs in a geometric context. In their papers [K1] and [KZ], Kontsevich and Zagier list various ways of how to define a period.

It is stated in their papers without reference that all these variants give the same definition. We give a proof of this statement in the Period Theorem 11.2.1.

9.1 First definition

We start with the simplest definition. In this section, let $k \subset \mathbb{C}$ be a subfield.

For this definition the following data is needed:

- X a smooth algebraic variety of dimension d, defined over k,
- D a divisor on X with normal crossings, also defined over k,
- $\omega \in \Gamma(X, \Omega^d_{X/k})$ an algebraic differential form of top degree,
- Γ a rational d-dimensional C^∞-chain on X^{an} with $\partial\Gamma$ on D^{an}, i.e.,

$$\Gamma = \sum_{i=1}^{n} \alpha_i \gamma_i$$

  with $\alpha_i \in \mathbb{Q}$, $\gamma_i : \Delta_d \to X^{\mathrm{an}}$ a C^∞-map for all i and $\partial\Gamma$ a chain on D^{an} as in Definition 2.2.2.

As before, we denote X^{an} the analytic space attached to $X(\mathbb{C})$.

Definition 9.1.1. Let $k \subset \mathbb{C}$ be a subfield.

1. Let (X, D, ω, Γ) as above. We will call the complex number
$$\int_\Gamma \omega = \sum_{i=1}^n \alpha_i \int_{\Delta_d} f_i^* \omega \ .$$
the *period (number)* of the quadruple (X, D, ω, Γ).

2. The *algebra of effective periods* $\mathbb{P}^{\text{eff}}_{\text{nc}} = \mathbb{P}^{\text{eff}}_{\text{nc}}(k)$ *over* k is the set of all period numbers for all (X, D, ω, Γ) defined over k.

3. The *period algebra* $\mathbb{P}_{\text{nc}} = \mathbb{P}_{\text{nc}}(k)$ *over* k is the set of numbers of the form $(2\pi i)^n\alpha$ with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{P}^{\text{eff}}_{\text{nc}}$.

Remark 9.1.2. 1. The subscript nc refers to the normal crossing divisor D in the above definition.

2. We will show a bit later (see Proposition 9.1.7) that $\mathbb{P}^{\text{eff}}_{\text{nc}}(k)$ is indeed an algebra.

3. Moreover, we will see in the next example that $2\pi i \in \mathbb{P}^{\text{eff}}_{\text{nc}}$. This means that $\mathbb{P}_{\text{nc}}$ is nothing but the localization
$$\mathbb{P}_{\text{nc}} = \mathbb{P}^{\text{eff}}_{\text{nc}}\left[\frac{1}{2\pi i}\right] \ .$$

4. This definition was motivated by Kontsevich's discussion of formal effective periods [K1, def. 20, p. 62]. For an extensive discussion of formal periods and their precise relation to periods see Chapter 12.

Example 9.1.3. Let $X = \mathbb{A}^1_{\mathbb{Q}}$ be the affine line, $\omega = \mathrm{d}t \in \Omega^1$. Let $D = V(t^3 - 2t)$. Let $\gamma : [0, 1] \to \mathbb{A}^1_{\mathbb{Q}}(\mathbb{C}) = \mathbb{C}$ be the line from 0 to $\sqrt{2}$. This is a singular chain with boundary in $D(\mathbb{C}) = \{0, \sqrt{2}, -\sqrt{t}\}$. Hence it defines a class in $H_1^{\text{sing}}(\mathbb{A}^1(\mathbb{C})^{\text{an}}, D^{\text{an}}, \mathbb{Q})$. We obtain the period
$$\int_\gamma \omega = \int_0^{\sqrt{2}} \mathrm{d}t = \sqrt{2} \ .$$

The same method works for all algebraic numbers.

Example 9.1.4. Let $X = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, $D = \emptyset$ and $\omega = \frac{1}{t}\mathrm{d}t$. We choose $\gamma : S^1 \to \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ the unit circle. It defines a class in $H_1^{\text{sing}}(\mathbb{C}^*, \mathbb{Q})$. We obtain the period
$$\int_{S^1} t^{-1}\mathrm{d}t = 2\pi i \ .$$
In particular, $\pi \in \mathbb{P}^{\text{eff}}_{\text{nc}}(k)$ for all k.

Example 9.1.5. Let $X = \mathbb{G}_m$, $D = V((t-2)(t-1))$, $\omega = t^{-1}\mathrm{d}t$, and γ the line from 1 to 2. We obtain the period

$$\int_1^2 t^{-1}\mathrm{d}t = \log(2)\ .$$

For more advanced examples, see Part IV.

Lemma 9.1.6. *Let (X, D, ω, Γ) as before. The period number $\int_\Gamma \omega$ depends only on the cohomology classes of ω in relative de Rham cohomology and of Γ in relative singular homology.*

Proof. The restriction of ω to the analytification D_j^{an} of some irreducible component D^j of D is a holomorphic d-form on a complex manifold of dimension $d-1$, hence zero. Therefore the integral $\int_\triangle \omega$ evaluates to zero for smooth singular simplices $\triangle$ that are supported on D. Now if Γ', Γ'' are two representatives of the same relative homology class, we have

$$\Gamma'_d - \Gamma''_d \sim \partial(\Gamma_{d+1})$$

modulo simplices living on some D_I^{an} for a smooth singular chain Γ of dimension $d+1$

$$\Gamma \in \mathrm{C}^\infty_{d+1}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

Using Stokes' theorem, we get

$$\int_{\Gamma'_d} \omega - \int_{\Gamma''_d} \omega = \int_{\partial(\Gamma_{d+1})} \omega = \int_{\Gamma_{d+1}} d\omega = 0,$$

since ω is closed. □

In the course of the chapter, we are also going to show the converse: every pair of relative cohomology classes gives rise to a period number.

Proposition 9.1.7. *The sets $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ and $\mathbb{P}_{\mathrm{nc}}(k)$ are k-algebras. Moreover, $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K) = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ if K/k is algebraic.*

Proof. Let (X, D, ω, Γ) and $(X', D', \omega', \Gamma')$ be two quadruples as in the definition of normal crossing periods.

By multiplying ω by an element of k, we obtain k-multiples of periods.

The product of the two periods is realized by the quadruple $(X \times X', D \times X' \cup X \times D', \omega \otimes \omega', \Gamma \times \Gamma')$.

Note that the quadruple $(\mathbb{A}^1, \{0,1\}, t, [0,1])$ has period 1. By multiplying with this factor, we do not change the period number of a quadruple, but we change its dimension. Hence we can assume that X and X' have the same dimension. The sum of their periods is then realized on the disjoint union $(X \cup X', D \cup D', \omega + \omega', \Gamma + \Gamma')$.

If K/k is finite algebraic, then we obviously have $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K)$. For the converse, consider a quadruple (X, D, ω, Γ) over K. We may also can view X as k-variety and write X_k for distinction. By Lemma 3.1.13 or more precisely its proof, ω can also be viewed as a differential form on X_k/k. The complex points $Y_k(\mathbb{C})$ consist of $[K : k]$ copies of the complex points $Y(\mathbb{C})$. Let Γ_k be the cycle Γ on one of them. Then the period of (X, D, ω, Γ) is the same as the period of $(X_k, D_k, \omega, \Gamma_k)$. This gives the converse inclusion.

If K/k is infinite, but algebraic, we obviously have $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K) = \bigcup_L \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(L)$ with L running through all fields $K \supset L \supset k$ finite over k. Hence, equality also holds in the general case. □

9.2 Periods for the category $(k, \mathbb{Q})-\mathrm{Vect}$

For a clean development of the theory of period numbers, it is of advantage to formalize the data. Recall from Section 5.1 the category $(k, \mathbb{Q})-\mathrm{Vect}$. Its objects are a pair of k-vector space V_k and $\mathbb{Q}$-vector space $V_\mathbb{Q}$ linked by an isomorphism $\phi_\mathbb{C} : V_k \otimes_k \mathbb{C} \to V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$. This is precisely what we need in order to define periods abstractly.

Definition 9.2.1. 1. Let $V = (V_k, V_\mathbb{Q}, \phi_\mathbb{C})$ be an object of $(k, \mathbb{Q})-\mathrm{Vect}$. The *period matrix* of V is the matrix of $\phi_\mathbb{C}$ in a choice of bases $v_1, \dots, v_n$ of V_k and $w_1, \dots, w_n$ of $V_\mathbb{Q}$, respectively. A complex number is a *period* of V if is an entry of a period matrix of V for some choice of bases. The set of periods of V together with the number 0 is denoted $\mathbb{P}(V)$. We denote by $\mathbb{P}\langle V\rangle$ the k-subvector space of $\mathbb{C}$ generated by the entries of the period matrix.

2. Let $\mathcal{C} \subset (k, \mathbb{Q})-\mathrm{Vect}$ be a subcategory. We denote by $\mathbb{P}(\mathcal{C})$ the set of periods for all objects in $\mathcal{C}$.

Remark 9.2.2. 1. The object $V = (V_k, V_\mathbb{Q}, \phi_\mathbb{C})$ gives rise to a bilinear map

$$V_k \times V_\mathbb{Q}^\vee \to \mathbb{C}\ , \quad (v, \lambda) \mapsto \lambda(\phi_\mathbb{C}^{-1}(v))\ ,$$

where we have extended $\lambda : V_\mathbb{Q} \to \mathbb{Q}$ $\mathbb{C}$-linearly to $V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} \to \mathbb{C}$. The periods of V are the numbers in its image. Note that it is a set, not a vector space in general. The period matrix depends on the choice of bases, but the vector space $\mathbb{P}\langle V\rangle$ does not.

2. The definition of $\mathbb{P}(\mathcal{C})$ does not depend on the morphisms. If the category has only one object, the second definition specializes to the first.

Lemma 9.2.3. *Let* $\mathcal{C} \subset (k, \mathbb{Q})-\mathrm{Vect}$ *be a subcategory.*

1. $\mathbb{P}(\mathcal{C})$ *is closed under multiplication by* k.

2. *If* $\mathcal{C}$ *is additive, then* $\mathbb{P}(\mathcal{C})$ *is a* k*-vector space.*

3. *If $\mathcal{C}$ is a tensor subcategory, then $\mathbb{P}(\mathcal{C})$ is a k-algbra.*

Proof. Multiplying a basis element w_i by an element α in k multiplies the periods by α. Hence the set is closed under multiplication by elements of k^*.

Let p be a period of V and p' a period of V'. Then $p+p'$ is a period of $V \oplus V'$. If $\mathcal{C}$ is additive, then $V, V' \in \mathcal{C}$ implies $V \oplus V' \in \mathcal{C}$. Moreover, pp' is a period of $V \otimes V'$. If $\mathcal{C}$ is a tensor subcategory of $(k, \mathbb{Q})-$Vect, then $V \otimes V'$ is also in $\mathcal{C}$. $\square$

Proposition 9.2.4. *Let $\mathcal{C} \subset (k, \mathbb{Q})-$Vect be a subcategory.*

1. *Let $\langle \mathcal{C} \rangle$ be the smallest full abelian subcategory of $(k, \mathbb{Q})-$Vect closed under subquotients and containing $\mathcal{C}$. Then $\mathbb{P}(\langle \mathcal{C} \rangle)$ is the abelian subgroup of $\mathbb{C}$ generated by $\mathbb{P}(\mathcal{C})$.*

2. *Let $\langle \mathcal{C} \rangle^{\otimes}$ be the smallest full abelian subcategory of $(k, \mathbb{Q})-$Vect closed under subquotients and tensor products and containing $\mathcal{C}$. Then $\mathbb{P}(\langle \mathcal{C} \rangle^{\otimes})$ is the (possibly non-unital) subring of $\mathbb{C}$ generated by $\mathbb{P}(\mathcal{C})$.*

Proof. The period algebra $\mathbb{P}(\mathcal{C})$ only depends on objects. Hence we can replace $\mathcal{C}$ by the full subscategory with the same objects without changing the period algebra.

Moreover, if $V \in \mathcal{C}$ and $V' \subset V$ in $(k, \mathbb{Q})-$Vect, then we can extend any basis for V' to a basis to V. In this form, the period matrix for V is block triangular with one of the blocks the period matrix of V'. This implies

$$\mathbb{P}(V') \subset \mathbb{P}(V) \ .$$

Hence, $\mathbb{P}(\mathcal{C})$ does not change, if we close it up under subobjects in $(k, \mathbb{Q})-$Vect. The same argument also implies that $\mathbb{P}(\mathcal{C})$ does not change if we close it up under quotients in $(k, \mathbb{Q})-$Vect.

After these reductions, the only thing missing to make $\mathcal{C}$ additive is closing it up under direct sums in $(k, \mathbb{Q})-$Vect. If V and V' are objects of $\mathcal{C}$, then the periods of $V \oplus V'$ are sums of periods of V and periods of V' (this is most easily seen in the pairing point of view in Remark 9.2.2). Hence closing the category up under direct sums amounts to passing from $\mathbb{P}(\mathcal{C})$ to the abelian group generated by it. It is automatically a k-vector space.

If V and V' are objects of $\mathcal{C}$, then the periods of $V \otimes V'$ are sums of products of periods of V and periods of V' (this is again most easily seen in the pairing point of view in Remark 9.2.2). Hence closing $\mathcal{C}$ up under tensor products (and their subquotients) amounts to passing to the ring generated by $\mathbb{P}(\mathcal{C})$. $\square$

So far, we fixed the ground field k. We now want to study the behaviour under change of fields.

Definition 9.2.5. Let K/k be a finite extension of subfields of $\mathbb{C}$. Let

$$\otimes_k K : (k, \mathbb{Q})-\mathrm{Vect} \to (K, \mathbb{Q})-\mathrm{Vect}\ , (V_k, V_{\mathbb{Q}}, \phi_{\mathbb{C}}) \mapsto (V_k \otimes_k K, V_{\mathbb{Q}}, \phi_{\mathbb{C}})$$

be the *extension of scalars.*

Lemma 9.2.6. *Let K/k be a finite extension of subfields of $\mathbb{C}$. Let $V \in (k, \mathbb{Q})-\mathrm{Vect}$. Then*

$$\mathbb{P}\langle V \otimes_k K\rangle = \mathbb{P}\langle V\rangle \otimes_k K\ .$$

Proof. The period matrix for V agrees with the period matrix for $V \otimes_k K$. On the left hand side, we pass to the K-vector space generated by its entries. On the right hand side, we first pass to the k-vector space generated by its entries, and then extend scalars. $\square$

Conversely, there is a *restriction of scalars* where we view a K-vector space V_K as a k-vector space.

Lemma 9.2.7. *Let K/k be a finite extension of subfields of $\mathbb{C}$. Then the functor $\otimes_k K$ has a right adjoint*

$$R_{K/k} : (K, \mathbb{Q})-\mathrm{Vect} \to (k, \mathbb{Q})-\mathrm{Vect}$$

For $W \in (K, \mathbb{Q})-\mathrm{Vect}$ we have

$$\mathbb{P}\langle W\rangle = \mathbb{P}\langle R_{K/k} W\rangle\ .$$

Proof. Choose a k-basis $e_1, \dots, e_n$ of K. We put

$$R_{K/k} : (K, \mathbb{Q})-\mathrm{Vect} \to (k, \mathbb{Q})-\mathrm{Vect}\ ,\ (W_K, W_{\mathbb{Q}}, \phi_{\mathbb{C}}) \mapsto (W_K, W_{\mathbb{Q}}^{[K:k]}, \psi_{\mathbb{C}})\ ,$$

where

$$\psi_{\mathbb{C}} : W_K \otimes_k \mathbb{C} = W_K \otimes_k K \otimes_K \mathbb{C} \cong (W_K \otimes_K \mathbb{C})^{[K:k]} \to (W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C})^{[K:k]}$$

maps elements of the form $w \otimes e_i$ to $\phi_{\mathbb{C}}(w \otimes e_i)$ in the i-component.

It is easy to check the universal property. We describe the unit and the counit. The natural map

$$V \to R_{K/k}(V \otimes_k K)$$

is given on the component V_k by the natural inclusion $V_k \to V_k \otimes K$. In order to describe it on the $\mathbb{Q}$-component, decompose $1 = \sum_{i=1}^n a_i e_i$ in K and put

$$V_{\mathbb{Q}} \to V_{\mathbb{Q}}^n \qquad v \mapsto (a_i v)_{i=1}^n\ .$$

The natural map

$$(R_{K/k} W) \otimes_k K \to W$$

is given on the K-component as the multiplication map

$$W_K \otimes_k K \to W_K$$

and on the $\mathbb{Q}$-component
$$W_{\mathbb{Q}}^n \to W_{\mathbb{Q}}$$
by summation.

This shows existence of the right adjoint. In particular, $R_{K/k}W$ is functorial and independent of the choice of basis.

In order to compute periods, we have to choose bases. Fix a $\mathbb{Q}$-basis $x_1, \dots, x_n$ of $W_{\mathbb{Q}}$. This also defines a $\mathbb{Q}$-basis for $W_{\mathbb{Q}}^n$ in the obvious way. Fix a K-basis $y_1, \dots, y_n$ of W_K. Multiplying by $e_1, \dots, e_n$, we obtain a k-basis of W_K. The entries of the period matrix of W are the coefficients of $\phi_{\mathbb{C}}(y_j)$ in the basis x_l The entries of the period matrix of $R_{K/k}W$ are the coefficients of $\phi_{\mathbb{C}}(e_i y_j) =$ in the basis x_l. Hence, the K-linear span of the former agrees with the k-linear span of the latter. □

Recall from Example 5.1.4 the object $L(\alpha) \in (k, \mathbb{Q})-\mathrm{Vect}$ for a complex number $\alpha \in \mathbb{C}^*$. It is given by the data $(k, \mathbb{Q}, \alpha)$. It is invertible for the tensor structure.

Definition 9.2.8. Let $L(\alpha) \in (k, \mathbb{Q})-\mathrm{Vect}$ be invertible. We call a pairing in $(k, \mathbb{Q})-\mathrm{Vect}$
$$V \times W \to L(\alpha)$$
perfect, if it is non-degenerate in the k- and $\mathbb{Q}$-components. Equivalently, the pairing induces an isomorphism
$$V \cong W^\vee \otimes L(\alpha)$$
where $\cdot^\vee$ denotes the dual in $(k, \mathbb{Q})-\mathrm{Vect}$.

Lemma 9.2.9. *Assume that*
$$V \times W \to L(\alpha)$$
is a perfect pairing. Then
$$\mathbb{P}\langle V, W, V^\vee, W^\vee\rangle^{\oplus,\otimes} \subset \mathbb{P}\langle V, W\rangle^{\oplus,\otimes}[\alpha^{-1}] \ .$$

Proof. The left hand side is the ring generated by $\mathbb{P}(V)$, $\mathbb{P}(W)$, $\mathbb{P}(V^\vee)$ and $\mathbb{P}(W^\vee)$. Hence we need to show that $\mathbb{P}(V^\vee)$ and $\mathbb{P}(W^\vee)$ are contained in the right hand side. This is true because $W^\vee \cong V \otimes L(\alpha^{-1})$ and $\mathbb{P}(V \otimes L(\alpha^{-1}) = \alpha^{-1}\mathbb{P}(V)$ □

9.3 Periods of algebraic varieties

9.3.1 Definition

Recall from Definition 8.1.1 the directed graph of effective pairs $\mathrm{Pairs}^{\mathrm{eff}}$. Its vertices are triples (X, D, j) with X a variety, D a closed subvariety and i an integer. The edges are not of importance for the consideration of periods.

Definition 9.3.1. Let (X, D, j) be a vertex of the diagram $\mathrm{Pairs}^{\mathrm{eff}}$.

1. The *set of periods* $\mathbb{P}(X, D, j)$ is the image of the period paring (see Definition and 5.3.1 and 5.5.4
$$\mathrm{per} : H^j_{\mathrm{dR}}(X, D) \times H^{\mathrm{sing}}_j(X^{\mathrm{an}}, D^{\mathrm{an}}) \to \mathbb{C}\ .$$
2. In the same situation, the *space of periods* $\mathbb{P}\langle X, D, j\rangle$ is the $\mathbb{Q}$-vector space generated by $\mathbb{P}(X, D, j)$.
3. Let S be a set of vertices in pairs(k). We define the *set of periods* $\mathbb{P}(S)$ as the union of the $\mathbb{P}(X, D, j)$ for (X, D, j) in S and the *k-space of periods* $\mathbb{P}\langle S\rangle$ as the sum of the $\mathbb{P}\langle X, D\rangle$ for $(X, D, j) \in S$.
4. The *effective period algebra* $\mathbb{P}^{\mathrm{eff}}(k)$ of k is defined as $\mathbb{P}(S)$ for S the set of (isomorphism classes of) all vertices (X, D, j).
5. The *period algebra* $\mathbb{P}(k)$ of k is defined as the set of complex numbers of the form $(2\pi i)^n\alpha$ with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{P}^{\mathrm{eff}}(k)$.

Remark 9.3.2. Note that $\mathbb{P}(X, D, j)$ is closed under multiplication by elements in k but not under addition. However, $\mathbb{P}^{\mathrm{eff}}(k)$ is indeed an algebra by Corollary 9.3.5. This means that $\mathbb{P}(k)$ is nothing but the localization
$$\mathbb{P}(k) = \mathbb{P}^{\mathrm{eff}}(k)\left[\frac{1}{2\pi i}\right]\ .$$
Passing to this localization is very natural from the point of view of motives: it corresponds to passing from periods of effective motives to periods of all mixed motives. For more details, see Chapter 10.

Example 9.3.3. Let $X = \mathbb{P}^n_k$. Then $(\mathbb{P}^n_k, \emptyset, 2j)$ has period set $(2\pi i)^j k^*$. The easiest way to see this is by computing the motive of $\mathbb{P}^n_k$, e.g., in Lemma 8.3.6. It is given by $\mathbf{1}(-j)$. By compatibility with tensor product, it suffices to consider the case $j = 1$ where the same motive can be defined from the pair $(\mathbb{G}_m, \emptyset, 1)$. It has the period $2\pi i$ by Example 9.1.4. The factor k^* appears because we may multiply the basis vector in de Rham cohomology by a factor in k^*.

Recall from Theorem 5.3.3 and Theorem 5.5.6 that we have an explicit desciption of the period isomorphism by integration.

Lemma 9.3.4. *There are natural inclusions $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}(k)$ and $\mathbb{P}_{\mathrm{nc}}(k) \subset \mathbb{P}(k)$.*

Proof. By definition, it suffices to consider the effective case. By Lemma 9.1.6, the period in $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ only depends on the cohomology class. By Theorem 3.3.19, the period in $\mathbb{P}^{\mathrm{eff}}(k)$ is defined by integration, i.e., by the formula in the definition of $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$. □

The converse inclusion is deeper, see Theorem 9.4.2.

9.3.2 First properties

Recall from Definition 5.4.2 that there is a functor

$$\mathrm{H} : \mathrm{Pairs}^{\mathrm{eff}} \to (k, \mathbb{Q})\mathrm{-Vect}$$

where the category $(k, \mathbb{Q})-\mathrm{Vect}$ was introduced in Section 5.1. By construction, we have

$$\begin{aligned}\mathbb{P}(X, D, j) &= \mathbb{P}(\mathrm{H}(X, D, j)),\\ \mathbb{P}\langle X, D, j\rangle &= \mathbb{P}\langle \mathrm{H}(X, D, j)\rangle,\\ \mathbb{P}^{\mathrm{eff}}(k) &= \mathbb{P}(\mathrm{H}(\mathrm{Pairs}^{\mathrm{eff}}))\ .\end{aligned}$$

This means that we can apply the abstract considerations of Section 5.1 to our periods algebras.

Corollary 9.3.5. *1. $\mathbb{P}^{\mathrm{eff}}(k)$ and $\mathbb{P}(k)$ are k-subalgebras of $\mathbb{C}$.*

2. If K/k is an algebraic extension of subfields of K, then $\mathbb{P}^{\mathrm{eff}}(K) = \mathbb{P}^{\mathrm{eff}}(k)$ and $\mathbb{P}(K) = \mathbb{P}(k)$.

3. If k is countable, then so is $\mathbb{P}(k)$.

Proof. It suffices to consider the effective case. The image of H is closed under direct sums because direct sums are realized by disjoint unions of effective pairs. As in the proof of Proposition 9.1.7, we can use $(\mathbb{A}^1, \{0, 1\}, 1)$ in order to shift the cohomological degree without changing the periods.

The image of H is also closed under tensor product. Hence its period set is a k-algebra by Lemma 9.2.3.

Let K/k be finite. For (X, D, i) over k, we have the base change (X_K, D_K, i) over K. By compatibility of the de Rham realization with base change (see Lemma 3.2.14), we have

$$\mathrm{H}(X, D, i) \otimes K = \mathrm{H}(X_K, D_K, j)\ .$$

By Lemma 9.2.6, this implies that the periods of (X, D, j) are contained in the periods of the base change. Hence $\mathbb{P}^{\mathrm{eff}}(k) \subset \mathbb{P}^{\mathrm{eff}}(K)$.

Conversely, if (Y, E, m) is defined over K, we may view it as defined over k via the map $\mathrm{Spec}K \to \mathrm{Spec}k$. We write (Y_k, E_k, m) in order to avoid confusion. Note that $Y_k(\mathbb{C})$ consists of $[K : k]$ many copies of $Y(\mathbb{C})$. Moreover, by Lemma 3.2.15, de Rham cohomology of Y/K agrees with de Rham cohomology of Y_k/k. Hence

$$\mathrm{H}(Y_k, E_k, m) = R_{K/k}\mathrm{H}(Y, E, m)$$

and their period sets agree by Lemma 9.2.7. Hence, we also have $\mathbb{P}^{\mathrm{eff}}(K) \subset \mathbb{P}^{\mathrm{eff}}(k)$.

Let k be countable. For each triple (X, D, j), the cohomologies $H^j_{\mathrm{dR}}(X)$ and $H_j^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}, \mathbb{Q})$ are countable. Hence, the image of period pairing is also countable. There are only countably many isomorphism classes of pairs (X, D, j), hence the set $\mathbb{P}^{\mathrm{eff}}(k)$ is countable. $\square$

9.4 The comparison theorem

We introduce two more variants of period algebras. Recall from Corollary 5.5.2 the functor

$$R\Gamma : K^-(\mathbb{Z}\mathrm{Sm}) \to D^+_{(k,\mathbb{Q})}$$

and

$$\mathrm{H}^i : K^-(\mathbb{Z}\mathrm{Sm}) \to (k, \mathbb{Q})\mathrm{-Vect}\ .$$

Definition 9.4.1.

- Let $\mathcal{C}(\mathrm{Sm})$ be the full abelian subcategory of $(k, \mathbb{Q})\mathrm{-Vect}$ closed under subquotients generated by $\mathrm{H}^i(X_\bullet)$ for $X_\bullet \in K^-(\mathbb{Z}\mathrm{Sm})$. Let $\mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{Sm}))$ be the *algebra of periods of complexes of smooth varieties.*
- Let $\mathcal{C}(\mathrm{SmAff})$ be the full abelian subcategory of $(k, \mathbb{Q})\mathrm{-Vect}$ closed under subquotients generated by $\mathrm{H}^i(X_\bullet)$ for $X_\bullet \in K^-(\mathbb{Z}\mathrm{SmAff})$ with SmAff the category of smooth affine varieties over k. Let $\mathbb{P}_{\mathrm{SmAff}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{SmAff}))$ be the *algebra of periods of complexes of smooth affine varieties.*

Theorem 9.4.2. *Let $k \subset \mathbb{C}$ be a subfield. Then all definitions of period algebras given so far agree:*

$$\mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}_{\mathrm{SmAff}}(k)$$

and

$$\mathbb{P}(k) = \mathbb{P}_{\mathrm{nc}}(k)\ .$$

Remark 9.4.3. This is a simple corollary of Theorem 8.2.20 and Corollary 8.2.21, once we will have discussed the formal period algebra, see Corollary 12.1.9. However, the argument does not use the full force of Nori's machine, hence we give the argument directly. Note that the key input is the same as the key input into Nori's construction: the existence of good filtrations.

Remark 9.4.4. We do not know whether $\mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$. The concrete definition of $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ only admits de Rham classes which are represented by a global differential form. This is true for all classes in the affine case, but not in general.

Proof. We are going to prove the identities on periods by showing that the subcategories of $(k, \mathbb{Q})\mathrm{-Vect}$ appearing in their definitions are the same.

Let $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$ (respectively, $\mathcal{C}(\mathrm{nc})$) be the full abelian subcategory closed under subquotients generated by $\mathrm{H}(X, D, j)$ for $(X, D) \in \mathrm{Pairs}^{\mathrm{eff}}$ (respectively

$\mathrm{H}^d(X,D)$ with X smooth affine of dimension d and D a divisor with normal crossings).

By definition
$$\mathcal{C}(\mathrm{nc}) \subset \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}) \ .$$
By the construction in Definition 3.3.6, we may compute any $\mathrm{H}(X,D,j)$ as $\mathrm{H}^j(C_\bullet)$ with $C_\bullet$ in $C^-(\mathbb{Z}\mathrm{Sm})$. Actually, the degree cohomology only depends on a bounded piece of $C_\bullet$. Hence
$$\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}) \subset \mathcal{C}(\mathrm{Sm}) \ .$$
We next show that
$$\mathcal{C}(\mathrm{Sm}) \subset \mathcal{C}(\mathrm{SmAff}) \ .$$
Let $X_\bullet \in C^-(\mathbb{Z}\mathrm{Sm})$. By Lemma 8.2.9, there is a rigidified affine cover $\tilde{U}_{X_\bullet}$ of $X_\bullet$. Let $C_\bullet = C_\bullet(\tilde{U}_{X_\bullet})$ be the total complex of the associated complex of Čech complexes (see Definition 8.2.10). By construction, $C_\bullet \in C^-(\mathbb{Z}\mathrm{SmAff})$. By the Mayer-Vietoris property, we have
$$\mathrm{H}(X_\bullet) = \mathrm{H}(C_\bullet).$$
We claim that $\mathcal{C}(\mathrm{SmAff}) \subset \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$. It suffices to consider bounded complexes because the cohomology of a bounded above complex of varieties only depends on a bounded quotient. Let X be smooth affine. Recall (see Proposition 8.2.2) that a very good filtration on X is a sequence of subvarieties
$$F_0X \subset F_1X \subset \dots F_nX = X$$
such that $F_jX \smallsetminus F_{j-1}X$ is smooth with F_jX of dimension j or $F_jX = F_{j-1}X$ of dimension less that j and cohomology of $(F_jX, F_{j-1}X)$ is concentrated in degree j. The boundary maps for the triples $F_{j-2}X \subset F_{j-1}X \subset F_jX$ define a complex $\tilde{R}(F_.X)$ in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$
$$\dots \to \mathrm{H}^{j-1}(F_{j-1}X, F_{j-2}X) \to \mathrm{H}^j(F_jX, F_{j-1}X) \to \mathrm{H}^{j+1}(F_{j+1}X, F_jX) \to \dots$$
whose cohomology agrees with $\mathrm{H}^\bullet(X)$.

Let $X_\bullet \in C^b(\mathbb{Z}\mathrm{SmAff})$. By Lemma 8.2.14, we can choose good filtration on all X_n in a compatible way. The double complex $\tilde{R}(F_.X)$ has the same cohomology as $X_\bullet$. By construction, it is a complex in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$, hence the cohomology is in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$.

Hence, we have now established that
$$\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}_{\mathrm{SmAff}}(k) \ .$$
We refine the argument in order to show that $\mathbb{P}_{\mathrm{SmAff}}(k) \subset \mathbb{P}_{\mathrm{nc}}(k)$. By the above computation, this will follow if periods of very good pairs are contained in $\mathbb{P}_{\mathrm{nc}}(k)$. We recall the construction of very good pairs (X,Y,n) by the direct

proof of Nori's Basic Lemma I in Section 2.5.1. We let $\tilde{X}$, D_0 and D_∞ be as in Lemma 2.5.7. In particular, there is a proper surjective map $\tilde{X} \setminus D_\infty \to X$ and $D_0 \setminus D_0 \cap D_\infty = \pi^{-1}Y$. Hence the periods of (X, Y, n) are the same as the periods of $(\tilde{X} \setminus D_0, D_\infty \setminus D_0 \cap D_\infty, n)$. The latter cohomology is Poincaré dual to the cohomology of the pair $(X', Y', n) = (\tilde{X} \setminus D_\infty, D_0 \setminus D_0 \cap D_\infty, n)$ by Theorem 2.4.5. In particular, all three are very good pairs with cohomology concentrated in degree n and free. Indeed, there is a natural pairing in $\mathcal{C}$

$$\mathrm{H}^d(X, Y) \times \mathrm{H}^d(X', Y') \to L((2\pi i)^d).$$

This is shown by the same arguments as in the proof of Lemma 8.3.7 but with the functor H instead of H^i_{Nori}. By Lemma 9.2.9, the periods of (X, Y) agree up to multiplication by $(2\pi i)^d$ with the periods of (X', Y'). We are now in the situation where X' is smooth affine of dimension n and Y' is a divisor with normal crossings. By Proposition 3.3.19, every de Rham cohomology class in degree n is represented by a global differential form on X. Hence all cohomological periods of (X', Y', n) are normal crossing periods in the sense of Definition 9.1.1. $\square$

Chapter 10

Categories of mixed motives

There are different candidates for the category of mixed motives over a field k of characteristic zero. The category of Nori motives of Chapter 8 is one of them. We review two more.

10.1 Geometric motives

We recall the definition of geometrical motives first introduced by Voevodsky, see [VSF] Chapter 5.

As before let $k \subset \mathbb{C}$ be a field (most of the time suppressed in the notation).

Definition 10.1.1 ([VSF] Chap. 5, Sect. 2.1)**.** The category of *finite correspondences* SmCor_k has as objects smooth k-varieties and as morphisms from X to Y the vector space of $\mathbb{Q}$-linear combinations of integral correspondences $\Gamma \subset X \times Y$ which are finite over X and dominant over a component of X.

The composition of $\Gamma : X \to Y$ and $\Gamma' : Y \to Z$ is defined by push-forward of the intersection of $\Gamma \times Z$ and $X \times \Gamma'$ in $X \times Y \times Z$ to $X \times Z$. The identity morphism is given by the diagonal. There is a natural covariant functor

$$\mathrm{Sm}_k \to \mathrm{SmCor}_k$$

which maps a smooth variety to itself and a morphism to its graph.

The category SmCor_k is additive, hence we can consider its homotopy category $K^b(\mathrm{SmCor}_k)$. The latter is triangulated.

Definition 10.1.2 ([VSF] Ch. 5, Defn. 2.1.1)**.** The category of *effective geometrical motives* $DM^{\mathrm{eff}}_{\mathrm{gm}} = DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the pseudo-abelian hull of the localization of $K^b(\mathrm{SmCor}_k)$ with respect to the thick subcategory generated by objects of the form

$$[X \times \mathbb{A}^1 \to X]$$

for all smooth varieties X and

$$[U \cap V \to U \amalg V \to X]$$

for all open covers $U \cup V = X$ for all smooth varieties X.

Remark 10.1.3. We think of $DM_{\mathrm{gm}}^{\mathrm{eff}}$ as the bounded derived category of the conjectural abelian category of effective mixed motives.

We denote by

$$M : \mathrm{SmCor}_k \to DM_{\mathrm{gm}}^{\mathrm{eff}}$$

the functor which views a variety as a complex concentrated in degree 0. By [VSF] Ch. 5 Section 2.2, it extends (non-trivially!) to a functor on the category of all k-varieties.

$DM_{\mathrm{gm}}^{\mathrm{eff}}$ is tensor triangulated such that

$$M(X) \otimes M(Y) = M(X \times Y)$$

for all smooth varieties X and Y. The unit of the tensor structure is given by

$$\mathbb{Q}(0) = M(\mathrm{Spec} k) \ .$$

The *Tate motive* $\mathbb{Q}(1)$ is defined by the equation

$$M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \ .$$

We write $M(n) = M \otimes \mathbb{Q}(1)^{\otimes n}$ for $n \geq 0$. By [VSF], Chap. 5 Section 2.2, the functor

$$(n) : DM_{\mathrm{gm}}^{\mathrm{eff}} \to DM_{\mathrm{gm}}^{\mathrm{eff}}$$

is fully faithful.

Definition 10.1.4. The category of *geometrical motives* DM_{gm} is the stabilization of $DM_{\mathrm{gm}}^{\mathrm{eff}}$ with respect to $\mathbb{Q}(1)$. Objects are of the form $M(n)$ for $n \in \mathbb{Z}$ with

$$\mathrm{Hom}_{DM_{\mathrm{gm}}}(M(n), M'(n')) = \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}}(M(n+N), M'(n'+N)) \qquad N \gg 0 \ .$$

Remark 10.1.5. We think of DM_{gm} as the bounded derived category of the conjectural abelian category of mixed motives.

The category DM_{gm} is rigid by [VSF], Chap. 5 Section 2.2, i.e., every object M has a strong dual $M^\vee$ such that

$$\mathrm{Hom}_{DM_{\mathrm{gm}}}(A \otimes B, C) = \mathrm{Hom}_{DM_{\mathrm{gm}}}(A, B^\vee \otimes C)$$
$$A^\vee \otimes B^\vee = (A \otimes B)^\vee$$
$$(A^\vee)^\vee = A$$

for all objects A, B, C.

Remark 10.1.6. Rigidity is a deep result. It depends on a moving lemma for cycles and computations in Voevodsky's category of motivic complexes.

Example 10.1.7. If X is smooth and projective of pure dimension d, then

$$M(X)^\vee = M(X)(-d)[-2d] \ .$$

10.2 Absolute Hodge motives

The notion of absolute Hodge motives was introduced by Deligne ([DMOS] Chapter II in the pure case), and independently by Jannsen in ([Ja1]). We follow the presentation of Jannsen, also used in our own extension to the triangulated setting ([Hu1]). We give a rough overview over the construction and refer to the literature for full details.

We fix a subfield $k \subset \mathbb{C}$ and an algebraic closure $\bar{k}/k$. Let $G_k = \mathrm{Gal}(\bar{k}/k)$. Let S be the set of embeddings $\sigma : k \to \mathbb{C}$ and $\bar{S}$ the set of embeddings $\bar{\sigma} : \bar{k} \to \mathbb{C}$. Restriction induces $\bar{S} \to S$.

Definition 10.2.1 ([Hu1] Defn. 11.1.1)**.** Let $\mathcal{MR} = \mathcal{MR}(k)$ be the additive category of *mixed realizations* with objects given by the following data:

- a bifiltered k-vector space A_{dR};
- for each prime l, a filtered $\mathbb{Q}_l$-vector space A_l with a continuous operation of G_k;
- for each prime l and each $\sigma \in S$, a filtered $\mathbb{Q}_l$-vector space $A_{\sigma,l}$;
- for each $\sigma \in S$, a filtered $\mathbb{Q}$-vector space A_σ;
- for each $\sigma \in S$, a filtered $\mathbb{C}$-vector space $A_{\sigma,\mathbb{C}}$;
- for each $\sigma \in S$, a filtered isomorphism

$$I_{\mathrm{dR},\sigma}; A_{\mathrm{dR}} \otimes_\sigma \mathbb{C} \to A_{\sigma,\mathbb{C}} \ ;$$

- for each $\sigma \in S$, a filtered isomorphism

$$I_{\sigma,\mathbb{C}} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to A_{\sigma,\mathbb{C}} \ ;$$

- for each $\sigma \in S$ and each prime l, a filtered isomorphism

$$I_{\bar{\sigma},l} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_l \to A_{\sigma,l} \ ;$$

- for each prime l and each $\sigma \in S$, a filtered isomorphism

$$I_{l,\sigma} : A_l \otimes_{\mathbb{Q}} \mathbb{Q}_l \to A_{\sigma,l} \ .$$

These data are subject to the following conditions:

- For each σ, the tuple $(A_\sigma, A_{\sigma,\mathbb{C}}, I_{\sigma,\mathbb{C}})$ is a mixed Hodge structure;
- For each l, the filtration on A_l is the *filtration by weights*: its graded pieces $\mathrm{gr}_n^W A_l$ extends to a model of finite type over $\mathbb{Z}$ which is pointwise pure of weight n in the sense of Deligne, i.e., for each closed point with residue field κ, the operation of Frobenius has eigenvalues $N(\kappa)^{n/2}$.

Morphisms of mixed realizations are morphisms of this data compatible with all filtrations and comparison isomorphisms.

The above has already used the notion of a Hodge structure as introduced by Deligne.

Definition 10.2.2 (Deligne [D4])**.** A *mixed Hodge structure* consists of the following data:

- a finite dimensional filtered $\mathbb{Q}$-vector space $(V_\mathbb{Q}, W_*)$;
- a finite dimensional bifiltered $\mathbb{C}$-vector space $(V_\mathbb{C}, W_*, F^*)$;
- a filtered isomorphism $I_\mathbb{C} : (V_\mathbb{Q}, W_*) \otimes \mathbb{C} \to (V_\mathbb{C}, W_*)$

sucht that for all $n \in \mathbb{Z}$ the induced triple $(\mathrm{gr}_n^W V_\mathbb{Q}, \mathrm{gr}_n^W V_\mathbb{C}, \mathrm{gr}_n^W I)$ satisfies

$$\mathrm{gr}_n^W V_C = \bigoplus_{p+q=n} F^p \mathrm{gr}_n^W V_\mathbb{C} \oplus \overline{F^q \mathrm{gr}_n V_\mathbb{C}}$$

with complex conjugation taken with respect to the $\mathbb{R}$-structure defined by $\mathrm{gr}_n^W V_\mathbb{Q} \otimes \mathbb{R}$.

A Hodge structure is called *pure of weight* n if W_* is concentrated in degree n. It is called *pure* if it is direct sum of pure Hodge structures of different weights.

A morphism of Hodge structures are morphisms of this data compatible with filtration and comparison isomorphism.

By [D4] this is an abelian category. All morphisms of Hodge structures are automatically strictly compatible with filtrations. This implies immediately:

Proposition 10.2.3 ([Hu1] Lemma 11.1.2)**.** *The category $\mathcal{MR}$ is abelian. Kernels and cokernels are computed componentwise.*

The notation is suggestive. If X is a smooth variety, then there is a natural mixed realization $H = H^*_{\mathcal{MR}}(X)$ with

- $H_{\mathrm{dR}} = H^*_{\mathrm{dR}}(X)$ algebraic de Rham cohomology as in Chapter 3 Section 3.1;

- $H_l = H^*(X_{\bar{k}}, \mathbb{Q}_l)$ the l-adic cohomology with its natural Galois operation;
- $H_\sigma = H^*(X \times_\sigma \mathrm{Spec}(\mathbb{C}), \mathbb{Q})$ singular cohomology;
- $H_{\sigma,\mathbb{C}} = H_\sigma \otimes \mathbb{C}$ and $H_{\sigma,l} = H_\sigma \otimes \mathbb{Q}_l$;
- $I_{\mathrm{dR},\sigma}$ is the period isomorphism of Definition 5.3.1 .
- $I_{l,\sigma}$ is induced by the comparison isomorphism between l-adic and singular cohomology over $\mathbb{C}$.

Remark 10.2.4. If we assume the Hodge or the Tate conjecture, then the functor $H^*_{\mathcal{MR}}$ is fully faithful on the category of Grothendieck motives (with homological or, under these assumptions equivalently, numerical equivalence). Hence it gives a linear algebra description of the conjectural abelian category of pure motives.

Jannsen ([Ja1] Theorem 6.11.1) extends the definition to singular varieties. A refined version of his construction is given in [Hu1]. We sum up its properties.

Definition 10.2.5 ([Hu2] Defn. 2.2.2)**.** Let C^+ be the category with objects given by a tuple of complexes in the additive categories in Definition 10.2.1 with filtered quasi-isomorphisms between them. The category of *mixed realization complexes* $C_{\mathcal{MR}}$ is the full subcategory of complexes with strict differentials and cohomology objects in $\mathcal{MR}$. Let $D_{\mathcal{MR}}$ be the localization of the homotopy category of $C_{\mathcal{MR}}$ (see [Hu1]) with respect to quasi-iosmorphisms (see [Hu1] 4.17).

By construction, there are natural cohomology functors:

$$H^i : C_{\mathcal{MR}} \to \mathcal{MR}$$

factoring over $D_{\mathcal{MR}}$.

Remark 10.2.6. One should think of $D_{\mathcal{MR}}$ as the derived category of $\mathcal{MR}$, even though this is false in a literal sense.

The main construction of [Hu1] is a functor from varieties to mixed realizations.

Theorem 10.2.7 ([Hu1] Section 11.2, [Hu2] Thm 2.3.1)**.** *Let* Sm_k *be the category of smooth varieties over* k*. There is a natural additive functor*

$$\tilde{R}_{\mathcal{MR}} : \mathrm{Sm}_k \to C_{\mathcal{MR}} \ ,$$

such that

$$H^i_{\mathcal{MR}}(X) = H^i(\tilde{R}_{\mathcal{MR}}(X)) \ .$$

This allows to extend $\tilde{R}$ to the additive category $\mathbb{Q}[\mathrm{Sm}_k]$ and even to the category of complexes $C^-(\mathbb{Q}[\mathrm{Sm}_k])$.

Remark 10.2.8. There is a subtle technical point here. The category C^+ is additive. Taking the total complex of a complex in C^+ gives again an object of C^+. That the subcategory $C_{\mathcal{MR}}$ is respected is a non-trivial statement, see [Hu2] Lemma 2.2.5.

Following Deligne and Jannsen, we can now define

Definition 10.2.9. An object $M \in \mathcal{MR}$ is called an *effective absolute Hodge motive* if it is a subquotient of an object in the image of

$$H^* \circ \tilde{R} : C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to \mathcal{MR} \ .$$

Let $\mathcal{MM}_{\mathrm{AH}}^{\mathrm{eff}} = \mathcal{MM}_{\mathrm{AH}}^{\mathrm{eff}}(k) \subset \mathcal{MR}$ be the category of all effective absolute Hodge motives over k. Let $\mathcal{MM}_{\mathrm{AH}} = \mathcal{MM}_{\mathrm{AH}}(k) \subset \mathcal{MR}$ be the full abelian tensor subcategory generated by $\mathcal{MM}^{\mathrm{eff}}$ and the dual of $\mathbb{Q}(-1) = H^2_{\mathcal{MR}}(\mathbb{P}^1)$. Objects in $\mathcal{MM}_{\mathrm{AH}}$ are called *absolute Hodge motives over k*.

Remark 10.2.10. The rationale behind this definition lies in Remark 10.2.4. Every mixed motive is supposed to be an iterated extension of pure motives. The latter are conjecturally fully described by their mixed realization. Hence, it remains to specify which extensions of pure motives are mixed motives.

Jannsen ([Ja1] Definition 4.1) does not use complexes of varieties but only single smooth varietes. It is not clear whether the two definitions agree, see also the discussion in [Hu1] Section 22.3. On the other hand, in [Hu1] Definition 22.13 the varieties were allowed to be singular. This is equivalent to the above by the construction in [Hu3] Lemma B.5.3 where every complex of varieties is replaced by complex of smooth varieties with the same cohomology.

Recall the abelian category $(k,\mathbb{Q})-\mathrm{Vect}$ from Definition 5.1.1.

Fix $\iota : k \to \mathbb{C}$. The projection

$$A \mapsto (A_{\mathrm{dR}}, A_\iota, I_{\iota,\mathbb{C}}^{-1} I_{\mathrm{dR},\iota})$$

defines a faithful functor

$$\mathcal{MR} \to (k,\mathbb{Q})-\mathrm{Vect} \ .$$

Recall the triangulated category $D^+_{(k,\mathbb{Q})}$ from Definition 5.2.1. The projection

$$K \mapsto (K_{\mathrm{dR}}, K_\iota, K_{\iota,\mathbb{C}}, I_{\mathrm{dR},\iota}, I_{\iota,\mathbb{C}})$$

defines a functor

$$C_{\mathcal{MR}} \to C^+_{(k,\mathbb{Q})}$$

which induces also a triangulated functor

$$\mathrm{forget} : D_{\mathcal{MR}} \to D^+_{(k,\mathbb{Q})} \ .$$

Lemma 10.2.11. *There is a natural transformation of functors*

$$K^-(\mathbb{Z}[\mathrm{Sm}_k]) \to D^+_{(k,\mathbb{Q})}$$

between forget $\circ\, R_{\mathcal{MR}}$ *and* $R\Gamma$.

Proof. This is true by construction of the dR- and σ-components of $R_{\mathcal{MR}}$ in [Hu1]. In fact, the definition of $R\Gamma$ is a simplified version of the construction given there. (They are *not* identical though because $\mathcal{MR}$ takes the Hodge and weight filtration into account.) □

10.3 Comparison functors

We now have three candidates for categories of mixed motives: the triangulated categories of geometric motives and the abelian categories of absolute Hodge motives and of Nori motives (see Chapter 8).

Theorem 10.3.1. *The functor $R_{\mathcal{MR}}$ factors via a chain of functors*

$$C^b(\mathbb{Q}[Sm_k]) \to DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}}) \subset D_{\mathcal{MR}}\ .$$

The proof will be given at the end of the section. The argument is a bit involved.

Theorem 10.3.2 ([Hu2], [Hu3])**.** *There is a tensor triangulated functor*

$$R_{\mathcal{MR}} : DM_{\mathrm{gm}} \to D_{\mathcal{MR}}$$

such that for smooth X

$$H^i R_{\mathcal{MR}}(X) = H^*_{\mathcal{MR}}(X)\ .$$

For all $M \in DM_{\mathrm{gm}}$, the objects $H^i R_{\mathcal{MR}}(M)$ are absolute Hodge motives.

Proof. This is the main result of [Hu2]. Note that there is a Corrigendum [Hu3]. The second assertion is [Hu2] Theorem 2.3.6. □

Proposition 10.3.3. *Let $k \subset \mathbb{C}$.*

1. *There is a faithful tensor functor*

   $$f : \mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MM}_{\mathrm{AH}}$$

   such that the functor $R_{\mathcal{MR}} : C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to D_{\mathcal{MR}}$ factors via $D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}})$.

2. *Every object in $\mathcal{MM}_{\mathrm{AH}}$ is a subquotient of an object in the image of $\mathcal{MM}_{\mathrm{Nori}}$.*

Proof. We want to use the universal property of Nori motives. Let $\iota : k \subset \mathbb{C}$ be the fixed embedding. The assignment $A \mapsto A_\iota$ (see Definition 10.2.1) is a fibre functor on the neutral Tannakian category $\mathcal{MM}_{\mathrm{AH}}$. We denote it H^*_{sing} because it agrees with singular cohomology of $X \otimes_k \mathbb{C}$ on $A = H^*_{\mathcal{MR}}(X)$.

We need to verify that the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of effective pairs from Chap. 8 can be represented in $\mathcal{MM}_{\mathrm{AH}}$ in a manner compatible with singular cohomology. More explicitly, let X be a variety and $Y \subset X$ a subvariety. Then $[Y \to X]$ is an object of DM_{gm}. Hence for every $i \geq 0$ there is

$$H^i_{\mathcal{MR}}(X,Y) = H^i R_{\mathcal{MR}}(X,Y) \in \mathcal{MM}_{\mathrm{AH}} \ .$$

By construction, we have

$$H^*_{\mathrm{sing}} H^i_{\mathcal{MR}}(X,Y) = H^i_{\mathrm{sing}}(X(\mathbb{C}), Y(\mathbb{C})) \ .$$

The edges in $\mathrm{Pairs}^{\mathrm{eff}}$ are also induced from morphisms in DM_{gm}. Moreover, the representation is compatible with the multiplicative structure on $\mathrm{Good}^{\mathrm{eff}}$.

By the universal property of Theorem 8.1.9, this yields a functor $\mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MR}$. It is faithful, exact and a tensor functor. We claim that it factors via $\mathcal{MM}_{\mathrm{AH}}$. As $\mathcal{MM}_{\mathrm{AH}}$ is closed under subquotients in $\mathcal{MR}$, it is enough to check this on generators. By Corollary 8.2.21, the category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is generated by objects of the form $H^i_{\mathrm{Nori}}(X,Y)$ for $X = W \setminus W_\infty$ with X smooth and Y a divisor with normal crossings. (In fact, it is generated by very good pairs; blow up the singularities without changing the motive by excision.) Let $Y_\bullet$ be the Čech nerve of the cover of Y by its normalization. This is the simplicial scheme described in detail in Section 3.3.6. Let

$$C_\bullet = \mathrm{Cone}(Y_\bullet \to X)[-1] \in C^-(\mathbb{Q}[\mathrm{Sm}_k]).$$

Then $H^i_{\mathcal{MR}}(X,Y) = H^i R_{\mathcal{MR}}(C_\bullet)$ is an absolute Hodge motive.

Consider $X_* \in C^b(\mathbb{Q}[\mathrm{Sm}_k])$. We apply Proposition 8.2.16 to $\mathcal{A} = \mathcal{MM}_{\mathrm{Nori}}$ and $\mathcal{A} = \mathcal{MM}_{\mathrm{AH}}$. Hence, there is $R_{\mathrm{Nori}}(X_*) \in D^b(\mathcal{MM}_{\mathrm{Nori}})$ such that the underlying vector space of $H^i R_{\mathrm{Nori}}(X_*)$ is singular cohomology. We claim that there is a natural morphism

$$f : R_{\mathrm{Nori}}(X_*) \to R_{\mathcal{MR}}(X_*).$$

It will automatically be a quasi-isomorphism because both compute singular cohomology of X_*.

We continue as in the proof of Proposition 8.2.16. We choose a rigidified affine cover $\tilde{U}_{X_*}$ of X_* and a very good filtration on the cover. This induces a very good filtration on $\mathrm{Tot}C_*(\tilde{U}_{X_*})$. This induces a double complex of very good pairs. Each very good pair may in turn be seen as complex with two entries. We apply $\tilde{R}_{\mathcal{MR}}$ to this triple complex and take the associated simple complex. On the one hand, the result is quasi-isomorphic to $R_{\mathcal{MR}}(X_*)$ because this is true in singular cohomology. On the other hand, it agrees with $fR_{\mathrm{Nori}}(X_*)$, also by construction.

Finally, we claim that every $M \in \mathcal{MM}_{\mathrm{AH}}$ it is subquotient of the image of a Nori motive. By definition of absolute Hodge motives it suffices to consider M of the form $H^iR_{\mathcal{MR}}(X_*)$ for $X_* \in C^b(\mathbb{Q}[\mathrm{Sm}_k])$. We have seen that $H^iR_{\mathcal{MR}}(X_*) = H^if(R_{\mathrm{Nori}}(X_*))$, hence M is in the image of f. $\square$

Remark 10.3.4. It is very far from clear whether the functor is also full or essentially surjective. The two properties are related because every object in $\mathcal{MM}_{\mathrm{AH}}$ is a subquotient of an object in the image of $\mathcal{MM}_{\mathrm{Nori}}$.

Theorem 10.3.5. *There is a functor*

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}})$$

such the composition

$$C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}})$$

agrees with the functor R_{Nori} of Proposition 8.2.16.

Proof. This is a result of Harrer, see [Ha]. $\square$

Proof of Theorem 10.3.1. We put together Theorem 10.3.5 and Theorem 10.3.3. $\square$

10.4 Weights and Nori motives

Let $k \subset \mathbb{C}$ be a subfield. We are now going to explore the connection between Grothendieck motives and pure Nori motives and weights.

Definition 10.4.1. Let $n \in \mathbb{N}_0$. An object $M \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is called *pure of weight n* if it is a subquotient of a motive of the form $H^n_{\mathrm{Nori}}(Y)$ with Y smooth and projective.

A motive is called *pure* if it is a direct sum of pure motives of some weights.

In particular, $H^*_{\mathrm{Nori}}(Y)$ is pure if Y is smooth and projective.

Definition 10.4.2.

1. The category of *effective Chow motives* $\mathrm{CHM}^{\mathrm{eff}}$ is given by the pseudo-abelian hull of the category with objects given by smooth, projective varieties and morphism form $[X]$ to $[Y]$ given by the Chow group $\mathrm{Ch}^{\dim X}(Y \times X)$ of algebraic cycles of codimension $\dim Y$ up to rational equivalence. The category of *Chow motives* CHM is given by the localization of the category of effective Chow motives with respect to the Lefschetz motive L which is the direct complement of $[\mathrm{Spec} k]$ in $\mathbb{P}^1$.

2. The category of *effective Grothendieck motives* $\mathrm{GRM}^{\mathrm{eff}}$ is given by the pseudo-abelian hull of the category with objects given by smooth, projective varieties and morphism form $[X]$ to $[Y]$ given by the group $A^{\dim X}(Y \times$

X) of algebraic cycles of codimension $\dim Y$ up to homological equivalence with respect to singular cohomology. The category of *Grothendieck motives* GRM is given by the localization of the category of effective Grothendieck motives with respect to the Lefschetz motive L.

In both cases, the composition is given by composition of correspondences.

Remark 10.4.3. There is a *contravariant* functor $X \mapsto [X]$ from the category of smooth, projective varieties over k to Chow or Grothendieck motives. It maps a morphism $f : Y \to X$ to the transpose of its graph Γ_f. The dimension of Γ_f is the same as the dimension of Y, hence it has codimension $\dim X$ in $X \times Y$. On the other hand, singular cohomology defines a well-defined *covariant* functor on Chow and Grothendieck motives. Note that it is not a tensor functor due to the signs in the Künneth formula.

This normalization is the original one, see e.g., [Man]. In recent years, it has also become common to use the covariant normalization instead, in particular in the case of Chow motives.

The category of Grothendieck motives is conjectured to be abelian and semi-simple. Jannsen has shown in [Ja2] that this is the case if and only if homological equivalence agrees with numerical equivalence.

Proposition 10.4.4. *Singular cohomology on* GRM *factors naturally via a faithful functor*

$$\mathrm{GRM} \to \mathcal{MM}_{\mathrm{Nori}}$$

whose image is contained in the category of pure Nori motives.

If the Hodge conjecture holds, then the inclusion is an equivalence of semi-simple abelian categories.

Proof. The opposite category of CHM is a full subcategory of the category of geometric motives DM_{gm} by [VSF, Chapter 5, Proposition 2.1.4]. Restricting the contravariant functor

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \xrightarrow{\bigoplus H^i} \mathcal{MM}_{\mathrm{Nori}}$$

to the subcategory yields a covariant functor

$$\mathrm{CHM} \to \mathcal{MM}_{\mathrm{Nori}} .$$

By definition, its image is contained in the category of pure Nori motives. Also by definition, a morphism in CHM is zero in GRM if it is zero in singular cohomology, and hence in $\mathcal{MM}_{\mathrm{Nori}}$. Therefore, the functor automatically factors via GRM. The induced functor then is faithful.

We now assume the Hodge conjecture. By [Ja1, Lemma 5.5], this implies that absolute Hodge cycles agree with cycles up to homological equivalence. Equivalently, the functor $\mathrm{GRM} \to \mathcal{MR}$ to mixed realizations is fully faithful. As it factors via $\mathcal{MM}_{\mathrm{Nori}}$, the inclusion $\mathrm{GRM} \to \mathcal{MM}_{\mathrm{Nori}}$ has to be full as well.

The endomorphisms of $[Y]$ for Y smooth and projective can be computed in $\mathcal{MR}$. Hence it is semi-simple because $H^*_{\mathcal{MR}}(Y)$ is polarizable, see [Hu1, Proposition 21.1.2 and 21.2.3]. This implies that its subquotients are the same as its direct summands. Hence, the functor from GRM to pure Nori motives is essentially surjective. $\square$

Proposition 10.4.5. *Every Nori motive $M \in \mathcal{MM}_{\mathrm{Nori}}$ carries a unique bounded increasing filtration $(W_nM)_{n\in\mathbb{Z}}$ inducing the weight filtration in $\mathcal{MR}$. Every morphism of Nori motives is strictly compatible with the filtration.*

Proof. As the functor $\mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MR}$ is faithful and exact, the filtration on $M \in \mathcal{MM}_{\mathrm{Nori}}$ is indeed uniquely determined by its image in M. Strictness of morphisms follows from the same property in $\mathcal{MR}$.

We turn to existence. Bondarko [Bo] constructed what he calls a weight structure on DM_{gm}. It induces a *weight filtration* on the values of any cohomological functor. We apply this to the functor to $\mathcal{MM}_{\mathrm{Nori}}$. In particular, the weight filtration on $H^n_{\mathrm{Nori}}(X,Y)$ is motivic for every vertex of $\mathrm{Pairs}^{\mathrm{eff}}$. The weight filtration on subquotients is the induced filtration, hence also motivic. As any object in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of some $H^n_{\mathrm{Nori}}(X,Y)$, this finishes the proof in the effective case. The non-effective case follows immediately by localization. $\square$

10.5 Periods of motives

Recall the chain of functors

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}}) \to D^b((k,\mathbb{Q})-\mathrm{Vect})$$

constructed in the last section.

Definition 10.5.1. 1. Let $\mathcal{C}(\mathrm{gm})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in DM_{\mathrm{gm}}$. Let $\mathbb{P}_{\mathrm{gm}} = \mathbb{P}(\mathcal{C}(\mathrm{gm})))$ be the *period algebra of geometric motives.*

2. Let $\mathcal{C}(\mathrm{Nori})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in \mathcal{MM}_{\mathrm{Nori}}$. Let $\mathbb{P}_{\mathrm{Nori}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{Nori}))$ be the *period algebra of Nori motives.*

3. Let $\mathcal{C}(\mathrm{AH})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in \mathcal{MM}_{\mathrm{AH}}$. Let $\mathbb{P}_{\mathrm{AH}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{AH}))$ be the *period algebra of absolute Hodge motives.*

Proposition 10.5.2. *We have*

$$\mathbb{P}(k) = \mathbb{P}_{\mathrm{gm}}(k) = \mathbb{P}_{\mathrm{Nori}}(k) = \mathbb{P}_{\mathrm{AH}}(k)\ .$$

Proof. From the functors between categories of motives, we have inclusions of subcategories of $(k, \mathbb{Q})-\text{Vect}$:

$$\mathcal{C}(\text{gm}) \subset \mathcal{C}(\text{Nori}) \subset \mathcal{C}(\text{AH}) .$$

Moreover, the category $\mathcal{C}(\text{Sm}_k)$ of Definition 9.4.1 is contained in $\mathcal{C}(\text{gm})$. By definition, we also have $\mathcal{C}(\text{AH}) = \mathcal{C}(\text{Sm}_k)$. Hence, all categories are equal. Finally recall, that $\mathbb{P}(k) = \mathbb{P}(\text{Sm}_k)$ by Theorem 9.4.2. □

This allows easily to translate information on motives into information on periods. Here is an example:

Corollary 10.5.3. *Let $\mathcal{X}$ be an algebraic space, or, more generally, a Deligne-Mumford stack over k. Then the periods of $\mathcal{X}$ are contained in $\mathbb{P}(k)$.*

Proof. Every Deligne-Mumford stack defines a geometric motive by work of Choudhury [Ch]. Their periods are therefore contained in the periods of geometric motives. □

Chapter 11

Kontsevich-Zagier Periods

This chapter follows closely the Diploma thesis of Benjamin Friedrich, see [Fr]. The results are due to him.

We work over $k = \mathbb{Q}$ or equivalently $\overline{\mathbb{Q}}$ throughout. Denote the integral closure of $\mathbb{Q}$ in $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$. Note that $\widetilde{\mathbb{Q}}$ is a field.

In this section, we sometimes use X_0, ω_0 etc. to denote objects over $\widetilde{\mathbb{Q}}$ and X, ω etc. for objects over $\mathbb{C}$.

11.1 Definition

Recall the notion of a $\widetilde{\mathbb{Q}}$-semialgebraic set from Definition 2.6.1.

Definition 11.1.1. Let

- $G \subseteq \mathbb{R}^n$ be an oriented compact $\widetilde{\mathbb{Q}}$-semi-algebraic set which is equidimensional of dimension d, and
- ω a rational differential d-form on $\mathbb{R}^n$ having coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.

Then we call the complex number $\int_G \omega$ a *naive period* and denote the set of all naive periods for all G and ω by $\mathbb{P}_{\mathrm{nv}}$.

This set $\mathbb{P}_{\mathrm{nv}}$ enjoys additional structure.

Proposition 11.1.2. *The set $\mathbb{P}_{nv}$ is a unital $\overline{\mathbb{Q}}$-algebra.*

Proof. Multiplicative structure: In order to show that $\mathbb{P}_{\mathrm{nv}}$ is closed under multiplication, we write

$$p_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{n_i}, \quad i = 1, 2$$

for the natural projections and obtain

$$\left(\int_{G_1} \omega_1\right) \cdot \left(\int_{G_2} \omega_2\right) = \int_{G_1 \times G_2} p_1^*\omega_1 \wedge p_2^*\omega_2 \in \mathbb{P}_{\mathrm{nv}}$$

by the Fubini formula.

Multiplication by $\overline{\mathbb{Q}}$: We find every $a \in \overline{\mathbb{Q}}$ as naive period with $G = [0,1] \subset \mathbb{R}$ with respect to the differential form adt. In particular, $1 \in \mathbb{P}_{\mathrm{nv}}$.

Combining the last two steps, we can shift the dimension of the set G in the definition of a period number. Let $\alpha = \int_G \omega$. Represent $1 = \int_{[0,1]} dt$ and $1\alpha = \int_{G\times[0,1]} \omega \wedge dt$.

Additive structure: Let $\int_{G_1} \omega_1$ and $\int_{G_2} \omega_2 \in \mathbb{P}_{\mathrm{nv}}$ be periods with domains of integration $G_1 \subseteq \mathbb{R}^{n_1}$ and $G_2 \subseteq \mathbb{R}^{n_2}$. Using the dimension shift described above, we may assume without loss of generality that $\dim G_1 = \dim G_2$. Using the inclusions

$$\begin{aligned} i_1 &: \mathbb{R}^{n_1} \cong \mathbb{R}^{n_1} \times \{1/2\} \times \{\underline{0}\} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \quad \text{and} \\ i_2 &: \mathbb{R}^{n_2} \cong \{\underline{0}\} \times \{-1/2\} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}, \end{aligned}$$

we can write $i_1(G_1) \cup i_2(G_2)$ for the disjoint union of G_1 and G_2. With the projections $p_j : \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_j}$ for $j = 1, 2$, we can lift ω_j on $\mathbb{R}^{n_j}$ to $p_j^*\omega_j$ on $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. For $q_1, q_2 \in \overline{\mathbb{Q}}$ we get

$$q_1\int_{G_1} \omega_1 + q_2\int_{G_2} \omega_2 = \int_{i_1(G_1)\cup i_2(G_2)} q_1 \cdot (1/2+t) \cdot p_1^*\omega_1 + q_2 \cdot (1/2-t) \cdot p_2^*\omega_2 \in \mathbb{P}_{\mathrm{nv}},$$

where t is the coordinate of the "middle" factor $\mathbb{R}$ of $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. This shows that $\mathbb{P}_{\mathrm{nv}}$ is a $\overline{\mathbb{Q}}$-vector space.

□

The Definition 11.1.1 was inspired by the one given in [KZ, p. 772]:

Definition 11.1.3 (Kontsevich-Zagier)**.** A *Kontsevich-Zagier period* is a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

We will show at the end of this section, that Kontsevich-Zagier periods agree with naive periods in definition 11.1.1, see Theorem 11.2.4.

Examples of naive periods are

- $\displaystyle\int_1^2 \frac{dt}{t} = \log(2)$,
- $\displaystyle\int_{x^2+y^2 \leq 1} dx\, dy = \pi$ and

- $\displaystyle\int_G \frac{dt}{s} = \int_1^2 \frac{dt}{\sqrt{t^3+1}}$ = elliptic integrals,
for $G := \{(t,s) \in \mathbb{R}^2 \,|\, 1 \le t \le 2,\, 0 \le s,\, s^2 = t^3+1\}$.

As a problematic example, we consider the following identity.

Proposition 11.1.4 (cf. [K1, p. 62]). *We have*

$$\int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1 \wedge dt_2}{(1-t_1)\,t_2} = \zeta(2). \tag{11.1}$$

Proof. This equality follows by a simple power series manipulation: For $0 \le t_2 < 1$, we have

$$\int_0^{t_2} \frac{dt_1}{1-t_1} = -\log(1-t_2) = \sum_{n=1}^{\infty} \frac{t_2^n}{n}.$$

Let $\epsilon > 0$. The power series $\sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}$ converges uniformly for $0 \le t_2 \le 1-\epsilon$ and we get

$$\int_{0 \le t_1 \le t_2 \le 1-\epsilon} \frac{dt_1\, dt_2}{(1-t_1)\,t_2} = \int_0^{1-\epsilon} \sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}\, dt_2 = \sum_{n=1}^{\infty} \frac{(1-\epsilon)^n}{n^2}.$$

Applying Abel's Theorem [Fi, XII, 438, 6°, p. 411] at $(*)$, using $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$ gives us

$$\int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1\, dt_2}{(1-t_1)\,t_2} = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} \frac{(1-\epsilon)^n}{n^2} \overset{(*)}{=} \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

□

Equation (11.1) is not a valid representation of $\zeta(2)$ as an integral for a naive period in our sense, because the pole locus $\{t_1 = 1\} \cup \{t_2 = 0\}$ of $\frac{dt_1 \wedge dt_2}{(1-t_1)\,t_2}$ is not disjoint with the domain of integration $\{0 \le t_1 \le t_2 \le 1\}$. But (11.1) gives a valid period integral according to the original definition Kontsevich-Zagier — see Definition 11.1.3. We will show in Example 14.1 how to circumvent directly this difficulty by a blow-up. The general blow-up procedure which makes this possible is used in the proof of Theorem 11.2.4. This argument shows that Kontsevich-Zagier periods and naive periods are the same.

11.2 Comparison of Definitions of Periods

Theorem 11.2.1 (Friedrich [Fr]).

$$\mathbb{P}^{\text{eff}}(\mathbb{Q}) = \mathbb{P}^{\text{eff}}_{\text{nc}}(\mathbb{Q}) = \mathbb{P}^{\text{eff}}_{nv} \quad \textit{and} \quad \mathbb{P}(\mathbb{Q}) = \mathbb{P}_{\text{nc}}(\mathbb{Q}) = \mathbb{P}_{nv},$$

The proof will take the rest of this section.

Lemma 11.2.2.

$$\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\mathbb{Q}) \subseteq \mathbb{P}^{\mathrm{eff}}_{nv}\ .$$

Proof. By definition its elements of $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\mathbb{Q})$ are of the form $\int_\gamma \omega$ where $\gamma \in H_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}, \mathbb{Q})$ with X_0 a smooth variety of dimension d and D_0 a divisor with normal crossings and $\omega_0 \in \Gamma(X_0, \Omega^d_{X_0})$.

We choose an embedding

$$X_0 \subseteq \underset{(x_0:\ldots:x_n)}{\mathbb{P}^n_{\mathbb{Q}}}$$

and equip $\mathbb{P}^n_{\mathbb{Q}}$ with coordinates as indicated. Lemma 2.6.5 provides us with a map

$$\psi : \mathbb{C}P^n \hookrightarrow \mathbb{R}^N$$

such that D^{an} and $\mathbb{C}P^n$ become $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$. Then, by Proposition 2.6.8, the cohomology class $\psi_*\gamma$ has a representative which is a rational linear combination of singular simplices Γ_i, each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.

As $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nv}}$ is a $\mathbb{Q}$-algebra by Proposition 11.1.2, it suffices and to prove

$$\int_{\psi^{-1}(\mathrm{Im}\Gamma_i)} \omega \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nv}}.$$

We drop the index i from now. Set $G = \mathrm{Im}\Gamma$. The claim will be clear as soon as we find a rational differential form ω' on $\mathbb{R}^N$ such that $\psi^*\omega' = \omega$, since then

$$\int_{\psi^{-1}(G)} \omega = \int_{\psi^{-1}(G)} \psi^*\omega' = \int_{G_i} \omega' \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nv}}.$$

After eventually applying a barycentric subdivision to Γ, we may assume w.l.o.g. that there exists a hyperplane in $\mathbb{C}P^n$, say $\{x_0 = 0\}$, which does not meet $\psi^{-1}(G)$. Furthermore, we may assume that $\psi^{-1}(G)$ lies entirely in U^{an} for U_0 an open affine subset of $D_0 \cap \{x_0 \neq 0\}$. (As usual, U^{an} denotes the complex analytic space associated to the base change to $\mathbb{C}$ of U.) The restriction of ω_0 to the open affine subset can be represented in the form (cf. [Ha2, II.8.4A, II.8.2.1, II.8.2A])

$$\sum_{|J|=d} f_J(x_0, \ldots, x_n)\, d\left(\frac{x_{j_1}}{x_0}\right) \wedge \ldots \wedge d\left(\frac{x_{j_d}}{x_0}\right)$$

with $f_J(x_1, \cdots, x_n) \in \mathbb{Q}(x_0, \cdots, x_n)$ being homogenous of degree zero. This expression defines a rational differential form on all of $\mathbb{P}^n_{\mathbb{Q}}$ with coefficients in $\mathbb{Q}$ and it does not have poles on $\psi^{-1}(G_i)$.

We construct the rational differential form ω' on $\mathbb{R}^N$ with coefficients in $\mathbb{Q}(i)$ as follows

$$\omega'_I := \sum_{|J|=d} f_J\left(1, \frac{y_{10} + iz_{10}}{y_{00} + iz_{00}}, \cdots, \frac{y_{n0} + iz_{n0}}{y_{00} + iz_{00}}\right) d\left(\frac{y_{j_10} + iz_{j_10}}{y_{00} + iz_{00}}\right) \wedge \ldots \wedge d\left(\frac{y_{j_d0} + iz_{j_d0}}{y_{00} + iz_{00}}\right),$$

where we have used the notation from the proof of Lemma 2.6.5. Using the explicit form of ψ given in this proof, we obtain

$$\begin{aligned}\psi^* f_J\left(1, \frac{y_{10}+iz_{10}}{y_{00}+iz_{00}}, \cdots, \frac{y_{n0}+iz_{n0}}{y_{00}+iz_{00}}\right) &= f_J\left(\frac{x_0\overline{x}_0}{|x_0|^2}, \frac{x_1\overline{x}_0}{|x_0|^2}, \ldots, \frac{x_n\overline{x}_0}{|x_0|^2}\right)\\ &= f_J(x_0, x_1, \ldots, x_n)\end{aligned}$$

and

$$\psi^* d\left(\frac{y_{j0}+iz_{j0}}{y_{00}+iz_{00}}\right) = d\left(\frac{x_j\overline{x}_0}{|x_0|^2}\right) = d\left(\frac{x_j}{x_0}\right).$$

This shows that $\psi^*\omega' = \omega$ and we are done. □

Lemma 11.2.3.

$$\mathbb{P}^{\mathrm{eff}}_{nv} \subseteq \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\overline{\mathbb{Q}})\ .$$

Proof. We will use objects over various base fields. We will use subscripts to indicate which base field is used: A 0 for $\widetilde{\mathbb{Q}}$, a 1 for $\overline{\mathbb{Q}}$, a subscript $\mathbb{R}$ for $\mathbb{R}$ and none for $\mathbb{C}$. Furthermore, we fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Let $\int_G \omega_{\mathbb{R}} \in \mathbb{P}_{\mathrm{nv}}$ be a naïve period with

- $G \subset \mathbb{R}^n$ an oriented $\widetilde{\mathbb{Q}}$-semi-algebraic set, equidimensional of dimension d, and
- $\omega_{\mathbb{R}}$ a rational differential d-form on $\mathbb{R}^n$ with coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.

The $\widetilde{\mathbb{Q}}$-semi-algebraic set $G \subset \mathbb{R}^n$ is given by polynomial inequalities and equalities. By omitting the inequalities but keeping the equalities in the definition of G, we see that G is supported on (the set of $\mathbb{R}$-valued points of) a variety $Y_{\mathbb{R}} \subseteq \mathbb{A}^n_{\mathbb{R}}$ of same dimension d. This variety $Y_{\mathbb{R}}$ is already defined over $\widetilde{\mathbb{Q}}$

$$Y_{\mathbb{R}} = Y_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}$$

for a variety $Y_0 \subseteq \mathbb{A}^n_{\widetilde{\mathbb{Q}}}$ over $\widetilde{\mathbb{Q}}$. Similarly, the boundary ∂G of G is supported on a variety $E_{\mathbb{R}}$, likewise defined over $\widetilde{\mathbb{Q}}$

$$E_{\mathbb{R}} = E_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}.$$

Note that E_0 is a divisor on Y_0. By eventually enlarging E_0, we may assume w.l.o.g. that E_0 contains the singular locus of Y_0. In order to obtain an abstract period, we need smooth varieties. The resolution of singularities according to Hironaka [Hi1] provides us with a Cartesian square

$$\begin{array}{ccc}\widetilde{E}_0 & \subseteq & \widetilde{Y}_0\\ \downarrow & & \downarrow \pi_0\\ E_0 & \subseteq & Y_0\end{array} \tag{11.2}$$

where

- $\widetilde{Y}_0$ is smooth and quasi-projective,
- π_0 is proper, surjective and birational, and
- $\widetilde{E}_0$ is a divisor with normal crossings.

In fact, π_0 is an isomorphism away from $\widetilde{E}_0$ since the singular locus of Y_0 is contained in E_0

$$\pi_{0|\widetilde{U}_0} : \widetilde{U}_0 \xrightarrow{\sim} U_0 \tag{11.3}$$

with $\widetilde{U}_0 := \widetilde{Y}_0 \setminus \widetilde{E}_0$ and $U_0 := Y_0 \setminus E_0$.

We apply the analytification functor to the base change to $\mathbb{C}$ of the map $\pi_0 : \widetilde{Y}_0 \to Y_0$ and obtain a projection

$$\pi_{\mathrm{an}} : \widetilde{Y}^{\mathrm{an}} \to Y^{\mathrm{an}}.$$

We want to show that the "strict transform" of G

$$\widetilde{G} := \overline{\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})} \subseteq \widetilde{Y}^{\mathrm{an}}$$

can be triangulated. Since $\mathbb{C}P^n$ is the projective closure of $\mathbb{C}^n$, we have $\mathbb{C}^n \subset \mathbb{C}P^n$ and thus get an embedding

$$Y^{\mathrm{an}} \subseteq \mathbb{C}^n \subset \mathbb{C}P^n.$$

We also choose an embedding

$$\widetilde{Y}^{\mathrm{an}} \subseteq \mathbb{C}P^m$$

for some $m \in \mathbb{N}$. Using Lemma 2.6.5, we may consider both Y^{an} and $\widetilde{Y}^{\mathrm{an}}$ as $\widetilde{\mathbb{Q}}$-semi-algebraic sets via some maps

$$\begin{aligned} \psi &: Y^{\mathrm{an}} \subset \mathbb{C}P^n \hookrightarrow \mathbb{R}^N, \quad \text{and} \\ \widetilde{\psi} &: \widetilde{Y}^{\mathrm{an}} \subseteq \mathbb{C}P^m \hookrightarrow \mathbb{R}^M. \end{aligned}$$

In this setting, the induced projection

$$\pi_{\mathrm{an}} : \widetilde{Y}^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

becomes a $\widetilde{\mathbb{Q}}$-semi-algebraic map. The composition of ψ with the inclusion $G \subseteq Y^{\mathrm{an}}$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic map; hence $G \subset \mathbb{R}^N$ is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4. Since E^{an} is also $\widetilde{\mathbb{Q}}$-semi-algebraic via ψ, we find that $G \setminus E^{\mathrm{an}}$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Again by Fact 2.6.4, $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}}) \subset \mathbb{R}^M$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Thus $\widetilde{G} \subset \mathbb{R}^M$, being the closure of a $\widetilde{\mathbb{Q}}$-semi-algebraic set, is $\widetilde{\mathbb{Q}}$-semi-algebraic. From Proposition 2.6.8, we see that $\widetilde{G}$ can be triangulated

$$\widetilde{G} = \cup_j \triangle_j, \tag{11.4}$$

where the $\triangle_j$ are (homeomorphic images of) d-dimensional simplices.

Our next aim is to define an algebraic differential form $\widetilde{\omega}_1$ replacing $\omega_{\mathbb{R}}$. We first make a base change in (11.2) from $\widetilde{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$ and obtain

$$\begin{array}{ccc} \widetilde{E}_1 & \subseteq & \widetilde{Y}_1 \\ \downarrow & & \downarrow \pi_1 \\ E_1 & \subseteq & Y_1\,. \end{array}$$

The differential d-form $\omega_{\mathbb{R}}$ can be written as

$$\omega_{\mathbb{R}} = \sum_{|J|=d} f_J(x_1,\ldots,x_n)\,dx_{j_1}\wedge\ldots\wedge dx_{j_d}, \tag{11.5}$$

where $x_1,\ldots,x_n$ are coordinates of $\mathbb{R}^n$ and $f_J \in \overline{\mathbb{Q}}(x_1,\ldots,x_n)$. We can use equation (11.5) to define a differential form ω_1 on $\mathbb{A}^n_{\overline{\mathbb{Q}}}$

$$\omega_{\mathbb{R}} = \sum_{|J|=d} f_J(x_1,\ldots,x_n)\,dx_{j_1}\wedge\ldots\wedge dx_{j_d},$$

where now $x_1,\ldots,x_n$ denote coordinates of $\mathbb{A}^n_{\overline{\mathbb{Q}}}$. The pole locus of ω_1 gives us a variety $Z_1 \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$. We set

$$\begin{aligned} X_1 &:= Y_1 \setminus Z_1, \quad D_1 := E_1 \setminus Z_1, \quad \text{and} \\ \widetilde{X}_1 &:= \pi_1^{-1}(X_1), \quad \widetilde{D}_1 := \pi_1^{-1}(D_1). \end{aligned}$$

The restriction $\omega_{1|X_1}$ of ω_1 to X_1 is a (regular) algebraic differential form on X_1; the pullback

$$\widetilde{\omega}_1 := \pi_1^*(\omega_{1|X_1})$$

is an algebraic differential form on $\widetilde{X}_1$.

We consider the complex analytic spaces $\widetilde{X}^{\mathrm{an}}$, $\widetilde{D}^{\mathrm{an}}$, Z^{an} associated to the base change to $\mathbb{C}$ of $\widetilde{X}_1$, $\widetilde{D}_1$, Z_1. Since $\omega_{\mathbb{R}}$ has no poles on G, we have $G \cap Z^{\mathrm{an}} = \emptyset$; hence $\widetilde{G} \cap \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}}) = \emptyset$. This shows $\widetilde{G} \subseteq \widetilde{X} = \widetilde{Y} \setminus \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}})$.

Since G is oriented, so is $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$, because π_{an} is an isomorphism away from E^{an}. Every d-simplex $\triangle_j$ in (11.4) intersects $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ in a dense open subset, hence inherits an orientation. As in the proof of Proposition 2.6.8, we choose orientation-preserving homeomorphisms from the standard d-simplex $\triangle_d^{\mathrm{std}}$ to $\triangle_j$

$$\sigma_j : \triangle_d^{\mathrm{std}} \longrightarrow \triangle_j.$$

These maps sum up to a singular chain

$$\widetilde{\Gamma} = \oplus_j\, \sigma_j \in \mathrm{C}_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}};\mathbb{Q}).$$

It might happen that the boundary of the singular chain $\widetilde{\Gamma}$ is not supported on $\partial\widetilde{G}$. Nevertheless, it will always be supported on $\widetilde{D}^{\mathrm{an}}$: The set $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ is oriented and therefore the boundary components of $\partial\triangle_j$ that do not belong to

$\partial\widetilde{G}$ cancel if they have non-zero intersection with $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$. Thus $\widetilde{\Gamma}$ gives rise to a singular homology class

$$\widetilde{\gamma} \in H_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}, \widetilde{D}^{\mathrm{an}}; \mathbb{Q}).$$

We denote the base change to $\mathbb{C}$ of ω_1 and $\widetilde{\omega}_1$ by ω and $\widetilde{\omega}$, respectively. Now

$$\begin{aligned}\int_G \omega_{\mathbb{R}} = \int_G \omega = \int_{G\cap U^{\mathrm{an}}} \omega \\ \overset{(11.3)}{=} \int_{\pi^{-1}(G\cap U^{\mathrm{an}})} \pi^*\omega = \int_{\widetilde{G}\cap\widetilde{U}^{\mathrm{an}}} \widetilde{\omega} \\ = \int_{\widetilde{G}} \widetilde{\omega} = \int_{\widetilde{\Gamma}} \widetilde{\omega} = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\overline{\mathbb{Q}})\end{aligned}$$

is a period for the quadruple $(\widetilde{X}_1, \widetilde{D}_1, \widetilde{\omega}_1, \widetilde{\gamma})$. □

Proof of Theorem 11.2.1. It suffices to consider the effective case. By Theorem 9.4.2, we have $\mathbb{P}^{\mathrm{eff}}(\mathbb{Q}) = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\mathbb{Q})$. By Corollary 9.3.5, this is also the same as $\mathbb{P}^{\mathrm{eff}}(\overline{\mathbb{Q}})$. The result now follows by combining Lemma 11.2.2 and Lemma 11.2.3. □

Now, we show that naive periods and Kontsevich-Zagier periods coincide:

Theorem 11.2.4.

$$\mathbb{P}^{\mathrm{eff}}_{KZ} = \mathbb{P}^{\mathrm{eff}}_{nv} = \mathbb{P}^{\mathrm{eff}},\ \mathbb{P}_{KZ} = \mathbb{P}_{nv} = \mathbb{P}.$$

Proof. We will use that $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nv}} = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}} = \mathbb{P}^{\mathrm{eff}}$ (see Theorem 11.2.1) and work with effective periods only. We partially follow ideas of Belkale and Brosnan [BB]. First we show that $\mathbb{P}^{\mathrm{eff}}_{\mathrm{KZ}} \subseteq \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}$: Assume we have given a period through an n-dimensional absolutely convergent integral $\int_\Delta \omega$, where $\omega = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}$ is a rational function defined over $\mathbb{Q}$ and Δ a $\mathbb{Q}$-semialgebraic region defined by inequalities $h_i \geq 0$. This defines a rational differential form ω on $\mathbb{A}^n$. We can extend ω to a rational differential form on $\mathbb{P}^n$ (also denoted by ω) by adding a homogenous variable x_0. The closure $\bar{\Delta}$ of Δ in $\mathbb{P}^n(\mathbb{R})$ is a compact semialgebraic region, defined by $H_i \geq 0$ for some homogenous polynomials H_i. Let $H = \prod_i H_i$. Now we use resolution of singularities and obtain a blow-up

$$\sigma : X \to \mathbb{P}^n,$$

such that we have the following properties:
1. σ is an isomorphism outside the union of the pole locus of ω and the zero sets of all polynomials H_i.
2. The strict transform of the zero locus of H is a normal crossing divisor in X.

3. Near each point $P \in X$, there are local algebraic coordinates $x_1, ..., x_n$ and integers e_j, f_j for each $j = 1, ..., n$, such that

$$H \circ \sigma = \mathrm{unit}_1 \times \prod_{j=1}^{n} x_j^{e_j},\ \sigma^*\omega = \mathrm{unit}_2 \times \prod_{j=1}^{n} x_j^{f_j}\, dx_1 \wedge \cdots \wedge dx_n.$$

Let $\tilde{\Delta}$ be the analytic closure of $\Delta \cap U$, where U is the set where σ is an isomorphism. Then $\tilde{\Delta}$ is compact, since it is a closed subset of the compact set $\sigma^{-1}(\bar{\Delta})$. The absolute convergence of $\int_\Delta \omega$ implies the local convergence of $\sigma^*\omega$ over regions $\{0 < x_i < \epsilon\}$ at point $P \in \tilde{\Delta}$. This is only possible, if all $f_j \geq 0$. Therefore, $\sigma^*\omega$ is regular (holomorphic) at the point P, and hence on the whole of $\tilde{\Delta}$.

Now we show that $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}} \subseteq \mathbb{P}^{\mathrm{eff}}_{\mathrm{KZ}}$: This argument is indicated in Kontsevich-Zagier [KZ, pg. 773]. First, note that naive periods in $\mathbb{P}^{\mathrm{eff}}_{\mathrm{KZ}}$ can also be defined with $\bar{\mathbb{Q}}$-coefficients and the polynomials involved can be replaced by algebraic functions without changing the set $\mathbb{P}^{\mathrm{eff}}_{\mathrm{KZ}}$. A proof is not given in loc. cit., but this can be achieved by using auxiliary variables and minimal polynomials as in the proof that $\sqrt{2} \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{KZ}}$. Assuming this, we now assume that we have given a smooth algebraic variety X of dimension n, a regular differential from ω of top degree (hence closed), a normal crossing divisor $D \subset X$, all this data defined over $\mathbb{Q}$, and a singular chain γ with boundary $\partial\gamma \subset D$. Now we can use the method of Lemma 11.2.2 and we can write

$$\int_\gamma \omega = \int_G \tilde{\omega},$$

where G is a $\tilde{\mathbb{Q}}$-semialgebraic subset of the required form, i.e., given by inequalities, and $\tilde{\omega}$ is a differential form with algebraic coefficients. $\square$

Chapter 12

Formal periods and the period conjecture

Following Kontsevich (see [K1]), we now introduce another algebra $\tilde{\mathbb{P}}(k)$ of *formal periods* from the same data we have used in order to define the actual period algebra of a field in Chapter 9. It comes with an obvious surjective map to $\mathbb{P}(k)$.

The first aim of the chapter is to give a conceptual interpretation of $\tilde{\mathbb{P}}(k)$ as the ring of algebraic functions on the torsor between two fibre functors on Nori motives: singular cohomomology and algebraic de Rham cohomology.

We then discuss the period conjecture from this point of view.

12.1 Formal periods and Nori motives

Definition 12.1.1. Let $k \subset \mathbb{C}$ be a subfield. The space of *effective formal periods* $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is defined as the $\mathbb{Q}$-vector space generated by symbols (X, D, ω, γ), where X is an algebraic variety over k, $D \subset X$ a subvariety, $\omega \in H^d_{\mathrm{dR}}(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ with relations

1. linearity in ω and γ;
2. for every $f : X \to X'$ with $f(D) \subset D'$
$$(X, D, f^*\omega', \gamma) = (X', D', \omega', f_*\gamma)$$
3. for every triple $Z \subset Y \subset X$
$$(Y, Z, \omega, \partial\gamma) = (X, Y, \delta\omega, \gamma)$$
with ∂ the connecting morphism for relative singular homology and δ the connecting morphism for relative de Rham cohomology.

We write $[X, D, \omega, \gamma]$ for the image of the generator. The vector space $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is turned into an algebra via

$$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup D' \times X, \omega \wedge \omega', \gamma \times \gamma')\ .$$

The space of *formal periods* is the localization $\tilde{\mathbb{P}}(k)$ of $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ with respect to $[\mathbb{G}_m, \{1\}, \frac{dX}{X}, S^1]$, where S^1 is the unit circle in $\mathbb{C}^*$.

Remark 12.1.2. This is modeled after Kontsevich [K1] Definition 20, but does not agree with it. We will discuss this point in more detail in Remark 12.1.7.

Theorem 12.1.3. *(Nori) Let $k \subset \mathbb{C}$ be subfield. Let $G_{\mathrm{mot}}(k)$ be the Tannakian dual of the category of Nori motives with $\mathbb{Q}$-coefficents (sic!), see Definition 8.1.6. Let $X = \mathrm{Spec}\tilde{\mathbb{P}}(k)$. Then X is naturally isomorphic to the torsor of isomorphisms between singular cohomology and algebraic de Rham cohomology on Nori motives. It has a natural torsor structure under the base change of $G_{\mathrm{mot}}(k, \mathbb{Q})$ to k (in the fpqc-topology on the category of k-schemes):*

$$X \times_k G_{\mathrm{mot}}(k, \mathbb{Q})_k \to X.$$

Remark 12.1.4. This was first formulated in the case $k = \mathbb{Q}$ without proof by Kontsevich as [K1, Theorem 6]. He attributes it to Nori.

Proof. Consider the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of Definition 8.1.1 and the representations $T_1 = H^*_{\mathrm{dR}}(-)$ and $T_2 = H^*(-, k)$ (sic!). Note that $H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ is dual to $H^d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$.

By the very definition, $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is the module $P_{1,2}(\mathrm{Pairs}^{\mathrm{eff}})$ of Definition 7.4.19. By Theorem 7.4.21, it agrees with the module $A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}})$ of Definition 7.4.2. We are now in the situation of Section 7.4 and apply its main result, Theorem 7.4.10. In particular,

$$A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}}) = A_{1,2}(\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}).$$

Recall that by Theorem 8.2.20, the diagram categories of $\mathrm{Pairs}^{\mathrm{eff}}$ and $\mathrm{Good}^{\mathrm{eff}}$ agree. This also shows that the modules

$$A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}}) = A_{1,2}(\mathrm{Good}^{\mathrm{eff}})$$

agree. From now on, we may work with the diagram $\mathrm{Good}^{\mathrm{eff}}$ which has the advantage of admitting a commutative product structure. The algebra structures on $A_{1,2}(\mathrm{Good}^{\mathrm{eff}}) = P_{1,2}(\mathrm{Good}^{\mathrm{eff}}) = \tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ agree.

We can apply the same considerations to the localized diagram Good. As in Proposition 7.2.5, localization on the level of diagrams or categories amounts to localization on the algebra. Hence,

$$A_{1,2}(\mathrm{Good}) = P_{1,2}(\mathrm{Good}) = \tilde{\mathbb{P}}(k)$$

and

$$X = \mathrm{Spec}A_{1,2}(\mathrm{Good}).$$

Also, by definition, $G_2(\text{Good})$ is the Tannakian dual of the category of Nori motives with k coefficients. By base change Lemma 6.5.6 it is the base change of the Tannaka dual of the category of Nori motives with $\mathbb{Q}$-coefficients. After these identifications, the operation

$$X \times_k G_{\text{mot}}(k,\mathbb{Q})_k \to X$$

is the one of Theorem 7.4.7.

By Theorem 7.4.10, it is a torsor because $\mathcal{MM}_{\text{Nori}}$ is rigid. □

Remark 12.1.5. There is a small subtlety here because our to fibre functors take values in different categories, $\mathbb{Q}-$Mod and $k-$Mod. As $H^*(X,Y,k) = H^*(X,Y,\mathbb{Q})\otimes_{\mathbb{Q}} k$ and $\tilde{\mathbb{P}}(k)$ already is a k-algebra, the algebra of formal periods does not change when replacing $\mathbb{Q}$-coefficients with k-coefficients.

We can also view X as torsor in the sense of Definition 1.7.9. The description of the torsor structure was discussed extensively in Section 7.4, in particular Theorem 7.4.10. In terms of period matrices, it is given by the formula in [K1]:

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P_{k\ell}^{-1} \otimes P_{\ell j}.$$

Corollary 12.1.6. *1. The algebra of effective formal periods $\tilde{\mathbb{P}}^{\text{eff}}(k)$ remains unchanged when we restrict in Definition 12.1.1 to (X,D,ω,γ) with X affine of dimension d, D of dimension $d-1$ and $X \smallsetminus D$ smooth, $\omega \in H^d_{\text{dR}}(X,D)$, $\gamma \in H_d(X(\mathbb{C}),D(\mathbb{C}),\mathbb{Q})$.*

2. $\tilde{\mathbb{P}}^{\text{eff}}(k)$ is generated as $\mathbb{Q}$-vector space by elements of the form $[X,D,\omega,\gamma]$ with X smooth of dimension d, D a divisor with normal crossings $\omega \in H^d_{\text{dR}}(X,D)$, $\gamma \in H_d(X(\mathbb{C}),D(\mathbb{C}),\mathbb{Q})$.

Proof. In the proof of Theorem 12.1.3, we have already argued that we can replace the diagram $\text{Pairs}^{\text{eff}}$ by the diagram Good^{eff}. The same argument also allows to replace it by $\text{VGood}^{\text{eff}}$.

By blowing up X, we get another good pair $(\tilde{X},\tilde{D},d)$. By excision, they have the same de Rham and singular cohomology as (X,D,d). Hence, we may identify the generators. □

Remark 12.1.7. We do not know whether it is enough to work only with formal periods of the form (X,D,ω,γ) with X smooth and D a divisor with normal crossings in Definition 12.1.1 as Kontsevich does in [K1, Definition 20]. By the Corollary, these symbols generate the algebra, but it is not clear to us if they also give all relations. Indeed, Kontsevich in loc. cit. only imposes the relation given by the connecting morphism of triples in an even more special case.

Moreover, Kontsevich considers differential forms of top degree rather than cohomology classes. They are automatically closed. He imposes Stokes' formula as an additional relation, hence this amounts to considering cohomology classes. Note, however, that not every de Rham class is of this form in general.

All formal effective periods (X, D, ω, γ) can be evaluated by "integrating" ω along γ. More precisely, recall (see Definition 5.4.1 the period pairing

$$H^d_{\mathrm{dR}}(X, D) \times H_d(X(\mathbb{C}), D(\mathbb{C})) \to \mathbb{C}$$

It maps $(\mathbb{G}_m, \{1\}, dX/X, S^1)$ to $2\pi i$.

Definition 12.1.8. Let

$$\mathrm{ev} : \tilde{\mathbb{P}}(k) \to \mathbb{C},$$

be the ring homomorphism induced by the period pairing. We denote by per the $\mathbb{C}$-valued point of $X = \operatorname{Spec} \tilde{\mathbb{P}}(k)$ defined by ev.

The elements in the image are precisely the element of the period algebra $\mathbb{P}(k)$ of Definition 9.3.1. By the results in Chapters 9, 10, and 11 (for $k = \mathbb{Q}$), it agrees with all other definitions of a period algebra. From this perspective, per is the $\mathbb{C}$-valued point of the torsor X of Theorem 12.1.3 comparing singular and algebraic de Rham cohomology. It is given by the period isomorphism per defined in Chapter 5.

The following statement of period number is a corollary from our previous results on formal periods.

Corollary 12.1.9. *The algebra $\mathbb{P}(k)$ is $\mathbb{Q}$-linearly generated by number of the form $(2\pi i)^j\alpha$ with $j \in \mathbb{Z}$, and α the period of (X, D, ω, γ) with X smooth affine, D a divisor with normal crossings, $\omega \in \Omega^d_X(X)$.*

This was also proved without mentioning motives as Theorem 9.4.2.

Proof. Recall that $2\pi i$ is itself a period of such a quadruple.

By Corollary 8.2.21, the category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is generated by motives of good pairs (X, Y, d) of the form $X = W \setminus W_\infty$, $Y = W_0 \setminus (W_\infty \cap W_0)$ with W smooth projective of dimension d, $W_0 \cup W_\infty$ a divisor with normal crossings, $X' = \setminus W_0$ affine. Hence, their periods generated $\mathbb{P}^{\mathrm{eff}}(k)$ as a $\mathbb{Q}$-vector space.

Let $Y' = W_\infty \setminus (W_0 \cap W_\infty)$. By Lemma 8.3.7, the motive $H^d_{\mathrm{Nori}}(X, Y)$ is dual to $H^d_{\mathrm{Nori}}(X', Y')(d)$. By Lemma 9.2.9, this implies that the periods of the first agree with the periods of the latter up to a factor $(2\pi i)^d$.

As X' is affine and Y' a divisor with normal crossings, $H^d_{\mathrm{dR}}(X', Y')$ is generated by $\Omega^d_{X'}(X')$ by Proposition 3.3.19. $\square$

Proposition 12.1.10. *Let K/k be algebraic. Then*

$$\tilde{\mathbb{P}}(K) = \tilde{\mathbb{P}}(k) \ ,$$

and hence also

$$\mathbb{P}(K) = \mathbb{P}(k) \ .$$

The second statement was already proved directly Corollary 9.3.5

Proof. It suffices to consider the case K/k finite. The general case follows by taking limits.

Generators of $\tilde{\mathbb{P}}(k)$ also define generators of $\tilde{\mathbb{P}}(K)$ by base change for the field extension K/k. The same is true for relations, hence we get a well-defined map $\tilde{\mathbb{P}}(k) \to \tilde{\mathbb{P}}(K)$.

We define a map in the opposite direction by viewing a K-variety as k-variety. More precisely, let (Y, E, m) be vertex of $\mathrm{Pairs}^{\mathrm{eff}}(K)$ and (Y_k, E_k, m) the same viewed as vertex of $\mathrm{Pairs}^{\mathrm{eff}}(k)$. As in the proof of Corollary 9.3.5, we have

$$\mathrm{H}(Y_k, E_k, m) = R_{K/k}\mathrm{H}(Y, E, m)$$

with $R_{K/k}$ as defined in Lemma 9.2.7. The same proof as in Lemma 9.2.7 (treating actual periods) also shows that the formal periods of (Y_k, E_k, m) agree with the formal periods (Y, E, m): $\square$

12.2 The period conjecture

We exlore the relation to transcendence questions from the point of view of of Nori motives and their periods. We only treat the case where $k/\mathbb{Q}$ is algebraic. For more general fields, see Ayoub's remarks in [Ay].

Recall that $\tilde{\mathbb{P}}(\mathbb{Q}) = \tilde{\mathbb{P}}(k) = \tilde{\mathbb{P}}(\bar{\mathbb{Q}})$ under this assumption.

Conjecture 12.2.1 (Kontsevich-Zagier)**.** *Let $k/\mathbb{Q}$ be an algebraic field extension contained in $\mathbb{C}$. The evaluation map (see Definition 12.1.8)*

$$\mathrm{ev} : \tilde{\mathbb{P}}(k) \to \mathbb{P}(k)$$

is bijective.

Remark 12.2.2. We have already seen that the map is surjective. Hence injectivity is the true issue. Equivalently, we can conjecture that $\tilde{\mathbb{P}}(k)$ is an integral domain and ev a generic point.

In the literature [A1, A2, Ay, BC, Wu], there are sometimes alternative formulations of this conjecture, called "Grothendieck conjecture". We will explain this a little bit more.

Definition 12.2.3. Let $M \in \mathcal{MM}_{\mathrm{Nori}}$ be a Nori motive. Let

$$X(M)$$

be the torsor of isomorphisms between singular and algebraic de Rham cohomology on the Tannaka category $\langle M, M^\vee \rangle^\otimes$ generated by M and

$$\tilde{\mathbb{P}}(M) = \mathcal{O}(X(M))$$

the associated ring of formal periods. If $M = H^*_{\mathrm{Nori}}(Y)$ for a variety Y, we also write $\tilde{\mathbb{P}}(Y)$.

Let $G_{\mathrm{mot}}(M)$ and $G_{\mathrm{mot}}(Y)$ be the Tannaka duals of the category with respect to singular cohomology.

These are the finite dimensional building blocks of $\tilde{\mathbb{P}}(k)$ and $G_{\mathrm{mot}}(k)$, respectively.

Remark 12.2.4. By Theorem 7.4.10, the space $X(M)$ is a $G_{\mathrm{mot}}(M)$-torsor. Hence they share all properties that can be tested after a faithfully flat base change. In particular, they have the same dimension. Moreover, $X(M)$ is smooth because $G(M)$ is a group scheme over a field of characteristic zero.

Analogous to [Ay] and [A2, Prop. 7.5.2.2 and Prop. 23.1.4.1], we can ask:

Conjecture 12.2.5 (Grothendieck conjecture for Nori motives)**.** *Let $k/\mathbb{Q}$ be an algebraic extension contained in $\mathbb{C}$ and $M \in \mathcal{MM}_{\mathrm{Nori}}(k)$. The following equivalent assertions are true:*

1. *The evaluation map*
$$\mathrm{ev} : \tilde{\mathbb{P}}(M) \to \mathbb{C}$$
is injective.
2. *The point ev_M of $\mathrm{Spec}\,\tilde{\mathbb{P}}(M)$ is a generic point, and $X(M)$ connected.*
3. *The space $X(M)$ is connected, and the transcendence degree of the subfield of $\mathbb{C}$ generated by the image of ev_M is the same as the dimension of $G_{\mathrm{mot}}(M)$.*

Proof of equivalence. Assume that ev is injective. Then $\tilde{\mathbb{P}}(M)$ is contained in the field $\mathbb{C}$, hence integral. The map to $\mathbb{C}$ factors via the residue field of a point. It ev is injective, this has to be the generic point. The subfield generated by $\mathrm{ev}(M)$ is isomorphic to the function field. Its transcendence degree is the dimension of the integral domain.

Conversely, if $X(M)$ is connected, then it $\tilde{\mathbb{P}}(M)$ is integral because it is already smooth. If ev factors the generic point, its function field embeds into $\mathbb{C}$ and hence $\tilde{\mathbb{P}}(M)$ does. If the subfield generated by the image of ev in $\mathbb{C}$ has the maximal possible transcendence degree, then ev has to be generic. $\square$

Lemma 12.2.6. *If Conjecture 12.2.5 is true for all M, then Conjecture 12.2.1 holds.*

Proof. By construction, we have
$$\tilde{\mathbb{P}}(k) = \mathrm{colim}_M \tilde{\mathbb{P}}(M).$$
Injectivity of the evaluation maps on the level of every M implies injectivity of the transition maps and injectivity of ev on the union. $\square$

Remark 12.2.7. The converse is not obvious. It amounts to asking whether $\tilde{\mathbb{P}}(M)$ is contained in $\tilde{\mathbb{P}}(k)$. In our description with generators and relations, this means that all relations are given by relations within the category $\langle M, M^\vee \rangle^\otimes$. This is not clear a priori. We have a conditional result in the pure case.

Proposition 12.2.8. *Assume that the Hodge conjecture holds for all varieties. Let M be a pure Nori motive. Then $\tilde{\mathbb{P}}(M)$ injects into $\tilde{\mathbb{P}}(k)$.*

Proof. The algebra $\tilde{\mathbb{P}}(M)$ is generated by classes (ω, γ) with $\omega \in H^*_{\mathrm{dR}}(M) \oplus H^*_{\mathrm{dR}}(M)^\vee$ and $\gamma \in H_*(M, \mathbb{Q}) \oplus H_*(M, \mathbb{Q})^\vee$ of the same cohomological degree. The relations are given by chains of morphisms and morphisms in the opposite direction

$$M \to M_1 \leftarrow M_2 \to \cdots \leftarrow M$$

in the tensor category generated by the direct sum of these Nori motives.

In $\tilde{\mathbb{P}}(k)$, the relations between these same generators are given by chains in the category of *all* Nori motives. A priori, there are more of these.

By Proposition 10.4.5, we have a weight filtration on the category of Nori motives. Morphisms between pure motives of different weights vanish. We choose our generators pure and we apply the weight filtration to the whole chain defining a relation. This implies that there are no relations between pure generators of different weights. The relations between pure generators of the same weight are already induced from relations of this fixed weight. We now apply the Hodge conjecture again and in a semi-simple category. The only relations are the ones given by the simple objects in the subcategory. □

The third version of Conjecture 12.2.5 is very close to the point of view taken originally by Grothendieck in the pure case. In order to understand the precise relation, we have to establish some properties first.

We specialize to the case $\tilde{\mathbb{P}}(Y)$ for Y smooth and projective. In this case, singular cohomology $H^*(Y, \mathbb{Q})$ carries a pure $\mathbb{Q}$-Hodge structure, see Definition 10.2.2. Recall that the Mumford-Tate group $\mathrm{MT}(V)$ of a polarizable pure Hodge structure V is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that Hodge representation $h : \mathbb{S} \to GL(V_\mathbb{R})$ factors via G as $h : \mathbb{S} \to G_\mathbb{R}$. Here, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Deligne torus. It is precisely the $\mathbb{Q}$-algebraic subgroup of $GL(V_\mathbb{R})$ that fixes all Hodge tensors in all tensor powers $\bigoplus V^{\otimes m} \otimes V^{\vee \otimes n}$ [M]. Alternatively, it can be understood as the Tannaka dual of the subcategory of the category of Hodge structures generated by V. The group $\mathrm{MT}(V)$ is a reductive $\mathbb{Q}$-algebraic group by [GGK, Chap. I].

Proposition 12.2.9. *Let $k = \bar{\mathbb{Q}}$ and let Y be smooth and projective. Assume that the Hodge conjecture holds for all powers of Y. Then $G_{\mathrm{mot}}(Y)$ is the same as the Mumford-Tate group of Y.*

Proof. By Proposition 10.4.4 the Tannaka subcategory of $\mathcal{MM}_{\mathrm{Nori}}$ generated by $H^*_{\mathrm{Nori}}(Y)$ agrees with the Tannaka subcategory of the category of Grothendieck motives GRM. Note that the statement of Proposition 10.4.4 assumes the full Hodge conjecture. The same argument also gives the statement on the subcategories under the weaker assumption. For the rest of the argument we refer to Lemme 7.2.2.1 and Remarque 23.1.4.2 of [A2]. It amounts to saying that equivalent Tannaka categories have isomorphic Tannaka duals. ☐

Corollary 12.2.10 (Period Conjecture)**.** *Let Y be a smooth, projective variety over $\mathbb{Q}$. Assume Conjecture 12.2.5 for powers of Y and the Hodge conjecture. Then every polynomial relation among the periods of Y are of motivic nature, i.e., they are induced by algebraic cycles (correspondences) in powers of Y.*

In the case of elliptic curves this was stated as conjecture by Grothendieck [Gro1].

Proof. By Conjecture 12.2.5 all $\mathbb{Q}$-linear relations between periods are induced by morphisms of Nori motives. Under the Hodge conjecture, the category of pure Nori motives is equivalent to the category of Grothendieck motives by Proposition 10.4.4. By definition of Grothendieck motives (Definition 10.4.2) this means that morphisms are induced from algebraic cycles.

Polynomial relations are induced from the tensor structure, hence powers of Y. ☐

Arnold [Ar, pg. 93] remarked in a footnote that this is related to a conjecture of Leibniz which he made in a letter to Huygens from 1691. Leibniz essentially claims that all periods of *generic* meromorphic 1-forms are transcendental. Of course, precisely the meaning of "generic" is the essential question. The conjecture of Leibniz can be rephrased in modern form as in [Wu]:

Conjecture 12.2.11 (Integral Conjecture of Leibniz)**.** *Any period integral of a rational algebraic 1-form ω on a smooth projective variety X over a number field k over a path γ with $\partial\gamma \subset D$ (the polar divisor of ω) which does not come from a proper mixed k-Hodge substructure $H \subsetneq H_1(X \setminus D)$ is transcendental.*

This is only a statement about periods of type $i = 1$, i.e., for $H^1(X, D)$ (or, by duality $H_1(X \setminus D)$) on curves. The Leibniz conjecture follows essentially from the period conjecture in the case $i = 1$, since the Hodge conjecture holds on $H^1(X) \otimes H^1(X) \subset H^2(X)$. This conjecture is still open. See also [BC] for strongly related questions.

Wüstholz [Wu] has related this problem to many other transcendance results. One can give transcendance proofs assuming this conjecture:

Example 12.2.12. Let us show that $\log(\alpha)$ is transcendental for $\alpha \neq 0, 1$ under the assumption of the Leibniz conjecture. One takes $X = \mathbb{P}^1$, and $\omega = d\log(z)$ and $\gamma = [1, \alpha]$. The polar divisor of ω is $D = \{0, \infty\}$, and the Hodge structure $H_1(X \setminus D) = H_1(\mathbb{C}^\times) = \mathbb{Z}(1)$ is irreducible as a Hodge structure. Hence, $\log(\alpha)$ is transcendental assuming Leibniz's conjecture.

There are also examples of elliptic curves in [Wu] related to Chudnovsky's theorem we mention below.

The third form of Conjecture 12.2.5 is also very useful in a computational sense. In this case, assuming the Hodge conjecture for all powers of Y, the motivic Galois group $G_{\mathrm{mot}}(Y)$ is the same as the *Mumford-Tate group* $\mathrm{MT}(Y)$ by Proposition 12.2.9.

André shows in [A2, Rem. 23.1.4.2]:

Corollary 12.2.13. *Let Y be a smooth, projective variety over $\mathbb{Q}$ and assume that the Hodge conjecture holds for all powers of Y. Then, assuming Grothendieck's conjecture,*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(Y) = \dim_{\mathbb{Q}} \mathrm{MT}(Y).$$

Proof. We view the right hand side as $G_{\mathrm{mot}}(Y_{\bar{\mathbb{Q}}})$ by Proposition 12.2.9. By [A2, Paragraph 7.6.4] it is of finite index in $G_{\mathrm{mot}}(Y)$, hence has the same dimension. It has also the same dimension as the torsor $\tilde{\mathbb{P}}(Y)$. Under Grothendieck's conjecture this is given by the transcendence degree of $\mathbb{P}(Y)$, see Conjecture 12.2.5. □

This corollary give a reasonable, completely unconditional testing conjecture for transcendence questions.

Example 12.2.14. (Tate motives) If the motive of Y is a Tate motives, e.g., $Y = \mathbb{P}^n$, then the conjecture is true, since $2\pi i$ is transcendant. The Mumford-Tate group is the 1-torus here. More generally, the conjecture holds for Artin-Tate motives, since the transcendance degree remains 1.

Example 12.2.15. (Elliptic curves) Let E be an elliptic curve over $\mathbb{Q}$. Then the Mumford-Tate group of E is either a 2-torus if E has complex multiplication, or $\mathrm{GL}_{2,\mathbb{Q}}$ otherwise (see [M]). Hence, the transcendence degree of $\mathbb{P}(E)$ is either 2 or 4. G. V. Chudnovsky [Ch] has proved that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(E) = 2$ if E is an elliptic curve with complex multiplication, and it is ≥ 2 for all elliptic curves over $\mathbb{Q}$. Note that in this situation we have actually 5 period numbers ω_1, ω_2, η_1, η_2 and π around (see Section 13.4 for more details), but they are related by Legendre's relation $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$, so that the transcendence degree cannot go beyond 4. Hence, it remains to show that the transcendence degree of the periods of an elliptic curve without complex multiplication is precisely 4, as predicted by the conjecture.

12.3 The case of 0-dimensional varieties

We go through all objects in the baby case of zero motives, i.e., the ones generated by 0-dimensional varieties.

Definition 12.3.1. Let $\mathrm{Pairs}^0 \subset \mathrm{Pairs}^{\mathrm{eff}}$ be the subdiagram of vertices (X, Y, n) with $\dim X = 0$. Let $\mathcal{MM}^0_{\mathrm{Nori}}$ be its diagram category with respect to the representation of $\mathrm{Pairs}^{\mathrm{eff}}$ given by singular cohomology with rational coefficients. Let $\mathrm{Var}^0 \subset \mathrm{Pairs}^0$ be the diagram defined by the opposite category of 0-dimensional k-varieties, or equivalently, the category of finite separable k-algebras.

If $\dim X = 0$, then $\dim Y = 0$ and X decomposes into a disjoint union of Y and $X \setminus Y$. Hence $H^*(X, Y, \mathbb{Q}) = H^*(X \setminus Y, \mathbb{Q})$ and it suffices to consider only vertices with $Y = \emptyset$. Moreover, all cohomology is concentrated in degree 0, and the pairs $(X, Y, 0)$ are all good and even very good. In particular, the multiplicative structure on Good restricts to the obvious multiplicative structure on Pairs^0 and Var^0.

We are always going to work with the multiplicative diagram Var^0 in the sequel.

Definition 12.3.2. Let $G^0_{\mathrm{mot}}(k)$ be the Tannaka dual of $\mathcal{MM}^0_{\mathrm{Nori}}$ and $\tilde{\mathbb{P}}^0(k)$ be the space of periods attached to $\mathcal{MM}^0_{\mathrm{Nori}}$.

The notation is a bit awkward because G^0 often denotes the connected component of unity of a group scheme G. Our $G^0_{\mathrm{mot}}(k)$ is very much *not* connected.

Our aim is to show that $G^0_{\mathrm{mot}}(k) = \mathrm{Gal}(\bar{k}/k)$ and $\tilde{\mathbb{P}}^0(k) \cong \bar{k}$ with the natural operation. In particular, the period conjecture (in any version) holds for 0-motives. This is essentially Grothendieck's treatment of Galois theory.

By construction of the coalgebra in Corollary 6.5.5, we have

$$A(\mathrm{Var}^0, H^0) = \mathrm{colim}_F \mathrm{End}(H^0|_F)^\vee \ ,$$

where F runs through a system of finite subdiagrams whose union is D.

We start with the case when F has a single vertex $\mathrm{Spec}K$, with K/k be a finite field extension, $Y = \mathrm{Spec}K$. The endomorphisms of the vertex are given by the elements of the Galois group $G = \mathrm{Gal}(K/k)$. We spell out $H^0(Y, \mathbb{Q})$. We have

$$Y(\mathbb{C}) = \mathrm{Mor}_k(\mathrm{Spec}\mathbb{C}, \mathrm{Spec}K) = \mathrm{Hom}_{k-\mathrm{alg}}(K, \mathbb{C})$$

the set of field embeddings of K into $\mathbb{C}$, viewed as a finite set with the discrete topology. Singular cohomology attaches a copy of $\mathbb{Q}$ to each point, hence

$$H^0(Y(\mathbb{C}), \mathbb{Q}) = \mathrm{Maps}(Y(\mathbb{C}), \mathbb{Q}) = \mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K, \mathbb{C}), \mathbb{Q}).$$

As always, this is contravariant in Y, hence covariant in fields. The left operation of the Galois group G on K induces a left operation on $H^0(Y(\mathbb{C}), \mathbb{Q})$.

Let K/k be Galois of degree d. We compute the ring of endomorphisms of H^0 on the single vertex $\mathrm{Spec}K$ (see Definition 6.1.8)

$$E = \mathrm{End}(H^0|_{\mathrm{Spec}K}).$$

By definition, these are the endomorphisms of $H^0(\mathrm{Spec}K, \mathbb{Q})$ commuting with the operation of the Galois group. The set $Y(\mathbb{C})$ has a simply transitive action of

G. Hence, $\mathrm{Maps}(Y(\mathbb{C}), \mathbb{Q})$ is a free $\mathbb{Q}[G]^{op}$-module of rank 1. Its commutator E is then isomorphic to $\mathbb{Q}[G]$. This statement already makes the algebra structure on E explicit.

The diagram algebra does not change when we consider the diagram $\mathrm{Var}^0(K)$ containing all vertices of the form A with $A = \bigoplus_{i=1}^n K_i$, $K_i \subset K$.

There are two essential cases: If $K' \subset K$ is a subfield, we have a surjective map $Y(\mathbb{C}) \to Y'(\mathbb{C})$. The compatibility condition with respect to this map implies that the value of the diagram endomorphism on K' is already determined by its value on K. If $A = K \bigoplus K$, then compatibility with the inclusion of the first and the second factor implies that the value of the diagram endomorphism on A is already determined by its value on K.

In more abstract language: The category $\mathrm{Var}^0(K)$ is equivalent to the category of finite G-sets. The algebra E is the group ring of the Galois group of this category under the representation $S \mapsto \mathrm{Maps}(S, \mathbb{Q})$.

Note that $K \otimes_k K = \bigoplus_\sigma K$, with σ running through the Galois group, is in $\mathrm{Var}^0(K)$. The category has fibre products. In the language of Definition 7.1.3, the diagram $\mathrm{Var}^0(K)$ has a commutative product structure (with trivial grading). By Proposition 7.1.5 and its proof, the diagram category is a tensor category, or equivalently, E carries a comultiplication.

We go through the construction in the proof of loc.cit. We start with an element of E and view it as an endomorphism of $H^0(Y \times Y(\mathbb{C}), \mathbb{Q}) \cong H^0(Y(\mathbb{C}), \mathbb{Q}) \otimes H^0(Y(\mathbb{C}), \mathbb{Q})$, hence as a tensor product of endomorphisms of $H^0(Y(\mathbb{C}), \mathbb{Q})$. The operation of $E = \mathbb{Q}[G]$ on $\mathrm{Maps}(Y(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})$ is determined by the condition that it has to be compatible with the diagonal map $Y(\mathbb{C}) \to Y(\mathbb{C}) \times Y(\mathbb{C})$. This amounts to the diagonal embedding $\mathbb{Q}[G] \to \mathbb{Q}[G] \otimes \mathbb{Q}[G]$.

Thus we have shown that $E = \mathbb{Q}[G]$ as bialgebra. This means that

$$G_{\mathrm{mot}}(Y) = \mathrm{Spec} E^\vee = G$$

as a constant monoid (even group) scheme over $\mathbb{Q}$.

Passing to the limit over all K we get

$$G^0_{\mathrm{mot}}(k) = \mathrm{Gal}(\bar{k}/k)$$

as proalgebraic group schemes of dimension 0. As a byproduct, we see that the monoid attachted to $\mathcal{MM}^0_{\mathrm{Nori}}$ is a group, hence the category is rigid.

We now turn to periods, again in the case K/k finite and Galois. Note that $H^0_{\mathrm{dR}}(\mathrm{Spec} K) = K$ and the period isomorphism

$$\begin{aligned} K \otimes_k \mathbb{C} &\to \mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K, \mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}, \\ v &\mapsto \qquad (f \mapsto f(v)) \end{aligned}$$

is the base change of the same map with values in K

$$K \otimes_k K \to \mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K, K), \mathbb{Q}) \otimes_{\mathbb{Q}} K.$$

In particular, all entries of the period matrix are in K. The space of formal periods of K is generated by symbols (ω, γ) where ω runs through a k-basis of K and γ through the set $\mathrm{Hom}_{k-\mathrm{alg}}(K, K)$ viewed as basis of a $\mathbb{Q}$-vector space. The relations coming from the operation of Galois group bring us down to a space of dimension $[K : k]$, hence the evaluation map is injective. Passing to the limit, we get

$$\tilde{\mathbb{P}}^0(k) = \bar{k}.$$

(We would get the same result by applying Proposition 12.1.10 and working only over $\bar{k}$.) The operation of $\mathrm{Gal}(\bar{k}/k)$ on $\tilde{\mathbb{P}}^0(k)$ is the natural one. More precisely, $g \in \mathrm{Gal}(\bar{k}/k)$ operates by applying g^{-1} because the operation is defined via γ, which is in the dual space. Note that the dimension of $\tilde{\mathbb{P}}^0(k)$ is also 0.

We have seen from general principles that the operation of $\mathrm{Gal}(\bar{k}/k)$ on $X^0(k) = \tilde{\mathbb{P}}^0(k)$ defines a torsor. In this case, we can trivialize it already over $\bar{k}$. We have

$$\mathrm{Mor}_k(\mathrm{Spec}\bar{k}, X^0(k)) = \mathrm{Hom}_{k-\mathrm{alg}}(\bar{k}, \bar{k}).$$

By Galois theory, the operation of $\mathrm{Gal}(\bar{k}/k)$ on this set is simply transitive.

When we apply the same discussion to the ground field $\bar{k}$, we get $G^0_{\mathrm{mot}}(\bar{k}) = \mathrm{Gal}(\bar{k}/\bar{k})$ and $\tilde{\mathbb{P}}^0(\bar{k}) = \bar{k}$. We see that the (formal) period algebra has not changed, but the motivic Galois group has. It is still true that $\mathrm{Spec}\bar{k}$ is a torsor under the motivic Galois group, but now viewed as $\bar{k}$-schemes, where both consist of a single point!