Periods and Nori Motives

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Part IV

Part IV

Examples

Chapter 13

Elementary examples

This chapter follows partly the Diploma thesis of Benjamin Friedrich, see [Fr].

13.1 Logarithms

In this section, we give one of the most simple examples for a cohomological period in the sense of Chap. 9. Let

$$X := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$$

be the affine line with the point 0 deleted and

$$D := \{1, \alpha\} \quad \text{with} \quad \alpha \neq 0, 1$$

a divisor on X. The singular homology of the pair $(X(\mathbb{C}), D(\mathbb{C})) = (\mathbb{C}^\times, \{1, \alpha\})$ is generated by a small loop σ turning counter-clockwise around 0 once and the interval $[1, \alpha]$. In order to compute the algebraic de Rham cohomology of (X, D), we first note that by Section 3.2, $H^\bullet_{\mathrm{dR}}(X, D)$ is the cohomology of the complex of global sections of the cone complex $\widetilde{\Omega}^\bullet_{X,D}$, since X is affine and the sheaves $\widetilde{\Omega}^p_{X,D}$ are quasi-coherent, hence acyclic for the global section functor. We spell out the complex $\Gamma(X, \widetilde{\Omega}^\bullet_{X,D})$ in detail

$$\begin{array}{c} 0 \\ \uparrow \\ \Gamma(X, \widetilde{\Omega}^1_{X,D}) = \Gamma\big(X, \Omega^1_X \oplus \bigoplus_j i_* \mathcal{O}_{D_j}\big) = \mathbb{Q}[t, t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} \\ \uparrow d \\ \Gamma(X, \mathcal{O}_X) = \mathbb{Q}[t, t^{-1}] \end{array}$$

and observe that the evaluation map

$$\begin{aligned} \mathbb{Q}[t,t^{-1}] &\twoheadrightarrow \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} \\ f(t) &\mapsto \big(f(1), f(\alpha)\big) \end{aligned}$$

is surjective with kernel

$$(t-1)(t-\alpha)\mathbb{Q}[t,t^{-1}] = \operatorname{span}_{\mathbb{Q}}\{t^{n+2} - (\alpha+1)t^{n+1} + \alpha t^n \mid n \in \mathbb{Z}\}.$$

Differentiation maps this kernel to

$$\operatorname{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1} \mid n \in \mathbb{Z}\}dt.$$

Therefore we get

$$\begin{aligned} H^1_{\mathrm{dR}}(X,D) &= \Gamma(X_0, \widetilde{\Omega}_{X,D}) \,/\, \Gamma(X, \mathcal{O}_X) \\ &= \mathbb{Q}[t,t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} \,/\, d(\mathbb{Q}[t,t^{-1}]) \\ &= \mathbb{Q}[t,t^{-1}]dt / \operatorname{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1}\}dt. \end{aligned}$$

By the last line, we see that the class of $t^n dt$ in $H^1_{\mathrm{dR}}(X,D)$ for $n \neq -1$ is linearly dependent of

- $t^{n-1}dt$ and $t^{n-2}dt$, and
- $t^{n+1}dt$ and $t^{n+2}dt$,

hence we see by induction that $\frac{dt}{t}$ and dt generate $H^1_{\mathrm{dR}}(X,D)$. Therefore, $H^1_{\mathrm{dR}}(X,D)$ is spanned by

$$\frac{dt}{t} \quad \text{and} \quad \frac{1}{\alpha-1}dt.$$

We obtain the following period matrix P for $H^1(X,D)$:

	$\frac{1}{\alpha-1}dt$	$\frac{dt}{t}$
$[1,\alpha]$	1	$\log\alpha$
σ	0	$2\pi i$

(13.1)

In Section 7.4.3 we have seen how the torsor structure on the periods of (X,D) is given by a triple coproduct Δ in terms of the matrix P:

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P^{-1}_{k\ell} \otimes P_{\ell j} \; .$$

The inverse period matrix in this example is given by:

$$P^{-1} = \begin{pmatrix} 1 & \frac{-\log\alpha}{2\pi i} \\ 0 & \frac{1}{2\pi i} \end{pmatrix}$$

and thus we get for the triple coproduct of the most important entry $\log(\alpha)$

$$\Delta(\log\alpha) = \log\alpha \otimes \tfrac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \tfrac{\log\alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \log\alpha \; . \tag{13.2}$$

13.2 More Logarithms

In this section, we describe a variant of the cohomological period in the previous section. We define

$$D_0 := \{1, \alpha, \beta\} \quad \text{with} \quad \alpha \neq 0, 1 \quad \text{and} \quad \beta \neq 0, 1, \alpha,$$

but keep $X := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$.

Then, $H_1^{\text{sing}}(X, D; \mathbb{Q})$ is generated by the loop σ from the first example and the intervals $[1, \alpha]$ and $[\alpha, \beta]$. Hence, the differential forms $\frac{dt}{t}$, dt and $2t\,dt$ give a basis of $H^1_{\text{dR}}(X, D)$: If they were linearly dependent, the period matrix P would not be of full rank

	$\frac{dt}{t}$	dt	$2t\,dt$
σ	$2\pi i$	0	0
$[1, \alpha]$	$\log \alpha$	$\alpha - 1$	$\alpha^2 - 1$
$[\alpha, \beta]$	$\log\left(\frac{\beta}{\alpha}\right)$	$\beta - \alpha$	$\beta^2 - \alpha^2$.

We observe that $\det P = 2\pi i(\alpha - 1)(\beta - \alpha)(\beta - 1) \neq 0$.

We have

$$P^{-1} = \begin{pmatrix} \frac{1}{2\pi i} & 0 & 0 \\ \frac{\log\beta(\alpha^2-1) - \log\alpha(\beta^2-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{\alpha+\beta}{(\alpha-1)(\beta-1)} & \frac{\alpha+1}{(\alpha-\beta)(\beta-1)} \\ \frac{-\log\beta(\alpha-1)+\log\alpha(\beta-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-\beta)(\beta-1)} \end{pmatrix},$$

and therefore we get for the triple coproduct for the entry $\log(\alpha)$:

$$\begin{aligned}
\Delta(\log\alpha) = {} & \log\alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i \\
& + (\alpha - 1) \otimes \frac{-\log\beta(\alpha^2 - 1) + \log\alpha(\beta^2 - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
& + (\alpha - 1) \otimes \frac{\alpha + \beta}{(\alpha - 1)(\beta - 1)} \otimes \log\alpha \\
& + (\alpha - 1) \otimes \frac{\alpha + 1}{(\alpha - \beta)(\beta - 1)} \otimes \log\left(\frac{\beta}{\alpha}\right) \\
& + (\alpha^2 - 1) \otimes \frac{\log\beta(\alpha - 1) - \log\alpha(\beta - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
& + (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - 1)(\beta - 1)} \otimes \log\alpha \\
& + (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - \beta)(\beta - 1)} \otimes \log\left(\frac{\beta}{\alpha}\right) \\
= {} & \log\alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \frac{\log\alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \log\alpha \ .
\end{aligned}$$

Compare this with Equation 13.2 !

13.3 Quadratic Forms

Let

$$\begin{array}{cccc} Q(\underline{x}): & \mathbb{Q}^3 & \longrightarrow & \mathbb{Q} \\ & \underline{x}=(x_0,x_1,x_2) & \mapsto & \underline{x}\,A\,\underline{x}^T \end{array}$$

be a quadratic form with $A \in \mathbb{Q}^{3\times 3}$ being a regular, symmetric matrix.

The zero-locus of $Q(\underline{x})$

$$\overline{X} := \{\underline{x} \in \mathbb{P}^2(\mathbb{Q}) \,|\, Q(\underline{x}) = 0\}$$

is a *quadric* or non-degenerate *conic*. We are interested in its affine piece

$$X := \overline{X} \cap \{x_0 \neq 0\} \subset \mathbb{Q}^2 \subset \mathbb{P}^2(\mathbb{Q}).$$

We show that we can assume $Q(\underline{x})$ to be of a particular nice form. A non-zero vector $v \in \mathbb{Q}^3$ is called *Q-anisotropic*, if $Q(v) \neq 0$. Since $\operatorname{char}\mathbb{Q} \neq 2$, there exist such vectors, just suppose the contrary:

$$\begin{array}{rcr} Q(1,0,0)=0 & \text{gives} & A_{11}=0, \\ Q(0,1,0)=0 & \text{gives} & A_{22}=0, \\ Q(1,1,0)=0 & \text{gives} & 2\cdot A_{12}=0 \end{array}$$

and A would be degenerate. In particular

$$Q(1,\lambda,0) = Q(1,0,0) + 2\lambda Q(1,1,0) + \lambda^2 Q(0,1,0)$$

will be different form zero for almost all $\lambda \in \mathbb{Q}$. Hence, we can assume that $(1,0,0)$ is anisotropic after applying a coordinate transformation of the form

$$x_0' := x_0, \quad x_1' := -\lambda x_0 + x_1, \quad x_2' := x_2.$$

After another affine change of coordinates, we can also assume that A is a diagonal matrix. An inspection reveals that we can choose this coordinate transformation such that the x_0-coordinate is left unaltered. (Just take for e_1 the anisotropic vector $(1,0,0)$ in the proof.) Such a transformation does not change the isomorphism type of X, and we can take X to be cut out by an equation of the form

$$ax^2 + by^2 = 1 \quad \text{for} \quad a,b \in \mathbb{Q}^\times$$

with affine coordinates $x := \frac{x_1}{x_0}$ and $y := \frac{x_2}{x_0}$. Since X is affine, the sheaves Ω_X^p are acyclic, hence we can compute its algebraic de Rham cohomology by

$$H^\bullet_{\mathrm{dR}}(X) = \mathrm{h}^\bullet\Gamma(X, \Omega_X^\bullet),$$

so we write down the complex $\Gamma(X, \Omega_X^\bullet)$ in detail

$$\begin{array}{l} 0 \\ \uparrow \\ \Gamma(X, \Omega_X^1) = \mathbb{Q}[x,y]/(ax^2+by^2-1)\{dx, dy\} \,/\, (axdx + bydy) \\ {\scriptstyle d}\uparrow \\ \Gamma(X, \mathcal{O}_X) = \mathbb{Q}[x,y]/(ax^2+by^2-1). \end{array}$$

Obviously, $H^1_{\mathrm{dR}}(X)$ can be presented with generators $x^n y^m dx$ and $x^n y^m dy$ for $m, n \in \mathbb{N}_0$ modulo numerous relations. Using $axdx + bydy = 0$, we get

- $y^m\,dy = d\,\frac{y^{m+1}}{m+1} \sim 0$
- $x^n\,dx = d\,\frac{x^{n+1}}{n+1} \sim 0$

$n \geq 1$
- $x^n y^m\,dy = \frac{-n}{m+1} x^{n-1} y^{m+1}\,dx + d\,\frac{x^n y^{m+1}}{m+1}$
  $\sim \frac{-n}{m+1} x^{n-1} y^{m+1}\,dx \quad \text{for} \quad n \geq 1, m \geq 0$
- $x^n y^{2m}\,dx = x^n \big(\frac{1-ax^2}{b}\big)^m\,dx \sim 0$
- $x^n y^{2m+1}\,dx = x^n \big(\frac{1-ax^2}{b}\big)^m y\,dx$
- $xy\,dx = \frac{-x^2}{2}\,dy + d\,\frac{x^2 y}{2}$
  $\sim \frac{by^2-1}{2a}\,dy$
  $= \frac{b}{2a} y^2\,dy - \frac{1}{2a}\,dy \sim 0$

$n \geq 2$
- $$\begin{aligned} x^n y\,dx &= \tfrac{-b}{a} x^{n-1} y^2\,dy + x^n y\,dx + \tfrac{b}{a} x^{n-1} y^2\,dy \\ &= \tfrac{-b}{a} x^{n-1} y^2\,dy + \tfrac{x^{n-1}y}{2a}\,d(ax^2+by^2-1) \\ &= \tfrac{-b}{a} x^{n-1} y^2\,dy + d\,\Big(\tfrac{(x^{n-1}y)(ax^2+by^2-1)}{2a}\Big) \\ &\sim \tfrac{-b}{a} x^{n-1} y^2\,dy \\ &= \Big(x^{n+1} - \tfrac{x^{n-1}}{a}\Big)\,dy \\ &= \Big(-(n+1)x^n y + \tfrac{n-1}{a} x^{n-2} y\Big)\,dx + d\,\Big(x^{n+1}y - \tfrac{x^{n-1}}{a} y\Big) \end{aligned}$$

$\Rightarrow\; x^n y\,dx \sim \frac{n-1}{(n+2)a} x^{n-2} y\,dx \quad \text{for} \quad n \geq 2.$

Thus we see that all generators are linearly dependent of $y\,dx$

$$H^1_{\mathrm{dR}}(X) = \mathrm{h}^1\Gamma(X, \Omega_X^\bullet) = \mathbb{Q}\; y\,dx.$$

What about the base change to $\mathbb{C}$ of X? We use the symbol $\sqrt{}$ for the principal branch of the square root. Over $\mathbb{C}$, the change of coordinates

$$u := \sqrt{a}x - i\sqrt{b}y, \quad v := \sqrt{a}x + i\sqrt{b}y$$

gives

$$\begin{aligned}
X =& \operatorname{Spec}\mathbb{C}[x,y]/(ax^2+by^2-1)\\
=& \operatorname{Spec}\mathbb{C}[u,v]/(uv-1)\\
=& \operatorname{Spec}\mathbb{C}[u,u^{-1}]\\
=& \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}.
\end{aligned}$$

Hence, the first singular homology group $H^{\text{sing}}_\bullet(X,\mathbb{Q})$ of X is generated by

$$\sigma : [0,1] \to X(\mathbb{C}), s \mapsto u = e^{2\pi i s},$$

i.e., a circle with radius 1 turning counter-clockwise around $u = 0$ once.

The period matrix consists of a single entry

$$\begin{aligned}
\int_\sigma y\,dx =& \int_\sigma \frac{v-u}{2i\sqrt{b}}\, d\,\frac{u+v}{2\sqrt{a}}\\
\overset{\text{Stokes}}{=}& \int_\sigma \frac{v\,du - u\,dv}{4i\sqrt{ab}}\\
=& \frac{1}{2i\sqrt{ab}} \int_\sigma \frac{du}{u}\\
=& \frac{\pi}{\sqrt{ab}}.
\end{aligned}$$

The denominator squared is nothing but the discriminant of the quadratic form Q

$$\operatorname{disc} Q := \det A \in \mathbb{Q}^\times/{\mathbb{Q}^\times}^2.$$

This is an important invariant, that distinguishes some, but not all isomorphism classes of quadratic forms. Since $\operatorname{disc} Q$ is well-defined modulo $(\mathbb{Q}^\times)^2$, it makes sense to write

$$H^1_{\text{dR}}(X) = \mathbb{Q}\,\frac{\pi}{\sqrt{\operatorname{disc} Q}} \subset H^1_{\text{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

13.4 Elliptic Curves

In this section, we give another well-known example for a cohomological period in the sense of Chap. 9.

An *elliptic curve* E is a one-dimensional non-singular complete and connected group variety over a field k, together with the origin 0, a k-rational point. An elliptic curve has genus $g = 1$, where the genus g of a smooth projective curve is defined as

$$g := \dim_k \Gamma(E, \Omega^1_E)\,.$$

We refer to the book [Sil] of Silverman for the theory of elliptic curves, but try to be self-contained in the following. For simplicity, we assume $k = \mathbb{Q}$. It can be shown, using the Riemann-Roch theorem that such an elliptic curve E can be given as the zero locus in $\mathbb{P}^2(\mathbb{Q})$ of a Weierstraß equation

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \tag{13.3}$$

with Eisenstein series coefficients $g_2 = 60G_4, g_3 = 140G_6$ and projective coordinates X, Y and Z.

By the classification of compact, oriented real surfaces, the base change of E to $\mathbb{C}$ gives us a complex torus E^{an}, i.e., an isomorphism

$$E^{\mathrm{an}} \cong \mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \tag{13.4}$$

in the complex analytic category with

$$\Lambda_{\omega_1,\,\omega_2} := \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$
$$\text{for } \omega_1, \omega_2 \in \mathbb{C} \text{ linearly independent over } \mathbb{R},$$

being a lattice of full rank. Thus, all elliptic curves over $\mathbb{C}$ are diffeomorphic to the standard torus $S^1 \times S^1$, but carry different complex structures as the parameter $\tau := \omega_2/\omega_1$ varies. We can describe the isomorphism (13.4) quite explicitly using periods. Let α and β be a basis of

$$H_1^{\mathrm{sing}}(E, \mathbb{Z}) = H_1^{\mathrm{sing}}(S^1 \times S^1, \mathbb{Z}) = \mathbb{Z}\,\alpha \,\oplus\, \mathbb{Z}\,\beta.$$

The $\mathbb{Q}$-vector space $\Gamma(E, \Omega_E^1)$ is spanned by the holomorphic differential

$$\omega = \frac{dX}{Y}.$$

The map

$$\begin{aligned} E^{\mathrm{an}} &\to \quad \mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \\ P &\mapsto \int_O^P \omega \;\text{ modulo } \Lambda_{\omega_1,\,\omega_2} \end{aligned} \tag{13.5}$$

then gives the isomorphism of Equation 13.4. Here $O = [0:1:0]$ denotes the group theoretic origin in E. The integrals

$$\omega_1 := \int_\alpha \omega \quad \text{and} \quad \omega_2 := \int_\beta \omega$$

are called the periods of E. Up to a $\mathbb{Z}$-linear change of basis, they are precisely the above generators of the lattice $\Lambda_{\omega_1,\,\omega_2}$.

The inverse map $\mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \to E^{\mathrm{an}}$ for the isomorphism (13.5) can be described in terms of the Weierstraß $\wp$-function of the lattice $\Lambda := \Lambda_{\omega_1,\,\omega_2}$

$$\wp(z) = \wp(z, \Lambda) := \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

and takes the form

$$\begin{aligned}\mathbb{C}/\Lambda_{\omega_1,\omega_2} &\to E^{\text{an}} \subset \mathbb{C}P_{\text{an}}{}^2\\ z &\mapsto [\wp(z) : \wp'(z) : 1], \Lambda_{\omega_1,\omega_2} \mapsto (0:1:0).\end{aligned}$$

Note that under the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_{\omega_1,\omega_2}$ any meromorphic function f on the torus $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$ lifts to a doubly-periodic function $\pi^* f$ on the complex plane $\mathbb{C}$ with periods ω_1 and ω_2

$$f(x + n\omega_1 + m\omega_2) = f(x) \quad \text{for all} \quad n, m \in \mathbb{Z} \quad \text{and} \quad x \in \mathbb{C}.$$

This example is possibly the origin of the "period" terminology.

The defining coefficients G_4, G_6 of E can be recovered from $\Lambda_{\omega_1,\omega_2}$ by the Eisenstein series

$$G_{2k} := \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \omega^{-2k} \quad \text{for} \quad k = 2, 3.$$

Therefore, the periods ω_1 and ω_2 determine the elliptic curve E uniquely. However, they are not invariants of E, since they depend on the chosen Weierstraß equation of E. A change of coordinates which preserves the shape of (13.3), must be of the form

$$X' = u^2 X, \quad Y' = u^3 Y, \quad Z' = Z \quad \text{for} \quad u \in \mathbb{Q}^\times.$$

In the new parametrization X', Y', Z', we have

$$\begin{aligned}&G_4' = u^4 G_4, \quad G_6' = u^6 G_6,\\ &\omega' = u^{-1}\omega\\ &\omega_1' = u^{-1}\omega_1 \quad \text{and} \quad \omega_2' = u^{-1}\omega_2.\end{aligned}$$

Hence, $\tau = \omega_2/\omega_1$ is a better invariant of the isomorphism class of E. The value of the j-function (a modular function)

$$j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = q^{-1} + 744 + 196884q + \cdots \ (q = \exp(2\pi i\tau)$$

on τ indeed distinguishes non-isomorphic elliptic curves E over $\mathbb{C}$:

$$E \cong E' \text{ if and only if } j(E) = j(E').$$

Hence, the moduli space of elliptic curves over $\mathbb{C}$ is the affine line.

A similar result holds over any algebraically closed field K of characteristic different from $2, 3$. For fields K that are not algebraically closed, the set of K-isomorphism classes of elliptic curves isomorphic over $\bar{K}$ to a fixed curve E/K is the Weil-Châtelet group of E [Sil], an infinite group for K a number field.

However, E has two more cohomological periods which are also called *quasi-periods*. In section 13.5, we will prove that the meromorphic differential form

$$\eta := X\frac{dX}{Y}$$

spans $H^1_{\mathrm{dR}}(E)$ together with $\omega = \frac{dX}{Y}$, i.e., modulo exact forms this form is a generator of $H^1(E, \mathcal{O}_E)$ in the Hodge decomposition. Like ω corresponds to dz under (13.5), η corresponds to $\wp(z)dz$. The quasi-periods then are

$$\eta_1 := \int_\alpha \eta, \quad \eta_2 := \int_\beta \eta.$$

We obtain the following period matrix for E:

$$\begin{array}{c|cc} & \frac{dX}{Y} & X\frac{dX}{Y} \\ \hline \alpha & \omega_1 & \eta_1 \\ \beta & \omega_2 & \eta_2 \end{array} \qquad (13.6)$$

Lemma 13.4.1. *One has the Legendre relation (negative determinant of period matrix)*

$$\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i.$$

Proof. Consider the Weierstraß ζ-function [Sil]

$$\zeta(z) := \frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

It satisfies $\zeta'(z) = -\wp(z)$. Since $\zeta'(z) = -\wp(z)$ and $\wp$ is periodic, we have that $\eta(w) = \zeta(z+w) - \zeta(z)$ is independent of z. Hence, the complex path integral counter-clockwise around the fundamental domain centered at some point $a \notin \Lambda_{\omega_1,\omega_2}$ yields

$$\begin{aligned} 2\pi i &= \int_a^{a+\omega_1} \zeta(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} \zeta(z)dz - \int_{a+\omega_2}^{a+\omega_1+\omega_2} \zeta(z)dz - \int_a^{a+\omega_2} \zeta(z)dz \\ &= \int_a^{a+\omega_2} \left(\zeta(z+\omega_1) - \zeta(z)\right) dz - \int_a^{a+\omega_1} \left(\zeta(z+\omega_2) - \zeta(z)\right) dz \\ &= \omega_2\eta_1 - \omega_1\eta_2, \end{aligned}$$

where $\eta_i = \eta(\omega_i)$. □

In the following two examples all four periods are calculated and yield Γ-values besides $\sqrt{\pi}$, π and algebraic numbers. Such period expressions for elliptic curves with complex multiplication are nowadays called the Chowla-Lerch-Selberg formula, after Lerch [L] and Chowla-Selberg [CS]. See also the thesis of B. Gross [Gr].

Example 13.4.2. Let E be the elliptic curve with $G_6 = 0$ and affine equation $Y^2 = 4X^3 - 4X$.Then one has [Wa]

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}}, \quad \omega_2 = i\omega_1,$$

and

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \quad \eta_2 = -i\eta_1.$$

E has complex multiplication with ring $\mathbb{Z}[i]$ (Gaußian integers).

Example 13.4.3. Look at the elliptic curve E with $G_4 = 0$ and affine equation $Y^2 = 4X^3 - 4$.Then one has [Wa]

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{3}B\left(\frac{1}{6}, \frac{1}{2}\right) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}, \quad \omega_2 = \rho\omega_1,$$

($\rho = \frac{-1+\sqrt{-3}}{2}$) and

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \quad \eta_2 = \rho^2\eta_1.$$

E has complex multiplication with ring $\mathbb{Z}[\rho]$ (Eisenstein numbers).

Both of these example have complex multiplication. As we have explained in Example 12.1.15, G. V. Chudnovsky [Ch] has proved that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(E) = 2$ if E is an elliptic curve with complex multiplication. This means that ω_1 and π are both transcendant and algebraically independent, and ω_2, η_1 and η_2 are algebraically dependent. The transcendance of ω_1 for all elliptic curves is a theorem of Th. Schneider [S]. Of course, the transcendance of π is Lindemann's theorem.

For elliptic without complex multiplication it is conjectured that the Legendre relation is the only algebraic relation among the 5 period numbers ω_1, ω_2, η_1, η_2 and π. But this is still open.

13.5 Periods of 1-forms on arbitrary curves

Let X be a smooth, projective curve of geometric genus g over k, where $k \subset \mathbb{C}$. We denote the associated analytic space by X^{an}.

In the classical literature, different types of meromorphic differential forms on X^{an} and their periods were considered. The survey of Messing [Me] gives a historical account, see also [GH, pg. 459]. In this section, we mention these notions, translate them into a modern language, and relate them to cohomological periods in the sense of Chap. 9, since the terminology is still used in many areas of mathematics, e.g., in transcendence theory.

A *meromorphic* 1-form ω on X^{an} is locally given by $f(z)dz$, where f is meromorphic. Any meromorphic function has poles in a discrete and finite set D in X^{an}. Using a local coordinate z at a point $P \in X^{\mathrm{an}}$, we can write $f(z) = z^{-\nu(P)} \cdot h(z)$, where h is holomorphic and $h(P) \neq 0$. In particular, a meromorphic 1-form is a section of the holomorphic line bundle $\Omega^1_{X^{\mathrm{an}}}(kD)$ for some integer $k \geq 0$. We say that ω has *logarithmic poles*, if $\nu(P) \leq 1$ at all points of D. A *rational* 1-form is a section of the line bundle $\Omega^1_X(kD)$ on X. In particular, we can speak of rational 1-forms defined over k, if X is defined over k.

Proposition 13.5.1. *Meromorphic* 1*-forms on* X^{an} *are the same as rational* 1*-forms on* X*.*

Proof. Since X is projective, and meromorphic 1-forms are section of the line bundle $\Omega^1_X(kD)$ for some integer $k \geq 0$, this follows from Serre's GAGA principle [Se1]. □

In the following, we will mostly use the analytic language of meromorphic forms.

Definition 13.5.2. A *differential of the first kind* on X^{an} is a holomorphic 1-form (hence closed). A *differential of the second kind* is a closed meromorphic 1-form with vanishing residues. A *differential of the third kind* is a closed meromorphic 1-form with at most logarithmic poles along some divisor $D^{\mathrm{an}} \subset X^{\mathrm{an}}$.

Note that forms of the second and third kind include forms of the first kind.

Theorem 13.5.3. *Any meromorphic* 1*-form* ω *on* X^{an} *can be written as*

$$\omega = df + \omega_1 + \omega_2 + \omega_3,$$

where df *is an exact form,* ω_1 *is of the first kind,* ω_2 *is of the second kind, and* ω_3 *is of the third kind. This decomposition is unique up to exact forms, if* ω_3 *is chosen not to be of second kind, and* ω_2 *not to be of the first kind.*

The first de Rham cohomology of X^{an} *is given by*

$$H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \cong \frac{1 - \textit{forms of the second kind}}{\textit{exact forms}}$$

The inclusion of differentials of the first kind into differentials of the second kind is given by the Hodge filtration

$$H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}) \subset H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}).$$

For differentials of the third kind, we note that

$$H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}\langle D^{\mathrm{an}}\rangle) \cong \frac{1 - \textit{forms of the third kind with poles in } D^{\mathrm{an}}}{\textit{exact forms} + \textit{forms of the first kind}}.$$

Proof. Let ω be a meromorphic 1-form on $X^{\rm an}$. The residue theorem states that the sum of the residues of ω is zero. Suppose that ω has poles in the finite subset $D \subset X^{\rm an}$. Then look at the exact sequence

$$0 \to H^0(X^{\rm an}, \Omega^1_{X^{\rm an}}) \to H^0(X^{\rm an}, \Omega^1_{X^{\rm an}}\langle D\rangle) \stackrel{\rm Res}{\to} \bigoplus_{P \in D} \mathbb{C} \stackrel{\Sigma}{\to} H^1(X^{\rm an}, \Omega^1_{X^{\rm an}}).$$

It shows that there exists a 1-form $\omega_3 \in H^0(X^{\rm an}, \Omega^1_{X^{\rm an}}(\log D))$ of the third kind which has the same residues as ω. In addition, the form $\omega - \omega_3$ is of the second kind, i.e., it has perhaps poles but no residues. Now, look at the meromorphic de Rham complex

$$\Omega^0_{X^{\rm an}}(*) \stackrel{d}{\longrightarrow} \Omega^1_{X^{\rm an}}(*)$$

of all meromorphic differential forms on $X^{\rm an}$ (with arbitrary poles along arbitrary divisors). The cohomology sheaves of it are given by [GH, pg. 457]

$$\mathcal{H}^0\Omega^\bullet_{X^{\rm an}}(*) = \mathbb{C}, \quad \mathcal{H}^1\Omega^\bullet_{X^{\rm an}}(*) = \bigoplus_{P \in X^{\rm an}} \mathbb{C}\ .$$

These isomorphisms are induced by the inclusion of constant functions and the residue map respectively. With the help of the spectral sequence abutting to $H^*(X^{\rm an}, \Omega^*_{X^{\rm an}}(*))$ [GH, pg. 458], one obtains an exact sequence

$$0 \to H^1_{\rm dR}(X^{\rm an}, \mathbb{C}) \to \frac{H^0(X^{\rm an}, \Omega^1_{X^{\rm an}}(*))}{\text{exact forms}} \stackrel{\rm Res}{\longrightarrow} \bigoplus_{P \in X^{\rm an}} \mathbb{C},$$

and the claim follows. □

Corollary 13.5.4. *In the algebraic category, if X is defined over $k \subset \mathbb{C}$, we have that*

$$H^1_{\rm dR}(X) \cong \frac{1 - \textit{rational forms of the second kind over } k}{\textit{exact forms}}$$

We can now define periods of differentials of the first, second, and third kind.

Definition 13.5.5. Periods of the n-th kind (n=1,2,3) in the sense of Definition 9.1.1 are periods of differentials ω of the n-th kind, i.e., integrals

$$\int_\gamma \omega\ ,$$

where γ is a closed path avoiding the poles of D for $n = 2$ and which is contained in $X \setminus D$ for $n = 3$.

Usually, in the literature periods of 1-forms of the first kind are called periods, and periods of 1-forms of the second kind and not of the first kind are called quasi-periods.

Theorem 13.5.6. *Let X be a smooth, projective curve over k as above.*

Periods of the second kind (and hence also periods of the first kind) are cohomological periods in the sense of 9.3.1 of the first cohomology group $H^1(X)$. Periods of the third kind with poles along D are periods of the cohomology group $H^1(U)$, where $U = X \setminus D$.

Every period of any smooth, quasiprojective curve U over k is of the first, second or third kind on a smooth compacification X of U.

Proof. The first assertion follows from the definition of periods of the n-th kind, since differentials of the n-th kind represent cohomology classes in $H^1(X)$ for $n = 1, 2$ and in $H^1(X \setminus D)$ for $n = 3$. If U is a smooth, quasiprojective curve over k, then we choose a smooth compactification X and the assertion follows from the exact sequence

$$0 \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}\langle D\rangle) \stackrel{\mathrm{Res}}{\to} \bigoplus_{P \in D} \mathbb{C} \stackrel{\Sigma}{\to} H^1(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}).$$

by Theorem 13.5.3. □

Examples 13.5.7. In the elliptic curve case of section 13.4, $\omega = \frac{dX}{Y}$ is 1-form of the first kind, and $\eta = X\frac{dX}{Y}$ a 1-form of the second kind, but not of the first kind. Some periods (and quasi-periods) of this sort were computed in the two Examples 13.4.2,13.4.3. For an example of the third kind, look at $X = \mathbb{P}^1$ and $D = \{0, \infty\}$ where $\omega = \frac{dz}{z}$ is a generator with period $2\pi i$. Compare this with section 13.1 where also logarithms occur as periods. For periods of differentials of the third kind on modular and elliptic curves see [Br].

Finally, let X be a smooth, projective curve of genus g defined over $\mathbb{Q}$. Then there is a $\mathbb{Q}$-basis $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g$ of $H^1_{\mathrm{dR}}(X)$, where the ω_i are of the first kind and the η_j of the second kind. One may choose a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for $H_1^{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z})$, such that after a change of basis over $\mathbb{Q}$, we have $\int_{\alpha_j} \omega_i = \delta_{ij}$ and $\int_{\beta_j} \eta_i = \delta_{ij}$.

The period matrix is then given by a block matrix:

	$\omega_\bullet$	$\eta_\bullet$
$\alpha_\bullet$	$\mathbb{I}$	τ'
$\beta_\bullet$	τ	$\mathbb{I}$

(13.7)

where, by Riemann's bilinear relations [GH, pg. 123], τ is a matrix in the *Siegel upper half space* $\mathbb{H}_g$ of symmetric complex matrices with positive definite imaginary part. In the example of elliptic curves, section 13.4 the matrix τ is the (1×1)-matrix given by $\tau = \omega_2/\omega_1 \in \mathbb{H}$.

Chapter 14

Multiple zeta values

This chapter follows partly the Diploma thesis of Benjamin Friedrich, see [Fr]. We study in some detail the very important class of periods called multiple zeta values (MZV). These are periods of mixed Tate motives.

14.1 A ζ-value

In Prop. 11.1.4, we saw how to write $\zeta(2)$ as a Kontsevich-Zagier period:

$$\zeta(2) = \int_{0 \leq x \leq y \leq 1} \frac{dx \wedge dy}{(1-x)\, y}.$$

The problem was that this identity did not give us a valid representation of $\zeta(2)$ as a naïve period, since the pole locus of the integrand and the domain of integration are not disjoint. We show how to circumvent this difficulty, as an example of Theorem 11.2.1.

First we define (often ignoring the difference between X and X^{an}),

$$\begin{aligned}
Y &:= \mathbb{A}^2 \quad \text{with coordinates } x \text{ and } y,\\
Z &:= \{x = 1\} \cup \{y = 0\},\\
X &:= Y \setminus Z,\\
D &:= (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}) \setminus Z,\\
\triangle &:= \{(x, y) \in Y \mid x, y \in \mathbb{R},\ 0 \leq x \leq y \leq 1\} \quad \text{a triangle in } Y, \quad \text{and}\\
\omega &:= \frac{dx \wedge dy}{(1-x)\, y},
\end{aligned}$$

thus getting

$$\zeta(2) = \int_{\triangle} \omega,$$

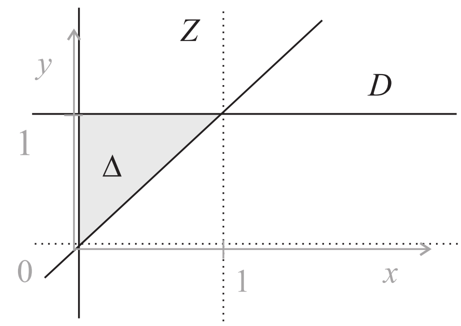


Figure 14.1: The configuration $Z, D, \triangle$

with $\omega \in \Gamma(X, \Omega^2_X)$ and $\partial\triangle \subset D \cup \{(0,0),(1,1)\}$, see Figure 14.1.

Now we blow up Y in the points $(0,0)$ and $(1,1)$ obtaining $\pi : \widetilde{Y} \to Y$. We denote the strict transform of Z by $\widetilde{Z}$, $\pi^*\omega_0$ by $\widetilde{\omega}$ and $\widetilde{Y} \setminus \widetilde{Z}$ by $\widetilde{X}$. The "strict transform" $\overline{\pi^{-1}(\triangle \setminus \{(0,0),(1,1)\})}$ will be called $\widetilde{\triangle}$ and (being $\widetilde{\mathbb{Q}}$-semi-algebraic hence triangulable — cf. Proposition 2.6.9) gives rise to a singular chain

$$\widetilde{\gamma} \in H_2^{\mathrm{sing}}(\widetilde{X}, \widetilde{D}; \mathbb{Q}).$$

Since π is an isomorphism away from the exceptional locus, this exhibits

$$\zeta(2) = \int_\triangle \omega = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}_a = \mathbb{P}$$

as a naïve period, see Figure 14.2.

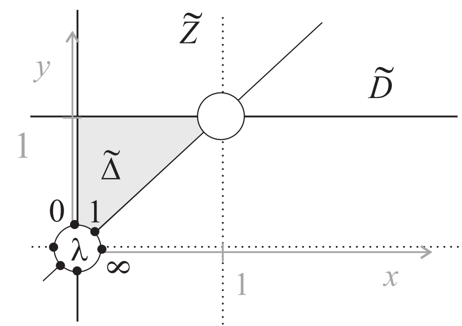


Figure 14.2: The configuration $\widetilde{Z}, \widetilde{D}, \widetilde{\triangle}$

We will conclude this example by writing out $\widetilde{\omega}$ and $\widetilde{\triangle}$ more explicitly. Note that $\widetilde{Y}$ can be described as the subvariety

$$\mathbb{A}^2_{\mathbb{Q}} \times \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \quad \text{with coordinates} \quad (\widetilde{x}, \widetilde{y}, \lambda_0 : \lambda_1, \mu_0 : \mu_1)$$

cut out by

$$\widetilde{x}\lambda_0 = \widetilde{y}\lambda_1 \quad \text{and} \quad (\widetilde{x}-1)\mu_0 = (\widetilde{y}-1)\mu_1.$$

With this choice of coordinates π takes the form

$$\begin{array}{rccc} \pi : & \widetilde{Y} & \to & Y \\ & (\widetilde{x}, \widetilde{y}, \lambda_0 : \lambda_1, \mu_0 : \mu_1) & \mapsto & (\widetilde{x}, \widetilde{y}) \end{array}$$

and we have $\widetilde{X} := \widetilde{Y} \setminus (\{\lambda_0 = 0\} \cup \{\mu_1 = 0\})$. We can embed $\widetilde{X}$ into affine space

$$\widetilde{X} \to \mathbb{A}^4_{\mathbb{Q}}$$
$$(\widetilde{x}, \widetilde{y}, \lambda_0 : \lambda_1, \mu_0 : \mu_1) \mapsto (\widetilde{x}, \widetilde{y}, \frac{\lambda_1}{\lambda_0}, \frac{\mu_0}{\mu_1})$$

and so have affine coordinates $\widetilde{x}$, $\widetilde{y}$, $\lambda := \frac{\lambda_1}{\lambda_0}$ and $\mu := \frac{\mu_0}{\mu_1}$ on $\widetilde{X}$.

Now, near $\pi^{-1}(0,0)$, the form $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = \frac{d\widetilde{x} \wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d(\lambda\widetilde{y}) \wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\lambda \wedge d\widetilde{y}}{1-\widetilde{x}},$$

while near $\pi^{-1}(1,1)$ we have

$$\widetilde{\omega} = \frac{d\widetilde{x} \wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x} \wedge d(\widetilde{y}-1)}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x} \wedge d(\mu(\widetilde{x}-1))}{(1-\widetilde{x})\,\widetilde{y}} = \frac{-d\widetilde{x} \wedge d\mu}{\widetilde{y}}.$$

The region $\widetilde{\triangle}$ is given by

$$\widetilde{\triangle} = \{(\widetilde{x}, \widetilde{y}, \lambda, \mu) \in \widetilde{X}(\mathbb{C}) \,|\, \widetilde{x}, \widetilde{y}, \lambda, \mu \in \mathbb{R},\ 0 \le \widetilde{x} \le \widetilde{y} \le 1,\ 0 \le \lambda \le 1,\ 0 \le \mu \le 1\}.$$

14.2 Definition of multiple zeta values

Recall that the Riemann ζ-function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

It has an analytic continuation to the whole complex plane with a simple pole at $s = 1$.

Definition 14.2.1. For integers $s_1, ..., s_r \ge 1$ with $s_1 \ge 2$ one defines the *multiple zeta values* (MZV)

$$\zeta(s_1, ..., s_r) := \sum_{n_1 > n_2 > ... > n_r \ge 1} n_1^{-s_1} \cdots n_r^{-s_r}.$$

The number $n = s_1 + \cdots + s_r$ is the *weight* of $\zeta(s_1, ..., s_r)$. The *length* is r.

Lemma 14.2.2. *$\zeta(s_1, ..., s_r)$ is convergent.*

Proof. Clearly, $\zeta(s_1, ..., s_r) \leq \zeta(2, 1, ..., 1)$. We use the formula

$$\sum_{n=1}^{m-1} n^{-1} \leq 1 + \log(m-1),$$

which is proved by comparing with the Riemann integral of $1/x$. Using induction, this implies that

$$\zeta(2, 1, ..., 1) \leq \sum_{n_1=1}^{\infty} n_1^{-2} \sum_{1 \leq n_r < \cdots < n_2 \leq n_1 - 1} n_2^{-1} \cdots n_r^{-1} \leq \sum_{n_1=1}^{\infty} \frac{(1+\log(n_1-1))^r}{n_1^2},$$

which is convergent. □

Lemma 14.2.3. *The positive even ζ-values are given by*

$$\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m},$$

where B_{2m} is a Bernoulli number*, defined via*

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

The first Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. All odd Bernoulli B_m numbers vanish for odd $m \geq 3$.

Proof. One uses the power series

$$x\cot(x) = 1 - 2\sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

The geometric series expansion gives

$$x\cot(x) = 1 - 2\sum_{n=1}^{\infty} \frac{\left(\frac{x}{n\pi}\right)^2}{1 - \left(\frac{x}{n\pi}\right)^2} = 1 - 2\sum_{m=1}^{\infty} \frac{x^{2m}}{\pi^{2m}} \zeta(2m).$$

On the other hand,

$$x\cot(x) = ix\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = ix\frac{e^{2ix} + 1}{e^{2ix} - 1} = ix + \frac{2ix}{e^{2ix} - 1} = ix + \sum_{m=0}^{\infty} B_m \frac{(2ix)^m}{m!}.$$

The claim then follows by comparing coefficients. □

Corollary 14.2.4. *$\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$.*

$\zeta(s)$ satisfies a functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Using it, one can show:

Corollary 14.2.5. *$\zeta(-m) = -\frac{B_{m+1}}{m+1}$ for $m \geq 1$. In particular, $\zeta(-2m) = 0$ for $m \geq 1$. These are called the trivial zeroes of $\zeta(s)$.*

Remark 14.2.6. J. Zhao has generalized the analytic continuation and the functional equation for multiple zeta values [Z2].

In the following, we want to study MZV as periods. They satisfy many relations. Already Euler knew that $\zeta(2,1) = \zeta(3)$. This can be shown as follows:

$$\begin{aligned}
\zeta(3) + \zeta(2,1) &= \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{1 \leq k < n} \frac{1}{n^2 k} = \sum_{1 \leq k \leq n} \frac{1}{n^2 k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k} \\
&= \sum_{k,n \geq 1} \frac{1}{n^2} \left(\frac{1}{k} - \frac{1}{n+k}\right) = \sum_{k,n \geq 1} \frac{1}{nk(n+k)} \\
&= \sum_{k,n \geq 1} \left(\frac{1}{n} + \frac{1}{k}\right) \frac{1}{(n+k)^2} = \sum_{k,n \geq 1} \frac{1}{n(n+k)^2} + \sum_{k,n \geq 1} \frac{1}{k(n+k)^2} \\
&= 2\zeta(2,1).
\end{aligned}$$

Other relations of this type are

$$\begin{aligned}
\zeta(2,1,1) &= \zeta(4), \\
\zeta(2,2) &= \frac{3}{4}\zeta(4), \\
\zeta(3,1) &= \frac{1}{4}\zeta(4), \\
\zeta(2)^2 &= \frac{5}{2}\zeta(4), \\
\zeta(5) &= \zeta(3,1,1) + \zeta(2,1,2) + \zeta(2,2,1) \\
\zeta(5) &= \zeta(4,1) + \zeta(3,2) + \zeta(2,3).
\end{aligned}$$

The last two relations are special cases of the sum relation:

$$\zeta(n) = \sum_{s_1 + \cdots + s_r = n} \zeta(s_1, ..., s_r).$$

It was conjectured by Zagier [Z] that the $\mathbb{Q}$-vector space Z_n of MZV of weight n has dimension d_n, where d_n is the coefficient of t^n in the power series

$$\sum_{n=0}^{\infty} d_n t^n = \frac{1}{1 - t^2 - t^3},$$

so that one has a recursion $d_n = d_{n-2} + d_{n-3}$. For example $d_4 = 1$, which can be checked using the above relations. By convention, $d_0 = 1$. This conjecture is still open, however one knows that d_n is an upper bound for $\dim_{\mathbb{Q}}(Z_n)$ [B1, Te]. It is also conjectured that the MZV of different weights are independent over $\mathbb{Q}$, so that the space of all MZV should be a direct sum

$$Z = \bigoplus_{n\geq 0} Z_n.$$

Hoffman [Hof] conjectured that all MZV containing only $s_i \in \{2,3\}$ form a basis of Z. Brown [B1] showed in 2010 that this set forms a generating set. Broadhurst et. al. [BBV] conjecture that the $\zeta(s_1, ..., s_r)$ with $s_i \in \{2,3\}$ a *Lyndon word* form a transcendence basis. A Lyndon word in two letters with an order, e.g. $2 < 3$, is a string in these two letters that is strictly smaller in lexicographic order than all of its circular shifts.

14.3 Kontsevich's integral representation

Define one-forms $\omega_0 := \frac{dt}{t}$ and $\omega_1 := \frac{dt}{1-t}$. We have seen that

$$\zeta(2) = \int_{0\leq t_1\leq t_2\leq 1} \omega_0(t_2)\omega_1(t_1).$$

In a similar way, we get that

$$\zeta(n) = \int_{0\leq t_1\leq\cdots\leq t_n\leq 1} \omega_0(t_n)\omega_0(t_{n-1})\cdots\omega_1(t_1).$$

We will now write this as

$$\zeta(n) = I(\underbrace{0\ldots 01}_{n}).$$

Definition 14.3.1. For $\epsilon_1, ..., \epsilon_n \in \{0,1\}$, we define the Kontsevich-Zagier periods

$$I(\epsilon_n \ldots \epsilon_1) := \int_{0\leq t_1\leq\cdots\leq t_n\leq 1} \omega_{\epsilon_n}(t_n)\omega_{\epsilon_{n-1}}(t_{n-1})\cdots\omega_{\epsilon_1}(t_1).$$

Note that this definition differs from parts of the literature in terms of the order, but it has the advantage that there is no sign in the following formula:

Theorem 14.3.2 (Attributed to Kontsevich by Zagier [Z])**.**

$$\zeta(s_1, ..., s_r) = I(\underbrace{0\ldots 01}_{s_1}\underbrace{0\ldots 01}_{s_2}\ldots\underbrace{0\ldots 01}_{s_r}).$$

In particular, the MZV are Kontsevich-Zagier periods.

Proof. We will define more generally

$$I(0;\epsilon_n\dots\epsilon_1;z) := \int_{0\le t_1\le\dots\le t_n\le z} \omega_{\epsilon_n}(t_n)\omega_{\epsilon_{n-1}}(t_{n-1})\cdots\omega_{\epsilon_1}(t_1)$$

for $0 \le z \le 1$. Then we show that

$$I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\dots\underbrace{0\dots01}_{s_r};z) = \sum_{n_1>n_2>\dots>n_r\ge 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

Convergence is always ok for $z < 1$, but at the end we will have it for $z = 1$ be Abel's theorem. We proceed by induction on $n = \sum_{i=1}^r s_i$. We start with $n = 1$:

$$I(0;1;z) = \int_0^z \omega_1(t) = \int_0^z \sum_{n\ge 0} t^n dt = \sum_{n\ge 0}\frac{z^{n+1}}{n+1} = \sum_{n\ge 1}\frac{z^n}{n}.$$

The induction step has two cases:

$$I(0;0\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\dots\underbrace{0\dots01}_{s_r};z) = \int_0^z \frac{dt_n}{t_n} I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\dots\underbrace{0\dots01}_{s_r};t_n)$$
$$= \int_0^z \frac{dt_n}{t_n} \sum_{n_1>n_2>\dots>n_r\ge 1} \frac{t_n^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}} = \sum_{n_1>n_2>\dots>n_r\ge 1} \frac{z^{n_1}}{n_1^{s_1+1}\cdots n_r^{s_r}}.$$

$$I(0;1\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\dots\underbrace{0\dots01}_{s_r};z) = \int_0^z \frac{dt_n}{1-t_n} I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\dots\underbrace{0\dots01}_{s_r};t_n)$$
$$= \int_0^z dt_n \sum_{m=0}^\infty t_n^m \sum_{n_1>n_2>\dots>n_r\ge 1} \frac{t_n^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}} = \sum_{m=0}^\infty \sum_{n_1>n_2>\dots>n_r\ge 1} \int_0^z dt_n \frac{t_n^{n_1+m}}{n_1^{s_1}\cdots n_r^{s_r}}$$
$$= \sum_{n_0>n_1>n_2>\dots>n_r\ge 1} \frac{z^{n_0}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

In the latter step we strictly use $z < 1$ to have convergence. It does not occur at the end of the induction, since the string starts with a 0. Convergence is proven by Abel's theorem at the end. □

14.4 Shuffle and Stuffle relations for MZV

In this section, we present a slightly more abstract viewpoint on multiple zeta values and their relations by looking only at the strings representing a MZV integral. It turns out that there are two types of multiplications on those strings, called the shuffle and stuffle products, which induce the usual multiplication on the integrals, but which have a different definition. Comparing both leads to all kind of relations between multiple zeta values. The reader may also consult [IKZ, Hof, HO, He] for more information.

A MZV can be represented via a tuple $(s_1, ..., s_r)$ of integers or a string

$$s = \underbrace{0\ldots01}_{s_1}\underbrace{0\ldots01}_{s_2}\cdots\underbrace{0\ldots01}_{s_r}$$

of 0's and 1's. There is a one-to-one correspondence between strings with a 0 on the left and a 1 on the right and all tuples $(s_1, ..., s_r)$ with all $s_i \geq 1$ and $s_1 \geq 2$. For any tuple $s = (s_1, ..., s_r)$, we denote the associated string by $\tilde{s}$. We will formalize the algebras arising from this set-up.

Definition 14.4.1 (Hoffman Algebra)**.** Let

$$\mathfrak{h} := \mathbb{Q}\langle x, y\rangle = \mathbb{Q} \oplus \mathbb{Q}x \oplus \mathbb{Q}y \oplus \mathbb{Q}xy \oplus \mathbb{Q}yx \oplus \cdots$$

be the free non-commutative graded algebra in two variables x, y (both of degree 1). There are subalgebras

$$\mathfrak{h}_1 := \mathbb{Q} \oplus \mathfrak{h}y, \quad \mathfrak{h}_0 := \mathbb{Q} \oplus x\mathfrak{h}y.$$

The generator in degree 0 is denoted by $\mathbb{I}$.

We will now identify x and y with 0 and 1, if it is convenient. For example any generator, i.e., a noncommutative word in x and y of length n can be viewed as a string $\epsilon_n \cdots \epsilon_1$ in the letters 0 and 1. With this identification, there is obviously an evaluation map such that

$$\zeta : \mathfrak{h} \longrightarrow \mathbb{R}, \quad \epsilon_n \cdots \epsilon_1 \mapsto I(\epsilon_n, ..., \epsilon_1)$$

holds on the generators of $\mathfrak{h}$. In addition, if s is the string

$$s = \epsilon_n \cdots \epsilon_1 = \underbrace{0\ldots01}_{s_1}\underbrace{0\ldots01}_{s_2}\cdots\underbrace{0\ldots01}_{s_r},$$

then we have $\zeta(s_1, ..., s_n) = \zeta(s)$ by Theorem 14.3.2.

We will now define two different multiplications

$$Ш, * : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h},$$

called shuffle and stuffle, such that ζ becomes a ring homomorphism in both cases.

Definition 14.4.2. Define the *shuffle permutations* for $r + s = n$ as

$$\Sigma_{r,s} := \{\sigma \in \Sigma_n \mid \sigma(1) < \sigma(2) < \cdots < \sigma(r), \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s)\}.$$

Define the action of $\sigma \in \Sigma_{r,s}$ on the set $\{1, 2, ..., n\}$ as

$$\sigma(x_1...x_n) := x_{\sigma^{-1}(1)}...x_{\sigma^{-1}(n)}.$$

The *shuffle product* is then defined as

$$x_1 ... x_r Ш x_{r+1} ... x_n := \sum_{\sigma \in \Sigma_{r,s}} \sigma(x_1 ... x_n).$$

Theorem 14.4.3. *The shuffle product* Ш *defines an associative, bilinear operation with unit* $\mathbb{I}$ *and hence an algebra structure on* $\mathfrak{h}$ *such that* ζ *is a ring homomorphism. It satisfies the recursive formula*

$$u Ш v = a(u' Ш v) + b(u Ш v'),$$

if $u = au'$ *and* $v = bv'$ *as strings.*

Proof. We only give a proof for the product formula $\zeta(\tilde{a} Ш \tilde{b}) = \zeta(a)\zeta(b)$; the rest is straightforward. Assume $a = (a_1, ..., a_r)$ is of weight m and $b = (b_1, ..., b_s)$ is of weight n. Then, by Fubini, the product $\zeta(a)\zeta(b)$ is an integral over the product domain

$$\Delta = \{0 \le t_1 \le \cdots \le t_m \le 1\} \times \{0 \le t_{m+1} \le \cdots \le t_{m+n} \le 1\}.$$

Ignoring subsets of measure zero,

$$\Delta = \coprod_{\sigma} \Delta_\sigma$$

indexed by all shuffles $\sigma \in \Sigma_{r,s}$, and where

$$\Delta_\sigma = \{(t_1, ..., t_{m+s} \mid 0 \le t_{\sigma^{-1}(1)} \le \cdots \le t_{\sigma^{-1}(n)} \le 1\}.$$

The proof follows then from the additivity of the integral. □

This induces binary relations as in the following examples.

Example 14.4.4. One has

$$(01)Ш(01) = 2(0101) + 4(0011)$$

and hence we have

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1).$$

In a similar way,

$$(01)Ш(001) = (010011) + 3(001011) + 9(000111) + (001101),$$

which implies that

$$\zeta(2)\zeta(3,1) = \zeta(2,3,1) + 3\zeta(3,2,1) + 9\zeta(4,1,1) + \zeta(3,1,2),$$

and

$$(01)Ш(011) = 3(01011) + 6(00111) + (01101)$$

implies that

$$\zeta(2)\zeta(2,1) = 3\zeta(2,2,1) + 6\zeta(3,1,1) + \zeta(2,1,2).$$

Definition 14.4.5. The *stuffle product*

$$* : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$$

is defined on tuples $a = (a_1, ..., a_r)$ and $b = (b_1, ..., b_s)$ as

$$\begin{aligned} a * b := &(a_1, ..., a_r, b_1, ..., b_s) + (a_1, ..., a_r + b_1, ..., b_s) \\ &+ (a_1, ..., a_{r-1}, b_1, a_r, b_2, ..., b_s) + (a_1, ..., a_{r-1} + b_1, a_r, b_2, ..., b_s) + \cdots \end{aligned}$$

The definition is made such that one has the formula $\zeta(a)\zeta(b) = \zeta(a * b)$ in the formula defining multiple zeta values.

Theorem 14.4.6. *The stuffle product $*$ defines an associative, bilinear multiplication on $\mathfrak{h}$ inducing an algebra $(\mathfrak{h}, *)$ with unit $\mathbb{I}$. One has $\zeta(a)\zeta(b) = \zeta(a*b)$ on tuples a and b. Furthermore, there is a recursion formula*

$$u * v = (a, u' * v) + (b, u * v') + (a, b, u' * v')$$

for tuples $u = (a, u')$ and $v = (b, v')$ with first entry a and b.

Proof. Again, we only give a proof for the product formula $\zeta(a)\zeta(b) = \zeta(a * b)$. Assume $a = (a_1, ..., a_r)$ is of weight m and $b = (a_{r+1}, ..., a_{r+s})$ is of weight n. The claim follows from a decomposition of the summation range:

$$\begin{aligned} &\zeta(a_1, ..., a_r)\zeta(a_{r+1}, ..., a_{r+s}) \\ &= \sum_{n_1 > n_2 > ... > n_r \geq 1} n_1^{-a_1} \cdots n_r^{-a_r} \cdot \sum_{n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_{r+1}^{-a_{r+1}} \cdots n_{r+s}^{-a_{r+s}} = \\ &= \sum_{n_1 > n_2 > ... > n_r > n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_1^{-a_1} \cdots n_r^{-a_r} n_{r+1}^{-a_{r+1}} \cdots n_{r+s}^{-a_{r+s}} \\ &+ \sum_{n_1 > n_2 > ... > n_r = n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_1^{-a_1} \cdots n_r^{-(a_r + a_{r+1})} \cdots n_{r+s}^{-a_{r+s}} \\ &+ \text{ etc.} \end{aligned}$$

where all terms in the stuffle set occur once. □

This induces again binary relations as in the following examples.

Example 14.4.7.

$$\begin{aligned} \zeta(2)\zeta(3, 1) &= \zeta(2, 3, 1) + \zeta(5, 1) + \zeta(3, 2, 1) + \zeta(3, 3) + \zeta(3, 1, 2) \\ \zeta(2)^2 &= 2\zeta(2, 2) + \zeta(4). \end{aligned}$$

More generally,

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(a + b) + \zeta(b, a), \text{ for } a, b \geq 2..$$

Since we have $\zeta(\tilde{a}Ш\tilde{b}) = \zeta(a * b)$ we can define the unary double-shuffle relation as

$$\zeta(\tilde{a}Ш\tilde{b} - a * b) = 0.$$

Example 14.4.8. We have $\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$ using shuffle and $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$ using the stuffle. Therefore one has

$$4\zeta(3,1) = \zeta(4).$$

In the literature [Hof, HO, IKZ, He] more relations were found, e.g., a modified version of this relation, called the *regularized double-shuffle relation*:

$$\zeta\left(\sum_{b\in(1)*a} b - \sum_{\tilde{c}\in(1)Ш\tilde{a}} c\right) = 0.$$

Example 14.4.9. Let $a = \widetilde{(2)} = (01)$. Then $(1)Ш(01) = (101) + 2(011)$ and $(1) * (2) = (1,2) + (3) + (2,1)$. Therefore, the corresponding relation is

$$\zeta(1,2) + 2\zeta(2,1) = \zeta(1,2) + \zeta(3) + \zeta(2,1), \text{ hence}$$

$$\zeta(2,1) = \zeta(3).$$

Like in this example, all non-convergent contributions cancel in the relation. It is conjectured that the regularized double-shuffle relation generates all relations among MZV. There are more relations: the sum theorem (mentioned above), the duality theorem, the derivation theorem and Ohno's theorem, which implies the first three [HO, He].

We will finish this subsection with some formulas mentioned by Brown [B1], mainly due to Broadhurst and Zagier:

$$\zeta(\underbrace{3,1,\ldots,3,1}_{2n}) = \frac{1}{2n+1}\zeta(\underbrace{2,2,\ldots,2}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

$$\zeta(\underbrace{2,\ldots,2}_{b},3,\underbrace{2,\ldots,2}_{a}) = \sum_{m+r=a+b+1} c_{m,r,a,b}\frac{\pi^{2m}}{(4m+1)!}\zeta(2r+1),$$

where $c_{m,r,a,b} = 2(-1)^r\left(\binom{2r}{2a+2} - \left(1-2^{-2r}\right)\binom{2r}{2b+1}\right) \in \mathbb{Q}$ $(m \geq 0, r \geq 1)$.

In the next section, we relate multiple zeta values to Nori motives and also to mixed Tate motives. This give a more conceptual embedding of such periods in the sense of Chapter 10, see in particular Section 10.4.

14.5 Multiple zeta values and moduli space of marked curves

In this short section, we indicate how one can relate multiple zeta values to Nori motives and to mixed Tate motives.

Multiple zeta values can also be seen as periods of certain cohomology groups of moduli spaces in such a way that they appear naturally as Nori motives. Recall that the moduli space $M_{0,n}$ of smooth rational curves with n marked points can be compactified to the space $\overline{M}_{0,n}$ of stable curves with n markings [K2]. Manin and Goncharov [GM] have observed the following.

Theorem 14.5.1. *For each convergent multiple zeta value $p = \zeta(s_1, ..., s_r)$ of weight $n = s_1 + ... + s_r$, one can construct divisors A, B in $\overline{M}_{0,n+3}$ such that p is a period of the cohomology group $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$.*

The group $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$ defines of course immediately a motive in Nori's sense.

Example 14.5.2. The fundamental example is $\zeta(2)$, which we already described in section 14.1. Here $\overline{M}_{0,5}$ is a compactification of

$$M_{0,5} = (\mathbb{P} \setminus \{0, 1, \infty\})^2 \setminus \text{ diagonal},$$

since $\overline{M}_{0,5}$ is the blow up $(0, 0)$, $(1, 1)$ and (∞, ∞) in $\mathbb{P}^1 \times \mathbb{P}^1$. This realizes $\zeta(2)$ as the integral

$$\zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_2} \frac{dt_2}{t_2} .$$

We leave it to the reader to make the divisors A and B explicit.

This viewpoint was very much refined in Brown's thesis [B3]. Recent related research for higher polylogarithms and elliptic polylogarithms can be found in [B4].

Levine [L2] has defined an abelian category as a full subcategory of the triangulated category of geometrical motives, see Chapter 10 for the notion of geometric motives. It is a full subcategory generated by the Tate objects $\mathbb{Q}(n)$. There is also a variant, called mixed Tate motives over $\mathbb{Z}$, see [Te, DG, B1]. The Theorem above implies:

Theorem 14.5.3 (Brown)**.** *Multiple zeta values together with $(2\pi i)^n$ are precisely all the periods of all mixed Tate motives over $\mathbb{Z}$.*

Proof. This is a result of Brown, see [B1, D3]. □

14.6 Multiple Polylogarithms

In this section, we study a variation of cohomology groups in a 2-parameter family of varieties over $\mathbb{Q}$, the so-called *double logarithm variation*, for which multiple polylogarithms appear as coefficients. This viewpoint gives more examples of Kontsevich-Zagier periods occuring as cohomological periods of canonical cohomology groups at particular values of the parameters. The degeneration of the parameters specializes such periods to simpler ones.

First define the *hyperlogarithm* as the iterated integral

$$\mathrm{I}_n(a_1,\ldots,a_n) := \int_{0\le t_1\le\cdots\le t_n\le 1} \frac{dt_1}{t_1-a_1}\wedge\cdots\wedge\frac{dt_n}{t_n-a_n}$$

with $a_1,\ldots,a_n\in\mathbb{C}$ (cf. [Z1, p. 168]). Note that, the order of terms here is different from the previous order, also in the infinite sum below.

These integrals specialize to the *multiple polylogarithm* (cf. [*loc. cit.*])

$$\mathrm{Li}_{m_1,\ldots,m_n}\left(\frac{a_2}{a_1},\cdots,\frac{a_n}{a_{n-1}},\frac{1}{a_n}\right) := (-1)^n\,\mathrm{I}_{\sum m_n}(a_1,\underbrace{0,\ldots,0}_{m_1-1},\ldots,a_n,\underbrace{0,\ldots,0}_{m_n-1}),$$

which is convergent if $1<|a_1|<\cdots<|a_n|$ (cf. [G3, 2.3, p. 9]). Alternatively, we can describe the multiple polylogarithm as a power series (cf. [G3, Thm. 2.2, p. 9])

$$\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \sum_{0<k_1<\cdots<k_n} \frac{x_1^{k_1}\cdots x_n^{k_n}}{k_1^{m_1}\cdots k_n^{m_n}} \quad\text{for}\quad |x_i|<1. \tag{14.1}$$

Of special interest to us will be the *dilogarithm* $\mathrm{Li}_2(x) = \sum_{k>0}\frac{x^k}{k^2}$ and the *double logarithm* $\mathrm{Li}_{1,1}(x,y) = \sum_{0<k<l}\frac{x^k y^l}{kl}$.

Remark 14.6.1. At first, the functions $\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ only make sense for $|x_i|<1$, but they can be analytically continued to multivalued meromorphic functions on $\mathbb{C}^n$ (cf. [Z1, p. 2]), for example $\mathrm{Li}_1(x) = -\log(1-x)$. One has $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$.

14.6.1 The Configuration

Let us consider the configuration

$$\begin{aligned}
&Y := \mathbb{A}^2 \quad \text{with coordinates } x \text{ and } y,\\
&Z := \{x=a\}\cup\{y=b\} \quad\text{with}\quad a\neq 0,1 \quad\text{and}\quad b\neq 0,1\\
&X := Y\setminus Z\\
&D := (\{x=0\}\cup\{y=1\}\cup\{x=y\})\setminus Z,
\end{aligned}$$

see Figure 14.3.

We denote the irreducible components of the divisor D as follows:

$$\begin{aligned} D_1 &:= \{x = 0\} \setminus \{(0, b)\}, \\ D_2 &:= \{y = 1\} \setminus \{(a, 1)\}, \quad \text{and} \\ D_3 &:= \{x = y\} \setminus \{(a, a), (b, b)\}. \end{aligned}$$

By projecting from Y onto the y- or x-axis, we get isomorphisms for the associated complex analytic spaces

$$D_1^{\text{an}} \cong \mathbb{C} \setminus \{b\}, \quad D_2^{\text{an}} \cong \mathbb{C} \setminus \{a\}, \quad \text{and} \quad D_3^{\text{an}} \cong \mathbb{C} \setminus \{a, b\}.$$

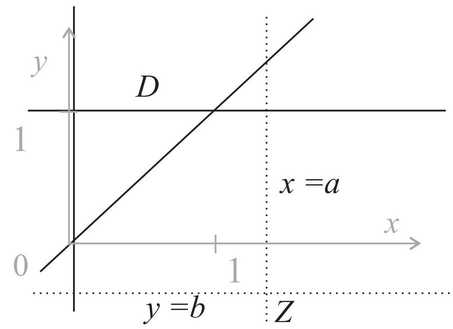


Figure 14.3: The algebraic pair (X, D)

14.6.2 Singular Homology

We can easily give generators for the second singular homology of the pair (X, D), see Figure 14.4.

- Let $\alpha : [0, 1] \to \mathbb{C}$ be a smooth path, which does not meet a or b. We define a "triangle"

$$\triangle := \{(\alpha(s), \alpha(t)) \mid 0 \le s \le t \le 1\}.$$

- Consider the closed curve in $\mathbb{C}$

$$C_b := \left\{ \frac{a}{b + \epsilon e^{2\pi i s}} \mid s \in [0, 1] \right\},$$

  which divides $\mathbb{C}$ into two regions: an inner one containing $\frac{a}{b}$ and an outer one. We can choose $\epsilon > 0$ small enough such that C_b separates $\frac{a}{b}$ from 0 to 1, i.e., such that 0 and 1 are contained in the outer region. This allows us to find a smooth path $\beta : [0,1] \to \mathbb{C}$ from 0 to 1 not meeting C_b. We define a "slanted tube"

$$S_b := \left\{ \left(\beta(t) \cdot (b + \epsilon e^{2\pi i s}), b + \epsilon e^{2\pi i s}\right) \mid s, t \in [0,1] \right\}$$

which winds around $\{y = b\}$ and whose boundary components are supported on D_1 (corresponding to $t = 0$) and D_3 (corresponding to $t = 1$). The special choice of β guarantees $S_b \cap Z(\mathbb{C}) = \emptyset$.

- Similarly, we choose $\epsilon > 0$ such that the closed curve

$$C_a := \left\{ \frac{b-1}{a - 1 - \epsilon e^{2\pi i s}} \mid s \in [0,1] \right\}$$

separates $\frac{b-1}{a-1}$ form 0 and 1. Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth path from 0 to 1 which does not meet C_a. We have a "slanted tube"

$$S_a := \left\{ \left(a + \epsilon e^{2\pi i s}, 1 + \gamma(t) \cdot (a + \epsilon e^{2\pi i s} - 1)\right) \mid s, t \in [0,1] \right\}$$

winding around $\{x = a\}$ with boundary supported on D_2 and D_3.

- Finally, we have a torus

$$T := \{(a + \epsilon e^{2\pi i s}, b + \epsilon e^{2\pi i t}) \mid s, t \in [0,1]\}.$$

The 2-form $ds \wedge dt$ defines an orientation on the unit square $[0,1]^2 = \{(s,t) \mid s, t \in [0,1]\}$. Hence the manifolds with boundary $\triangle$, S_b, S_a, T inherit an orientation, and since they can be triangulated, they give rise to smooth singular chains. By abuse of notation we will also write $\triangle$, S_b, S_a, T for these smooth singular chains. The homology classes of $\triangle$, S_b, S_a and T will be denoted by γ_0, γ_1, γ_2 and γ_3, respectively.

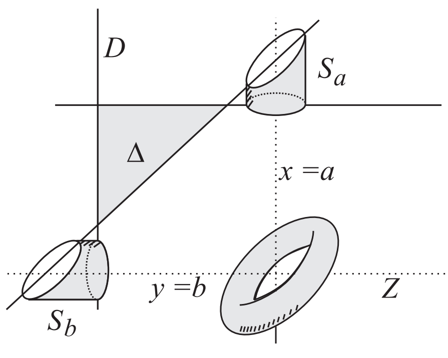


Figure 14.4: Generators of $H_2^{\mathrm{sing}}(X, D; \mathbb{Q})$

An inspection of the long exact sequence in singular homology will reveal that $\gamma_0, \ldots, \gamma_3$ form a system generators (see the following proof)

$$H_2^{\text{sing}}(D, \mathbb{Q}) \longrightarrow H_2^{\text{sing}}(X, \mathbb{Q}) \longrightarrow H_2^{\text{sing}}(X, D, \mathbb{Q}) \longrightarrow$$
$$H_1^{\text{sing}}(D, \mathbb{Q}) \xrightarrow{i_1} H_1^{\text{sing}}(X, \mathbb{Q}) .$$

Proposition 14.6.2. *With notation as above, we have for the second singular homology of the pair* (X, D)

$$H_2^{\text{sing}}(X, D; \mathbb{Q}) = \mathbb{Q}\,\gamma_0 \oplus \mathbb{Q}\,\gamma_1 \oplus \mathbb{Q}\,\gamma_2 \oplus \mathbb{Q}\,\gamma_3.$$

Proof. For $c := a$ and $c := b$, the inclusion of the circle $\{c + \epsilon e^{2\pi i s} \mid s \in [0, 1]\}$ into $\mathbb{C} \setminus \{c\}$ is a homotopy equivalence, hence the product map $T \hookrightarrow X(\mathbb{C})$ is also a homotopy equivalence. This shows

$$H_2^{\text{sing}}(X, \mathbb{Q}) = \mathbb{Q}\,T,$$

while $H_1^{\text{sing}}(X, \mathbb{Q})$ has rank two with generators

- one loop winding counterclockwise around $\{x = a\}$ once, but not around $\{y = b\}$, thus being homologous to both $\partial S_a \cap D_2(\mathbb{C})$ and $-\partial S_a \cap D_3(\mathbb{C})$, and
- another loop winding counterclockwise around $\{y = b\}$ once, but not around $\{x = a\}$, thus being homologous to $\partial S_b \cap D_1(\mathbb{C})$ and $-\partial S_b \cap D_3(\mathbb{C})$.

In order to compute the Betti-numbers b_i of D, we use the spectral sequence for the closed covering $\{D_i\}$

$$E_2^{p,q} := \begin{array}{cccccc} \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \bigoplus_{i=1}^3 H^1_{\text{dR}}(D_i, \mathbb{C}) & 0 & 0 & \cdots \\ \cdots & 0 & \text{Ker}\delta & \text{Coker}\delta & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{array} \quad \Rightarrow \quad E_\infty^n := H^n_{\text{dR}}(D, \mathbb{C}),$$

where

$$\delta : \bigoplus_{i=1}^{3} H^0_{\text{dR}}(D_i, \mathbb{C}) \longrightarrow \bigoplus_{i<j} H^0_{\text{dR}}(D_{ij}, \mathbb{C}).$$

Note that this spectral sequence degenerates. Since D is connected, we have $b_0 = 1$, i.e.,

$$1 = b_0 = \text{rank}_{\mathbb{C}} E^0_\infty = \text{rank}_{\mathbb{C}} E_2^{0,0} = \text{rank}_{\mathbb{C}}\text{Ker}\delta.$$

Hence

$$\begin{aligned} \text{rank}_{\mathbb{C}}\text{Coker}\delta &= \text{rank}_{\mathbb{C}}\text{codomain}\,\delta - \text{rank}_{\mathbb{C}}\text{domain}\,\delta + \text{rank}_{\mathbb{C}}\text{Ker}\delta \\ &= (1 + 1 + 1) - (1 + 1 + 1) + 1 = 1, \end{aligned}$$

and so

$$\begin{aligned} b_1 &= \mathrm{rank}_{\mathbb{C}} E_\infty^1 = \mathrm{rank}_{\mathbb{C}} E_2^{1,0} + \mathrm{rank}_{\mathbb{C}} E_2^{0,1} \\ &= \sum_{i=1}^{3} \mathrm{rank}_{\mathbb{C}} H^1_{\mathrm{dR}}(D_i, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} \mathrm{Coker}\delta \\ &= \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{b\}, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{a\}, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{a, b\}, \mathbb{C}) + 1 \\ &= (1 + 1 + 2) + 1 = 5. \end{aligned}$$

We can easily specify generators of $H_1^{\mathrm{sing}}(D, \mathbb{Q})$ as follows

$$H_1^{\mathrm{sing}}(D,\mathbb{Q}) = \mathbb{Q}\cdot(\partial S_b \cap D_1) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_2) \oplus \mathbb{Q}\cdot(\partial S_b \cap D_3) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_3) \oplus \mathbb{Q}\cdot\partial\triangle.$$

Clearly $b_2 = \mathrm{rank}_{\mathbb{C}} H_2^{\mathrm{sing}}(D, \mathbb{Q}) = 0$. Now we can compute $\mathrm{Ker} i_1$ and obtain

$$\mathrm{Ker} i_1 = \mathbb{Q}\cdot\partial\triangle \oplus \mathbb{Q}\cdot(\partial S_b \cap D_1(\mathbb{C}) + \partial S_b \cap D_3(\mathbb{C})) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_2(\mathbb{C}) + \partial S_a \cap D_3(\mathbb{C})).$$

This shows finally

$$\mathrm{rank}_{\mathbb{Q}} H_2^{\mathrm{sing}}(X, D; \mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}} H_2^{\mathrm{sing}}(X, \mathbb{Q}) + \mathrm{rank}_{\mathbb{Q}} \mathrm{Ker} i_1 = 1 + 3 = 4.$$

From these explicit calculations we also derive the linear independence of $\gamma_0 = [\triangle]$, $\gamma_1 = [S_b]$, $\gamma_2 = [S_a]$, $\gamma_3 = [T]$ and Proposition 14.6.2 is proved. □

14.6.3 Smooth Singular Homology

Recall the definition of smooth singular cohomology (cf. Theorem 2.2.5). With the various sign conventions made so far, the boundary map $\delta : \mathrm{C}_2^\infty(X, D; \mathbb{Q}) \to \mathrm{C}_1^\infty(X, D; \mathbb{Q})$ is given by

$$\delta : \mathrm{C}_2^\infty(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{3} \mathrm{C}_1^\infty(D_i, \mathbb{Q}) \oplus \bigoplus_{i<j} \mathrm{C}_0^\infty(D_{ij}, \mathbb{Q}) \to \mathrm{C}_1^\infty(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{3} (D_i, \mathbb{Q})$$

$$(\underset{\emptyset}{\sigma}, \underset{1}{\sigma_1}, \underset{2}{\sigma_2}, \underset{3}{\sigma_3}, \underset{12}{\sigma_{12}}, \underset{13}{\sigma_{13}}, \underset{23}{\sigma_{23}}) \mapsto$$

$$(\partial\sigma + \sigma_1 \underset{\emptyset}{+} \sigma_2 + \sigma_3, -\partial\sigma_1 + \underset{1}{\sigma_{12}} + \sigma_{13}, -\partial\sigma_2 - \underset{2}{\sigma_{12}} + \sigma_{23}, -\partial\sigma_3 - \underset{3}{\sigma_{13}} - \sigma_{23}).$$

Thus the following elements of $\mathrm{C}_2^\infty(X, D; \mathbb{Q})$ are cycles

- $\Gamma_0 := (\underset{\emptyset}{\triangle}, -\partial\triangle \underset{1}{\cap} D_1(\mathbb{C}), -\partial\triangle \underset{2}{\cap} D_2(\mathbb{C}), -\partial\triangle \underset{3}{\cap} D_3(\mathbb{C}), \underset{12}{D_{12}(\mathbb{C})}, \underset{13}{-D_{13}(\mathbb{C})}, \underset{23}{D_{23}(\mathbb{C})}),$
- $\Gamma_1 := (\underset{\emptyset}{S_b}, -\partial S_b \underset{1}{\cap} D_1(\mathbb{C}), \underset{2}{0}, -\partial S_b \underset{3}{\cap} D_3(\mathbb{C}), \underset{12}{0}, \underset{13}{0}, \underset{23}{0}),$
- $\Gamma_2 := (\underset{\emptyset}{S_a}, \underset{1}{0}, -\partial S_a \underset{2}{\cap} D_2(\mathbb{C}), \underset{3}{0}, -\partial S_a \underset{12}{\cap} D_3(\mathbb{C}), \underset{13}{0}, \underset{23}{0})$ and
- $\Gamma_3 := (\underset{\emptyset}{T}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0}).$

Under the isomorphism $H_2^\infty(X, D; \mathbb{Q}) \xrightarrow{\sim} H_2^{\mathrm{sing}}(X, D; \mathbb{Q})$ the classes of these cycles $[\Gamma_0]$, $[\Gamma_1]$, $[\Gamma_2]$, $[\Gamma_3]$ are mapped to γ_0, γ_1, γ_2, γ_3, respectively.

14.6.4 Algebraic de Rham cohomology and period matrix of (X, D)

Recall the definition of the complex $\widetilde{\Omega}^{\bullet}_{X,D}$. We consider

$$\Gamma(X, \widetilde{\Omega}^2_{X,D}) = \Gamma(X, \Omega^2_X) \oplus \bigoplus_{i=1}^{3} \Gamma(D_i, \Omega^1_{D_i}) \oplus \bigoplus_{i<j} \Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$$

together with the following cycles of $\Gamma(X, \widetilde{\Omega}^2_{X,D})$

- $\omega_0 := (\underset{\emptyset}{\frac{dx\wedge dy}{(x-a)(y-b)}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,
- $\omega_1 := (\underset{\emptyset}{0}, \underset{1}{\frac{-dy}{y-b}}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,
- $\omega_2 := (\underset{\emptyset}{0}, \underset{1}{0}, \underset{2}{\frac{-dx}{x-a}}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$, and
- $\omega_3 := (\underset{\emptyset}{0}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{1})$.

By computing the (transposed) period matrix $P_{ij} := \langle \Gamma_j, \omega_i \rangle$ and checking its non-degeneracy, we will show that $\omega_0, \ldots, \omega_3$ span $H^2_{\mathrm{dR}}(X, D)$.

Proposition 14.6.3. *Let X and D be as above. Then the second algebraic de Rham cohomology group $H^2_{\mathrm{dR}}(X, D)$ of the pair (X, D) is generated by the cycles $\omega_0, \ldots, \omega_3$ considered above.*

Proof. Easy calculations give us the (transposed) period matrix P:

	Γ_0	Γ_1	Γ_2	Γ_3
ω_0	1	0	0	0
ω_1	$\mathrm{Li}_1(\frac{1}{b})$	$2\pi i$	0	0
ω_2	$\mathrm{Li}_1(\frac{1}{a})$	0	$2\pi i$	0
ω_3	?	$2\pi i \mathrm{Li}_1(\frac{b}{a})$	$2\pi i \log\left(\frac{a-b}{1-b}\right)$	$(2\pi i)^2$.

For example,

- $P_{1,1} = \langle \Gamma_1, \omega_1 \rangle = \int_{-\partial S_b \cap D_1(\mathbb{C})} \frac{-dy}{y-b}$
  $= \int_{|y-b|=\epsilon} \frac{dy}{y-b}$
  $= 2\pi i$,
- $P_{3,3} = \langle \Gamma_3, \omega_3 \rangle = \int_T \frac{dx}{x-a} \wedge \frac{dy}{y-b}$
  $= \left(\int_{|x-a|=\epsilon} \frac{dx}{x-a}\right) \cdot \left(\int_{|y-b|=\epsilon} \frac{dy}{y-b}\right)$ by Fubini
  $= (2\pi i)^2$,

- $P_{1,0} = \langle \Gamma_0, \omega_1 \rangle = \int_{-\partial\triangle \cap D_1(\mathbb{C})} \frac{-dy}{y-b}$
$= \int_0^1 \frac{-\alpha(t)}{\alpha(t)-b}$
$= -[\log(\alpha(t)) - b]_0^1$
$= -\log\left(\frac{1-b}{-b}\right)$
$= -\log\left(1 - \frac{1}{b}\right)$
$= \mathrm{Li}_1\left(\frac{1}{b}\right), \quad \text{and}$

- $P_{3,1} = \langle \Gamma_1, \omega_3 \rangle = \int_{S_b} \frac{dx}{x-a} \wedge \frac{dy}{y-b}$
$= \int_{[0,1]^2} \frac{d\left(\beta(t)\cdot\left(b+\epsilon e^{2\pi i s}\right)\right)}{\beta(t)\cdot(b+\epsilon e^{2\pi i s})-a} \wedge \frac{d\left(b+\epsilon e^{2\pi i s}\right)}{\epsilon e^{2\pi i s}}$
$= \int_{[0,1]^2} \frac{b+\epsilon e^{2\pi i s}}{\beta(t)\cdot(b+\epsilon e^{2\pi i s})-a} d\beta(t) \wedge 2\pi i ds$
$= -\int_0^1 \left[\frac{a\log\left(\beta(t)\cdot\left(b+\epsilon e^{2\pi i s}\right)-a\right)-2\pi i\beta(t)bs}{\beta(t)\cdot(-\beta(t)b+a)} \right]_0^1 d\beta(t)$
$= -2\pi i \int_0^1 \frac{d\beta(t)}{\beta(t)-\frac{a}{b}}$
$= -2\pi i \left[\log\left(\beta(t) - \frac{a}{b}\right)\right]_0^1$
$= -2\pi i \log\left(\frac{1-\frac{a}{b}}{-\frac{a}{b}}\right)$
$= -2\pi i \log\left(1 - \frac{a}{b}\right)$
$= 2\pi i \,\mathrm{Li}_1\left(\frac{b}{a}\right).$

Obviously the period matrix P is non-degenerate and so Proposition 14.6.3 is proved. □

What about the entry $P_{3,0}$?

Proposition 14.6.4. $P_{3,0} = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right)$.

For the proof we need to show that $\langle \Gamma_0, \omega_3 \rangle = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right)$, where $\mathrm{Li}_{1,1}(x, y)$ is an analytic continuation of the double logarithm defined for $|x|, |y| < 1$ in Subsection 14.6.

Lemma 14.6.5. *The integrals*

$$\mathrm{I}_2^{\alpha}\left(\frac{1}{xy}, \frac{1}{y}\right) = \int_{0 \le s \le t \le 1} \frac{d\alpha(s)}{\alpha(s) - \frac{1}{xy}} \wedge \frac{d\alpha(t)}{\alpha(t) - \frac{1}{y}}$$

with $\alpha : [0,1] \to \mathbb{C}$ a smooth path from 0 to 1, and $\frac{1}{xy}, \frac{1}{b} \in \mathbb{C} \setminus \mathrm{Im}\alpha$, defined above on page 276, provide a genuine analytic continuation of $\mathrm{Li}_{1,1}(x, y)$ *to a multivalued function which is defined on* $\{(x, y) \in \mathbb{C}^2 \mid x, y \neq 0, xy \neq 1, y \neq 1\}$.

Proof. We describe this analytic continuation in detail. Our approach is similar to the one taken in [G3, 2.3, p. 9], but differs from that in [Z2a, p. 7].

Let $B^{\mathrm{an}} := (\mathbb{C}\setminus\{0,1\})^2$ be the parameter space and choose a point $(a,b) \in B^{\mathrm{an}}$. For $\epsilon > 0$ we denote by $D_\epsilon(a,b)$ the polycylinder

$$D_\epsilon(a,b) := \{(a,b) \in B^{\mathrm{an}} \,|\, |a'-a| < \epsilon, |b'-b| < \epsilon\}.$$

If $\alpha : [0,1] \to \mathbb{C}$ is a smooth path from 0 to 1 passing through neither a nor b, then there exists an $\epsilon > 0$ such that $\mathrm{Im}\alpha$ does not meet any of the discs

$$D_{2\epsilon}(a) := \{a' \in \mathbb{C} \,|\, |a'-a| < 2\epsilon\}, \quad \text{and}$$
$$D_{2\epsilon}(b) := \{b' \in \mathbb{C} \,|\, |b'-b| < 2\epsilon\}.$$

Hence the power series (14.2) below

$$\begin{aligned}\frac{1}{\alpha(s)-a'}\frac{1}{\alpha(t)-b'} &= \frac{1}{\alpha(s)-a}\frac{1}{1-\frac{a'-a}{\alpha(s)-a}}\frac{1}{\alpha(t)-b}\frac{1}{1-\frac{b'-b}{\alpha(t)-b}} \\ &= \sum_{k,l=0}^{\infty} \underbrace{\frac{1}{(\alpha(s)-a)^{k+1}(\alpha(t)-b)^{l+1}}}_{c_{k.l}}(a'-a)^k(b'-b)^l \quad (14.2)\end{aligned}$$

has coefficients $c_{k,l}$ satisfying

$$|c_{k,l}| < \left(\frac{1}{2\epsilon}\right)^{k+l+2}.$$

In particular, (14.2) converges uniformly for $(a',b') \in D_\epsilon(a,b)$ and we see that the integral

$$\begin{aligned}\mathrm{I}_2^\alpha(a',b') &:= \int_{0\le s\le t\le 1} \frac{d\alpha(s)}{\alpha(s)-a'} \wedge \frac{d\alpha(t)}{\alpha(t)-b'} \\ &= \sum_{k,l=0}\left(\int_{0\le s\le t\le 1} \frac{d\alpha(s)}{(\alpha(s)-a)^{k+1}} \wedge \frac{d\alpha(t)}{(\alpha(t)-b)^{l+1}}\right)(a'-a)^k(b'-b)^l\end{aligned}$$

defines an analytic function of $D_\epsilon(a,b)$. In fact, by the same argument we get an analytic function I_2^α on all of $(\mathbb{C}\setminus \mathrm{Im}\alpha)^2$.

Now let $\alpha_r : [0,1] \to \mathbb{C}\setminus(D_{2\epsilon}(a) \cup D_{2\epsilon}(b))$ with $r \in [0,1]$ be a smooth homotopy of paths from 0 to 1, i.e. $\alpha_r(0) = 0$ and $\alpha_r(1) = 1$ for all $r \in [0,1]$. We show

$$\mathrm{I}_2^{\alpha_0}(a',b') = \mathrm{I}_2^{\alpha_1}(a',b') \quad \text{for all} \quad (a',b') \in D_\epsilon(a,b).$$

Define a subset $\Gamma \subset \mathbb{C}^2$

$$\Gamma := \{(\alpha_r(s), \alpha_r(t)) \,|\, 0 \le s \le t \le 1, r \in [0,1]\}.$$

The boundary of Γ is built out of five components (each being a manifold with boundary)

- $\Gamma_{s=0} := \{(0, \alpha_r(t)) \,|\, r,t \in [0,1]\}$,

- $\Gamma_{s=t} := \{(\alpha_r(s), \alpha_r(s)) \mid r, s \in [0,1]\}$,
- $\Gamma_{t=1} := \{(\alpha_r(s), 1) \mid r, s \in [0,1]\}$,
- $\Gamma_{r=0} := \{(\alpha_0(s), \alpha_0(t) \mid 0 \le s \le t \le 1\}$,
- $\Gamma_{r=1} := \{(\alpha_1(s), \alpha_1(t) \mid 0 \le s \le t \le 1\}$.

Let $(a', b') \in D_\epsilon(a, b)$. Since the restriction of $\frac{dx}{x-a'} \wedge \frac{dy}{y-b'}$ to $\Gamma_{s=0}$, $\Gamma_{s=t}$ and $\Gamma_{t=1}$ is zero, we get by Stokes' theorem

$$\begin{aligned} 0 = \int_\Gamma 0 &= \int_\Gamma d\frac{dx}{x-a'} \wedge \frac{dy}{y-b'} \\ &= \int_{\partial\Gamma} \frac{dx}{x-a'} \wedge \frac{dy}{y-b'} \\ &= \int_{\Gamma_{r=1}-\Gamma_{r=0}} \frac{dx}{x-a'} \frac{dy}{y-b'} \\ &= \mathrm{I}_2^{\alpha_1}(a', b') - \mathrm{I}_2^{\alpha_0}(a', b'). \end{aligned}$$

For each pair of smooth paths $\alpha_0, \alpha_1 : [0,1] \to \mathbb{C}$ from 0 to 1, we can find a homotopy α_r relative to $\{0,1\}$ between both paths. Since $\mathrm{Im}\alpha_r$ is compact, we also find a point $(a, b) \in B^{\mathrm{an}} = (\mathbb{C} \setminus \{0,1\})^2$ and an $\epsilon > 0$ such that $\mathrm{Im}\alpha_r$ does not meet $D_{2\epsilon}(a, b)$ or $D_{2\epsilon}(a, b)$. Then $\mathrm{I}_2^{\alpha_0}$ and $\mathrm{I}_2^{\alpha_1}$ must agree on $D_\epsilon(a, b)$. By the identity principle for analytic functions of several complex variables [Gun, A, 3, p. 5], the functions $\mathrm{I}_2^{\alpha}(a', b')$, each defined on $(\mathbb{C} \setminus \mathrm{Im}\alpha)^2$, patch together to give a multivalued analytic function on $B^{\mathrm{an}} = (\mathbb{C} \setminus \{0,1\})^2$.

Now assume $1 < |b| < |a|$, then we can take $\alpha = \mathrm{id} : [0,1] \to \mathbb{C}$, $s \mapsto s$, and obtain

$$\mathrm{I}_2^{\mathrm{id}}(a, b) = \mathrm{I}_2(a, b) = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{y}\right),$$

where $\mathrm{Li}_{1,1}(x, y)$ is the double logarithm defined for $|x|, |y| < 1$ in Subsection 14.6. Thus we have proved the lemma. □

Definition 14.6.6 (Double logarithm)**.** We call the analytic continuation from Lemma 14.6.5 the *double logarithm* as well and continue to use the notation $\mathrm{Li}_{1,1}(x, y)$.

The period matrix P is thus given by:

	Γ_0	Γ_1	Γ_2	Γ_3
ω_0	1	0	0	0
ω_1	$\mathrm{Li}_1(\frac{1}{b})$	$2\pi i$	0	0
ω_2	$\mathrm{Li}_1(\frac{1}{a})$	0	$2\pi i$	0
ω_3	$\mathrm{Li}_{1,1}(\frac{b}{a}, \frac{1}{b})$	$2\pi i \mathrm{Li}_1(\frac{b}{a})$	$2\pi i \log\left(\frac{a-b}{1-b}\right)$	$(2\pi i)^2$.

14.6.5 Varying parameters a and b

The homology group $H_2^{\text{sing}}(X, D; \mathbb{Q})$ of the pair (X, D) carries a $\mathbb{Q}$-MHS $(W_\bullet, F^\bullet)$. The weight filtration is given in terms of the $\{\gamma_j\}$:

$$W_p H_2^{\text{sing}}(X, D; \mathbb{Q}) = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}\gamma_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p = -2, -1 \\ \mathbb{Q}\gamma_0 \oplus \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p \geq 0, \end{cases}$$

The Hodge filtration is given in terms of the $\{\omega_i^*\}$:

$$F^p H_2^{\text{sing}}(X, D; \mathbb{Q}) = \begin{cases} \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* \oplus \mathbb{C}\omega_3^* & \text{for} \quad p \leq -2 \\ \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* & \text{for} \quad p = -1 \\ \mathbb{C}\omega_0^* & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1. \end{cases}$$

This $\mathbb{Q}$-MHS resembles very much the $\mathbb{Q}$-MHS considered in [G1, 2.2, p. 620] and [Z2a, 3.2, p. 6]. Nevertheless a few differences are note-worthy:

- Goncharov defines the weight filtration slightly different, for example his lowest weight is -6.
- The entry $P_{3,2} = 2\pi i \log\left(\frac{a-b}{1-b}\right)$ of the period matrix P differs by $(2\pi i)^2$, or put differently, the basis $\{\gamma_0, \gamma_1, \gamma_2 - \gamma_3, \gamma_3\}$ is used.

Up to now, the parameters a and b of the configuration (X, D) have been fixed. By varying a and b, we obtain a family of configurations: Equip $\mathbb{A}^2_{\mathbb{C}}$ with coordinates a and b and let

$$B := \mathbb{A}^2_{\mathbb{C}} \setminus (\{a = 0\} \cup \{a = 1\} \cup \{b = 0\} \cup \{b = 1\})$$

be the parameter space. Take another copy of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates x and y and define total spaces

$$\underline{X} := \underset{(a,b,x,y)}{(B \times \mathbb{A}^2_{\mathbb{C}})} \setminus (\{x = a\} \cup \{y = b\}), \quad \text{and}$$

$$\underline{D} := \text{“}B \times D\text{”} = \underline{X} \cap (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}).$$

We now have a projection

$$\begin{array}{ccc} \underline{D} & \hookrightarrow & \underline{X} \\ & \searrow & \downarrow \pi \\ & & B \end{array} \qquad \begin{array}{c} (a, b, x, y) \\ \downarrow \\ (a, b) \end{array},$$

whose fiber over a closed point $(a, b) \in B$ is precisely the configuration (X, D) for the parameter choice a, b. π is a flat morphism. The assignment

$$(a, b) \mapsto (V_{\mathbb{Q}}, W_\bullet, F^\bullet),$$

where

$$V_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}\{s_0, \ldots, s_3\},$$

$$V_{\mathbb{C}} := \mathbb{C}^4 \quad \text{with standard basis } e_0, \ldots, e_3,$$

$$s_0 := \begin{pmatrix} 1 \\ \mathrm{Li}\left(\frac{1}{b}\right) \\ \mathrm{Li}_1\left(\frac{1}{a}\right) \\ \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right) \end{pmatrix},\ s_1 := \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i \mathrm{Li}_1\left(\frac{b}{a}\right) \end{pmatrix},\ s_2 := \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 2\pi i \log\left(\frac{a-b}{1-b}\right) \end{pmatrix},\ s_3 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix},$$

$$W_p V_{\mathbb{Q}} = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}s_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}s_1 \oplus \mathbb{Q}s_2 \oplus \mathbb{Q}s_3 & \text{for} \quad p = -2, -1 \\ V_{\mathbb{Q}} & \text{for} \quad p \geq 0, \quad \text{and} \end{cases}$$

$$F^p V_{\mathbb{C}} = \begin{cases} V_{\mathbb{C}} & \text{for} \quad p \leq -2 \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \text{for} \quad p = -1 \\ \mathbb{C}e_0 & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1 \end{cases}$$

defines a good unipotent variation of $\mathbb{Q}$-MHS on B^{an}. Note that the Hodge filtration $F^\bullet$ does not depend on $(a, b) \in B^{\mathrm{an}}$.

One of the main characteristics of good unipotent variations of $\mathbb{Q}$-MHS is that they can be extended to a compactification of the base space (if the complement is a divisor with normal crossings).

The algorithm for computing these extensions, so called *limit mixed $\mathbb{Q}$-Hodge structures*, can be found for example in [H, 7, p. 24f] and [Z2b, 4, p. 12].

In a first step, we extend the variation to the divisor $\{a = 1\}$ minus the point $(1, 0)$ and then in a second step we extend it to the point $(1, 0)$. In particular, we assume that a branch has been picked for each entry P_{ij} of P. We will follow [Z2b, 4.1, p. 14f] very closely.

First step: Let σ be the loop winding counterclockwise around $\{a = 1\}$ once, but not around $\{a = 0\}$, $\{b = 0\}$ or $\{b = 1\}$. If we analytically continue an entry P_{ij} of P along σ we possibly get a second branch of the same multivalued function. In fact, the matrix resulting from analytic continuation of every entry along σ will be of the form

$$P \cdot T_{\{a=1\}},$$

where

$$T_{\{a=1\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the *monodromy matrix* corresponding to σ. The *local monodromy logarithm* is defined as

$$N_{\{a=1\}} = \frac{\log T_{\{a=1\}}}{2\pi i} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{-1}{n}\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} - T_{\{a=1\}}\right)^n$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-1}{2\pi i} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

We want to extend our $\mathbb{Q}$-MHS along the tangent vector $\frac{\partial}{\partial a}$, i.e. we introduce a local coordinate $t := a-1$ and compute the *limit period matrix*

$$P_{\{a=1\}} := \lim_{t\to 0} P\cdot e^{-\log(t)\cdot N_{\{a=1\}}}$$
$$= \lim_{t\to 0}\begin{pmatrix} 1 & 0 & 0 & 0\\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0\\ \mathrm{Li}_1\left(\frac{1}{1+t}\right) & 0 & 2\pi i & 0\\ \mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right) & 2\pi i\mathrm{Li}_1\left(\frac{b}{1+t}\right) & 2\pi i\log\left(\frac{1-b+t}{1-b}\right) & (2\pi i)^2\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{\log(t)}{2\pi i} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$
$$= \lim_{t\to 0}\begin{pmatrix} 1 & 0 & 0 & 0\\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0\\ \mathrm{Li}_1\left(\frac{1}{1+t}\right)+\log(t) & 0 & 2\pi i & 0\\ \mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right)+\log\left(\frac{1-b+t}{1-b}\right)\cdot\log(t) & 2\pi i\mathrm{Li}_1\left(\frac{b}{1+t}\right) & 2\pi i\log\left(\frac{1-b+t}{1-b}\right) & (2\pi i)^2\end{pmatrix}$$
$$\overset{(*)}{=} \begin{pmatrix} 1 & 0 & 0 & 0\\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0\\ 0 & 0 & 2\pi i & 0\\ -\mathrm{Li}_2\left(\frac{1}{1-b}\right) & 2\pi i\mathrm{Li}_1(b) & 0 & (2\pi i)^2\end{pmatrix}.$$

Here we used at $(*)$

- $P_{\{a=1\}2,0} = \lim_{t\to 0}\mathrm{Li}_1\left(\frac{1}{1+t}\right)+\log(t)$
  $= \lim_{t\to 0} -\log\left(1-\frac{1}{1+t}\right)+\log(t)$
  $= \lim_{t\to 0} -\log(t)+\log(1+t)+\log(t)$
  $= 0,$ and

- $P_{\{a=1\}3,0} = \lim_{t\to 0}\mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right)+\log\left(\frac{1-b+t}{1-b}\right)\cdot\log(t)$
  $= \mathrm{Li}_{1,1}\left(b,\frac{1}{b}\right)$ by L'Hospital
  $= -\mathrm{Li}_2\left(\dfrac{1}{1-b}\right).$

The vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS on $\{a = 1\} \setminus \{(1,0)\}$ are now given by the columns of the limit period matrix

$$s_0 = \begin{pmatrix} 1 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) \\ 0 \\ -\mathrm{Li}_2\left(\frac{1}{1-b}\right) \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i \mathrm{Li}_1(b) \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix}.$$

The weight and Hodge filtration of the limit $\mathbb{Q}$-MHS can be expressed in terms of the s_j and the standard basis vectors e_i of $\mathbb{C}^4$. This gives us a variation of $\mathbb{Q}$-MHS on the divisor $\{a = 1\} \setminus \{(1,0)\}$. This variation is actually (up to signs) an extension of Deligne's famous *dilogarithm variation* considered for example in [Kj, 4.2, p. 38f]. In loc. cit. the geometric origin of this variation is explained in detail.

Second step: We now extend this variation along the tangent vector $\frac{-\partial}{\partial b}$ to the point $(1,0)$, i.e. we write $b = -t$ with a local coordinate t. Let σ be the loop in $\{a = 1\} \setminus \{(1,0)\}$ winding counterclockwise around $(1,0)$ once, but not around $(1,1)$. Then the monodromy matrix corresponding to σ is given by

$$T_{(1,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

hence the local monodromy logarithm is given by

$$N_{(1,0)} = \frac{\log T_{(1,0)}}{2\pi i} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we get for the limit period matrix

$$\begin{aligned} P_{(1,0)} &:= \lim_{t\to 0} P_{\{a=1\}} \cdot e^{-\log(t)\cdot N_{(1,0)}} \\ &= \lim_{t\to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) & 2\pi i\,\mathrm{Li}_1(-t) & 0 & (2\pi i)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-\log(t)}{2\pi i} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \lim_{t\to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) - \log(t) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) - \mathrm{Li}_1(-t)\cdot\log(t) & 0 & 0 & (2\pi i)^2 \end{pmatrix} \\ &\overset{(*)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\zeta(2) & 0 & 0 & (2\pi i)^2 \end{pmatrix}. \end{aligned}$$

We remark that in the last matrix we see a decomposition into two (2×2)-blocks, one consisting of a Tate motive, the other involving $\zeta(2)$.

Here we used at $(*)$

- $$\begin{aligned} {P_{(1,0)}}_{1,0} &= \lim_{t\to 0} \mathrm{Li}_1\left(\tfrac{-1}{t}\right) - \log(t) \\ &= \lim_{t\to 0} -\log\left(1+\tfrac{1}{t}\right) - \log(t) \\ &= \lim_{t\to 0} -\log(1+t) + \log(t) - \log(t) \\ &= 0, \quad \text{and} \end{aligned}$$

- $$\begin{aligned} {P_{(1,0)}}_{3,0} &= \lim_{t\to 0} -\mathrm{Li}_2\left(\tfrac{1}{1+t}\right) - \mathrm{Li}_1(-t)\cdot\log(t) \\ &= \lim_{t\to 0} \mathrm{Li}_2\left(\tfrac{1}{1+t}\right) + \log(1+t)\cdot\log(t) \\ &= -\mathrm{Li}_2(1) \quad \text{by L'Hospital} \\ &= -\zeta(2). \end{aligned}$$

As in the previous step, the vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS are given by the columns of the limit period matrix $P_{(1,0)}$ and weight and Hodge filtrations by the formulae in subsection 14.6.5.

So we obtained $-\zeta(2)$ as a "period" of a limiting $\mathbb{Q}$-MHS.

Chapter 15

Miscellaneous periods: an outlook

In this chapter, we collect several other important examples of periods in the literature for the convenience of the reader.

15.1 Special values of L-functions

The Beilinson conjectures give a formula for the values (more precisely, the leading coefficients) of L-functions of motives at integral points. We sketch the formulation in order to explain that these numbers are periods.

In this section, fix the base field $k = \mathbb{Q}$. Let $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. For any prime p, let $I_p \subset \Gamma$ be the inertia group. Let $\mathrm{Fr}_p \in \Gamma/I_p$ be the Frobenius.

Let M be a mixed motive, i.e., an object in the conjectural $\mathbb{Q}$-linear abelian category of mixed motives over $\mathbb{Q}$. For any prime l, it has an l-adic realization M_l which is a finite dimensional $\mathbb{Q}_l$-vector space with a continuous operation of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Definition 15.1.1. Let M as above, p a prime and l a prime different from p. We put

$$P_p(M,t)_l = \det(1 - \mathrm{Fr}_p t | M_l^{I_p}) \in \mathbb{Q}_l[t] \ .$$

It is conjectured that $P_p(M,t)_l$ is in $\mathbb{Q}[t]$, and independent of l. We denote this polynomial by $P_p(M,t)$.

Example 15.1.2. Let $M = H^i(X)$ for smooth projective variety over $\mathbb{Q}$ with good reduction at p. Then the conjecture holds by the Weil conjectures proved by Deligne. In the special case $X = \mathrm{Spec}(\mathbb{Q})$, we get

$$P_p(H(\mathrm{Spec}\mathbb{Q}), t) = 1 - t \ .$$

In the special case $X = \mathbb{P}^1$, $i = 1$, we get

$$P_p(H^1(\mathbb{P}^1), t) = 1 - pt \ .$$

Remark 15.1.3. There is a sign issue with the operation of Fr_p depending on the normalization of $\mathrm{Fr} \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ and whether it operates via geometric or arithmetic Frobenius. We refrain from working out all the details.

Definition 15.1.4. Let M be as above. We put

$$L(M, s) = \prod_{p \text{ prime}} \frac{1}{P_p(M, p^{-s})}$$

as function in the variable $s \in \mathbb{C}$. For $n \in \mathbb{Z}$, let

$$L(M, n)^*$$

be the leading coefficient of the Laurent expansion of $L(M, s)$ around n.

We conjecture that the infinite product converges for $\mathrm{Re}(s)$ big enough and that the function has a meromorphic continuation to all of $\mathbb{C}$.

Example 15.1.5. Let $M = H^i(X)$ for X a smooth projective variety over $\mathbb{Q}$. Then convergence follows from the Riemann hypothesis part of the Weil conjectures. (Note that X has good reduction at almost all p. It suffices to consider these. Then the zeros of $P_p(M, t)$ are known to have absolute value $p^{-\frac{i}{2}}$.

Analytic continuation is a very deep conjecture. It holds for all 0-dimensional X. Indeed, for any number field K, we have

$$L(H^0(\mathrm{Spec}K), s) = \zeta_K(s)$$

where $\zeta_K(s)$ is the Dedekind ζ-function. For $M = H^1(E)$ with E an elliptic curve over $\mathbb{Q}$, we have

$$L(H^1(E), s) = L(E, s) \ .$$

Analytic continuation holds, because E is modular.

Example 15.1.6. Let M be as above, $\mathbb{Q}(-1) = H^2(\mathbb{P}^1)$ be the Lefschetz motive. We put $M(-1) = M \otimes \mathbb{Q}(-1)$. Then

$$L(M(-1), s) = L(M, s - 1)$$

by the formula for $P_p(\mathbb{Q}(-1), t)$ above.

Hence, the Beilinson conjecture about $L(M,s)$ at $n \in \mathbb{Z}$ can be reduced to the Beilinson conjecture about $L(M(-n),s)$ at 0.

Conjecture 15.1.7 (Beilinson [Be3])**.** *Let M be as above. Then the vanishing order of $L(M,s)$ at $s=0$ is given by*

$$\dim H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1)) - \dim H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),$$

where $H_{\mathcal{M},f}$ is unramified motivic cohomology.

For a conceptional discussion of unramified motivic cohomology and a comparison of the different possible definitions, see Scholbach's discussion in [Sch2].

In particular, we assume that unramified motivic cohomology is finite dimensional.

This conjecture is known for example when $M = H^0(\mathrm{Spec}K)(n)$ with K a number field, $n \in \mathbb{Z}$ or when $M = H^1(E)$ with E an elliptic curve with Mordell-Weil rank at most 1.

Definition 15.1.8. We call M *special* if the motivic cohomology groups

$$H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),\ H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),\ H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1)),\ H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1))$$

all vanish.

We are only going to state the Beilinson conjecture for special motives. In this case it is also known as Deligne conjecture. This suffices:

Proposition 15.1.9 (Scholl, [Scho])**.** *Let M be a motive as above. Assume all motivic cohomology groups over $\mathbb{Q}$ are finite-dimensional. Then there is a special motive M' such that*

$$L(M,0)^* = L(M',0)$$

and the Beilinson conjecture for M is equivalent to the Beilinson conjecture for M'.

Conjecture 15.1.10 (Beilinson [Be3], Deligne [D1])**.** *Let M be a special motive. Let M_B be its Betti-realization and M_{dR} its de Rham realization.*

1. *$L(M,0)$ is defined and non-zero.*
2. *The composition*

$$M_B^+ \otimes \mathbb{C} \to M_B \otimes \mathbb{C} \xrightarrow{\mathrm{per}} M_{\mathrm{dR}} \otimes \mathbb{C} \to M_{\mathrm{dR}} \otimes \mathbb{C}/F^0 M_{\mathrm{dR}} \otimes \mathbb{C}$$

   is an isomorphism. Here M_B^+ denotes the invariants under complex conjugation and $F^0 M_{\mathrm{dR}}$ denotes the 0-step of the Hodge filtration.
3. *Up to a rational factor, the value $L(M,0)$ is given by the determinant of the above isomorphism in any choice of rational basis of M_B^+ and M_{dR}.*

Corollary 15.1.11. *Assume the Beilinson conjecture holds. Let M be a motive. Then $L(M,0)^*$ is a period number.*

Proof. By Scholl's reduction, it suffices to consider the case M special. The matrix of the morphism in the conjecture is a block in the matrix of

$$\mathrm{per} : M_B \otimes \mathbb{C} \to M_{\mathrm{dR}} \otimes \mathbb{C} .$$

All its entries are periods. Hence, the same is true for the determinant. □

15.2 Feynman periods

Standard procedures in quantum field theory (QFT) lead to loop amplitudes associated to certain graphs [BEK, MWZ2]. Although the foundations of QFT via path integrals are mathematically non-rigorous, Feynman and others have set up the so-called Feynman rules as axioms, leading to a mathematically precise definition of *loop integrals (or, amplitudes)*.

These are defined as follows. Associated to a graph G one defines the integral as

$$I_G = \frac{\prod_{j=1}^n \Gamma(\nu_j)}{\Gamma(\nu - \ell D/2)} \int_{\mathbb{R}^{D\ell}} \frac{\prod_{r=1}^{\ell} dk_r}{i\pi^{D/2}} \prod_{j=1}^{n} (-q_j^2 + m_j^2)^{-\nu_j}.$$

Here, D is the dimension of space-time (usually, but not always, $D = 4$), n is the number of internal edges of G, $\ell = h_1(G)$ is the loop number, ν_j are integers associated to each edge, ν is the sum of all ν_j, the m_j are masses, the q_j are combinations of external momenta and internal loop momenta k_r, over which one has to integrate [MWZ2, Sect. 2]. All occurring squares are scalar products in D-dimensional Minkowski space. The integrals usually do not converge in D-space, but standard renormalization procedures in physics, e.g. dimensional regularization, lead to explicit numbers as coefficients of Laurent series. In dimensional regularization, one views the integrals as analytic meromorphic functions in the paramter $\epsilon \in \mathbb{C}$ where $D = 4 - 2\epsilon$. The coefficients of the resulting Laurent expansion in the variable ϵ are then the relevant numbers. By a theorem of Belkale-Brosnan [BB] and Bogner-Weinzierl [BW], such numbers are periods, if all moments and masses in the formulas are rational numbers.

A process called Feynman-Schwinger trick [BEK] transforms the above integral into a period integral

$$I_G = \int_\sigma f\omega$$

with

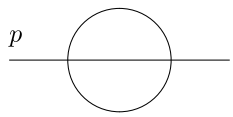
$$f = \frac{\prod_{j=1}^n x_j^{\nu_j - 1} \mathcal{U}^{\nu-(\ell+1)D/2}}{\mathcal{F}^{\nu-\ell D/2}}, \quad \omega = \sum_{j=1}^{n} (-1)^j dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Here, $\mathcal{U}$ and $\mathcal{F}$ are homogenous graph polynomials of Kirchhoff type [MWZ2, Sect. 2], with only $\mathcal{F}$ depending on kinematical invariants, and σ is the standard real simplex in $\mathbb{P}^{n-1}(\mathbb{C})$. Since σ is a compact subset of $\mathbb{P}^{n-1}(\mathbb{C})$, this is almost a representation of I_G as a naive period, and it is indeed one as a Kontsevich-Zagier period, provided the external momenta p_i are rational numbers. The differential form $f\omega$ has poles along σ, but there is a canonical blow-up process to resolve this problem [BEK, MWZ2]. The period which emerges is the period of the relative cohomology group

$$H^n(P \setminus Y, B \setminus (B \cap Y)),$$

where P is a blow-up of projective space in linear coordinate subspaces, Y is the strict transform of the singularity set of the integrand, and B is the strict transform of the standard algebraic simplex $\Delta^{n-1} \subset \mathbb{P}^{n-1}$ [MWZ2, Sect. 2]. It is thus immediate that I_G is a Kontsevich-Zagier period, if it is convergent, and provided that all masses and momenta involved are rational. If I_G is not convergent, then, by a theorem of Belkale-Brosnan [BB] and Bogner-Weinzierl [BW], the same holds under these assumptions for the coefficients of the Laurent expansion in renormalization.

Example 15.2.1. A very popular graph with a divergent amplitude is the two-loop sunset graph



The corresponding amplitude in D dimensions is

$$\Gamma(3-D)\int_\sigma \frac{(x_1x_2+x_2x_3+x_3x_1)^{3-\frac{3}{2}D}(x_1dx_2\wedge dx_3 - x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2)}{(-x_1x_2x_3p^2+(x_1m_1^2+x_2m_2^2+x_3m_3^2)(x_1x_2+x_2x_3+x_3x_1))^{3-D}},$$

where σ is the real 2-simplex in $\mathbb{P}^2$.

In $D=4$, this integral does not converge. One may, however, compute the integral in $D=2$ and study its dependence on the momentum p as an inhomogenous differential equation [MWZ1]. There is an obvious family of elliptic curves involved in the equations of the denominator of the integral, which gives rise to the homogenous Picard-Fuchs equation [MWZ1]. Then, a trick of Tarasov allows to compute the $D=4$ situation from that, see [MWZ1]. The extension of mixed Hodge structures

$$0 \to \mathbb{Z}(-1) \to H^2(P\setminus Y, B\setminus B\cap Y) \to H^2(P\setminus Y) \to 0$$

arising from this graph is already quite complicated [MWZ1, BV], as there are three different weights involved. The corresponding period functions when the momentum p varies are given by elliptic dilogarithm functions [BV, ABW]. There are generalizations to higher loop banana graphs [BKV].

In the literature, there are many more concrete examples of such periods, see the work of Broadhurst-Kreimer [BK] and subsequent work. Besides multiple zeta values, there are for examples graphs G where the integral is related to periods of K3 surfaces [BS].

15.3 Algebraic cycles and periods

In this section, we want to show how algebraic cycles in (higher) Chow groups give rise to Kontsevich-Zagier periods. Let us start with an example.

Example 15.3.1. Assume that $k \subset \mathbb{C}$, and let X be a smooth, projective curve of genus g, and $Z = \sum_{i=1}^k a_i Z_i \in CH^1(X)$ be a non-trivial zero-cycle on X with degree 0, i.e., $\sum_i a_i = 0$. Then we have a sequence of cohomology groups

$$0 \to H^1(X^{\mathrm{an}}) \to H^1(X^{\mathrm{an}} \backslash |Z|) \to H^2_{|Z|}(X^{\mathrm{an}}) \cong \bigoplus_i \mathbb{Z}(-1) \xrightarrow{\Sigma} H^2(X^{\mathrm{an}}, \mathbb{Z}) \cong \mathbb{Z}(-1) .$$

The cycle Z defines a non-zero vector $(a_1, ..., a_k) \in \bigoplus_i \mathbb{Z}(-1)$ mapping to zero in $H^2(X^{\mathrm{an}}, \mathbb{Z})$. Hence, by pulling back, we obtain an extension

$$0 \to H^1(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-1) \to 0 .$$

The extension class of this sequence in the category of mixed Hodge structures is known to be the *Abel-Jacobi class* of Z [C]. One can compute it in several ways. For example, one can choose a continuous chain γ with $\partial\gamma = \sum_i a_i Z_i$ and a basis $\omega_1, ..., \omega_g$ of holomorphic 1-forms on X^{an}. Then the vector

$$\left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right)$$

defines the Abel-Jacobi class in the *Jacobian*

$$\mathrm{Jac}(X) = \frac{H^1(X^{\mathrm{an}}, \mathbb{C})}{F^1 H^1(X^{\mathrm{an}}, \mathbb{C}) + H^1(X^{\mathrm{an}}, \mathbb{Z})} \cong \frac{H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}})^\vee}{H_1(X^{\mathrm{an}}, \mathbb{Z})} .$$

If X and the cycle Z are both defined over k, then obviously the Abel-Jacobi class is defined by g period integrals in $\mathbb{P}^{\mathrm{eff}}(k)$. In the case of smooth, projective curves, the Abel-Jacobi map

$$\mathrm{AJ}^1 : CH^1(X)_{\mathrm{hom}} \to \mathrm{Jac}(X)$$

gives an isomorphism when $k = \mathbb{C}$.

One can generalize this construction to Chow groups. Let X be a smooth, projective variety over $k \subset \mathbb{C}$, and $Z \in CH^q(X)$ a cycle which is homologous to zero. Then the Abel-Jacobi map

$$\mathrm{AJ}^q : CH^q(X)_{\mathrm{hom}} \longrightarrow \frac{H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q + H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z})} \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-q), H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z}))\ ,$$

As in the example above, the cycle Z defines an extension of mixed Hodge structures

$$0 \to H^{2q-1}(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-q) \to 0\ ,$$

where E is a subquotient of $H^{2q-1}(X^{\mathrm{an}} \setminus |Z|)$. The Abel-Jacobi class is given by period integrals

$$\left(\int_\gamma \omega_1, ..., \int_\gamma \omega_g \right)$$

in *Griffiths' Intermediate Jacobian*

$$\begin{aligned} \mathrm{J}^q(X) &= \frac{H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q H^{2q-1}(X^{\mathrm{an}}, \mathbb{C}) + H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z})} \\ &\cong \frac{F^q H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})^\vee}{H_{2q-1}(X^{\mathrm{an}}, \mathbb{Z})}\ . \end{aligned}$$

Even more general, one may use Bloch's *higher Chow groups* [Bl]. Higher Chow groups are isomorphic to motivic cohomology in the smooth case by a result of Voevodsky. In the general case, they only form a Borel-Moore homology theory and not a cohomology theory [VSF]. Then the Abel-Jacobi map becomes

$$\mathrm{AJ}^{q,n} : CH^q(X,n)_{\mathrm{hom}} \longrightarrow \frac{H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q + H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{Z})} \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-q), H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{Z}))\ ,$$

There are explicit formulas for $\mathrm{AJ}^{q,n}$ in [KLM, KLM2, Wei] on the level of complexes. This shows that the higher Abel-Jacobi class is defined by period integrals which define numbers in $\mathbb{P}^{\mathrm{eff}}(k)$.

In analogy with the classical Chow groups, Spencer Bloch has found an explicit description of the extension of mixed Hodge structures associated to a cycle $Z \in CH^q(X,n)_{\mathrm{hom}}$. This is explained in [DS, Scho2]. The periods associated to this mixed Hodge structures can then be viewed as the periods associated to Z.

Let us describe this construction. We let $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$. For varying n, this defines a cosimplicial object with face and degeneracy maps obtained by using the natural coordinate t on $\mathbb{P}^1$. Faces are given by setting $t_i = 0$ or $t_i = \infty$. By definition, a cycle Z in a higher Chow group $CH^q(X,n)$ is a subvariety of $X \times \square^n$ meeting all faces $F = X \times \square^m \subset X \times \square^n$ for $m < n$ properly, i.e., in codimension q. By looking at the normalized cycle complex, we may assume that Z has zero intersection with all faces of $X \times \square^n$. Removing the support of Z, let $U := X \times \square^n \setminus |Z|$, and define ∂U to be the union of the intersection of U

with the codimension 1 faces of $X \times \Box^n$. Then one obtains an exact sequence [DS, Lemma A.2]

$$0 \to H^{2q-n-1}(X^{\mathrm{an}}) \to H^{2q-1}(U^{\mathrm{an}}, \partial U^{\mathrm{an}}) \to H^{2q-1}(U^{\mathrm{an}}) \to H^{2q-1}(\partial U^{\mathrm{an}}) ,$$

which can be pulled back to an extension E if Z is homologous to zero:

$$0 \to H^{2q-n-1}(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-q) \to 0 .$$

Hence, E is a subquotient of the mixed Hodge structure $H^{2q-1}(U^{\mathrm{an}}, \partial U^{\mathrm{an}})$. This works for any cohomology satisfying certain axioms, see [DS]. In particular, applying it to singular or de Rham cohomology, we obtain an extension inside the category of Nori motives.

For the category of Nori motives, extension groups are not known in general, and have only been computed in the situation of 1-motives [AB]. The extension groups of any abelian category $\mathrm{MM}(k)$ of mixed motives over k are conjecturally supposed to be Adams eigenspaces of algebraic K-groups, or, equivalently, motivic cohomology groups. For example, one expects that

$$\mathrm{Ext}^1_{\mathrm{MM}(k)}(\mathbb{Q}(-q), H^{2q-n-q}(X)) = H^{2q-n}_M(X, \mathbb{Q}(-q)) = K_n(X)^{(q)}_{\mathbb{Q}}$$

for a smooth, projective variety X.

15.4 Periods of homotopy groups

In this section, we want to explain the periods associated to fundamental groups and higher homotopy groups.

The topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)$ of an algebraic variety X (defined over $k \subset \mathbb{C}$) with base point a carries a MHS in the following sense.

First, look at the group algebra $\mathbb{Q}\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)$, and the augmentation ideal $I := \mathrm{Ker}(\mathbb{Q}\pi_1^{\mathrm{top}}(X, a) \to \mathbb{Q})$. Then the Malcev-type object

$$\hat{\pi}_1(X(\mathbb{C}), a)_{\mathbb{Q}} := \lim_{n\to\infty} \mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$$

should carry an Ind-MHS, as we will explain now. Beilinson observed that each finite step $\mathbb{Q}\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)/I^{n+1}$ can be obtained as a MHS of a certain algebraic variety defined over the same field k. This was known to experts for some time, and later worked out in [DG].

Theorem 15.4.1. *Let M be any connected complex manifold and $a \in M$ a point. Then there is an isomorphism*

$$H_n(\underbrace{M \times \cdots \times M}_{n}, D; \mathbb{Q}) \cong \mathbb{Q}\pi_1^{\mathrm{top}}(M, a)/I^{n+1},$$

and $H_k(\underbrace{M \times \cdots \times M}_{n}, D; \mathbb{Q}) = 0$ for $k < n$. Here $D = \cup D_i$ is a divisor, where $D_0 = \{a\} \times M^{n-1}$, $D_{n+1} = M^{n-1} \times \{a\}$, and, for $1 \leq i \leq n-1$, $D_i = M^{i-1} \times \Delta \times M^{n-i-1}$ with $\Delta \subset M \times M$ the diagonal.

Proof. The proof in loc. cit., which we will not give here, proceeds by induction on n, using the first projection $p_1 : M^n \to M$ and the Leray spectral sequence. □

In the framework of Nori motives, one can thus see that $\hat{\pi}_1(X,a)_\mathbb{Q}$ immediately carries the structure of an Ind-Nori motive over k, since the Betti realization is obvious. Deligne-Goncharov [DG] and F. Brown [B2, B1] work instead within the framework of the abelian category of mixed Tate motives over $\mathbb{Q}$ of Levine [L2]. From this it follows, that $\hat{\pi}_1(\mathbb{P}^1 \setminus \{0,1,\infty\},a)_\mathbb{Q}$ is an Ind-mixed Tate motive over $\mathbb{Q}$ (in fact, over $\mathbb{Z}$ as explained in [B1]). There is also a description of the de Rham realization in [DG, B2, B1]. In particular, Brown showed that each MZV occurs as a period of this Ind-MHS [B2, B1, D3], as we explained in Section 14.5.

Theorem 15.4.2. *Every multiple zeta value occurs as a period of $\hat{\pi}_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\},a)_\mathbb{Q}$. Furthermore, every multiple zeta value is a polynomial with $\mathbb{Q}$-coefficients in multiple zate values with only* 2 *and* 3 *as entries.*

Proof. See [B1, B2]. □

The proof of this theorem also implies that every mixed Tate motive over $\mathbb{Z}$ occurs as a finite subquotient of the Ind-motive $\hat{\pi}_1(\mathbb{P}^1 \setminus \{0,1,\infty\},a)_\mathbb{Q}$.

Let us now look at higher homotopy groups $\pi_n(X^{\mathrm{an}})$ for $n \geq 2$ of an algebraic variety X over $k \subset \mathbb{C}$. They carry a MHS rationally by a theorem of Morgan [Mo] and Hain [H]:

Theorem 15.4.3. *The homotopy groups $\pi_n(X^{\mathrm{an}}) \otimes \mathbb{Q}$ of a simply connected and smooth projective variety over $\mathbb{C}$ carry a functorial mixed Hodge structure for $n \geq 2$.*

This theorem has a natural extension to the non-compact case using logarithmic forms, and to the singular case using cubical hyperresolutions, see [PS] and [Na].

Example 15.4.4. Let X be a simply connected, smooth projective 3-fold over $\mathbb{C}$. Then the MHS on $\pi_3(X^{\mathrm{an}})^\vee$ is given by an extension

$$0 \to H^3(X^{\mathrm{an}},\mathbb{Q}) \to \mathrm{Hom}(\pi_3(X^{\mathrm{an}}),\mathbb{Q}) \to \mathrm{Ker}\left(S^2H^2(X^{\mathrm{an}},\mathbb{Q}) \to H^4(X^{\mathrm{an}},\mathbb{Q})\right) \to 0$$

Carlson, Clemens, and Morgan [CCM] prove that this extension is given by the Abel-Jacobi class of a certain codimension 2 cycle $Z \in CH^2_{\mathrm{hom}}(X)$, and the extension class of this MHS in the sense of [C] is given by the Abel-Jacobi class

$$\mathrm{AJ}^2(Z) \in J^2(X) = \frac{H^3(X^{\mathrm{an}},\mathbb{C})}{F^2 + H^3(X^{\mathrm{an}},\mathbb{Z})} .$$

The proof of Morgan uses the theory of Sullivan [Su]. In the simply connected case, there is a differential graded Lie algebra $L(X,x)$ over $\mathbb{Q}$, concentrated in degrees 0, -1, ..., such that

$$H_*(L(X,x)) \cong \pi_{*+1}(X^{\mathrm{an}}) \otimes \mathbb{Q}.$$

One can then use the cohomological description of $L(X, x)$ and Deligne's mixed Hodge theory, to define the MHS on homotopy groups using a complex defined over k. We would like to mention that one can try to make this construction motivic in the Nori sense. At least for affine varieties, this was done in [Ga], see also [CG, pg. 22]. In [G4], a description of periods of homotopy groups is given in terms of Hodge correlators. This is not well understood yet.

From the approach in [Ga], one can see, at least in the affine case, that the periods of the MHS on $\pi_n(X^{\mathrm{an}})$ are defined over k, i.e., are contained in $\mathbb{P}^{\mathrm{eff}}(k)$, when X is defined over k, since all motives involved in the construction are defined over k.

15.5 Non-periods

The question whether a given transcendental complex number is a period number in $\mathbb{P}^{\mathrm{eff}}(\mathbb{Q})$, i.e., is a Kontsevich-Zagier period, is very difficult to answer in general, even though we know that there are only countably many of them. For example, we expect (but do not know) that the Euler number e is not a period. Also $1/\pi$ and Euler's γ are presumably not effective periods, although no proof is known.

When Kontsevich-Zagier wrote their paper, the situation was like at the beginning of the 19th century for the study of algebraic and trancendental numbers. It took a lot of effort to prove that Liouville numbers $\sum_i 10^{-i!}$, e (Hermite) and π (Lindemann) were transcendental.

In 2008, M. Yoshinaga [Y] first wrote down a non-period $\alpha = 0.77766444...$ in 3-adic expansion

$$\alpha = \sum_{i=1}^{\infty} \epsilon_i 3^{-i} \ .$$

We will now explain how to define this number, and why it is not a period. First, we have to explain the notions of computable and elementary computable numbers.

Computable numbers and equivalent notions of computable (i.e., equivalently, partial recursive) functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$ were introduced by Turing [T], Kleene and Church around 1936 following the ideas from Gödel's famous paper [G], see the references in [Kl] . We refer to [Bri] for a modern treatment of such notions which is intended for mathematicians.

The modern theory of computable functions starts with the notion of certain classes $\mathcal{E}$ of functions $f : \mathbb{N}_0 \to \mathbb{N}_0$. For each class $\mathcal{E}$ there is then a notion of $\mathcal{E}$-computable real numbers. In the following definition we follow [Y], but this was defined much earlier, see for example [R, Spe].

Definition 15.5.1. A real number $\alpha > 0$ is called *$\mathcal{E}$-computable*, if there are sequences $a(n)$, $b(n)$, $c(n)$ in $\mathcal{E}$, such that

$$\left| \frac{a(n)}{b(n)+1} - \alpha \right| < \frac{1}{k}, \text{ for all } n \geq c(k) \ .$$

The set of $\mathcal{E}$-computable numbers, including 0 and closed under $\alpha \mapsto -\alpha$, is denoted by $\mathbb{R}_{\mathcal{E}}$.

Some authors use the bound 2^{-k} instead of $\frac{1}{k}$. This leads to an equivalent notion only for classes $\mathcal{E}$ which contain exponentials $n \mapsto 2^n$.

If $\mathcal{E} = \text{comp}$ is the class of *Turing computable* [T], or equivalently Kleene's *partial recursive functions* [Kl], then $\mathbb{R}_{\text{comp}}$ is the set of *computable real numbers*. Computable complex numbers $\mathbb{C}_{\text{comp}}$ are those complex numbers where the real- and imaginary part are computable reals.

Theorem 15.5.2. *$\mathbb{R}_{\text{comp}}$ is a countable subfield of $\mathbb{R}$, and $\mathbb{C}_{\text{comp}} = \mathbb{R}_{\text{comp}}(i)$ is algebraically closed.*

One can think of computable numbers as the set of all numbers that can be accessed with a computer.

There are some important levels of hierarchies inside the set of computable reals

$$\mathbb{R}_{\text{low-elem}} \subsetneq \mathbb{R}_{\text{elem}} \subsetneq \mathbb{R}_{\text{comp}} \ ,$$

induced by the elementary functions of Kalmár (1943) [Ka], and the lower elementary functions of Skolem (1962) [Sk]. There is also the related Grzegorczyk hierarchy [Gr]. In order to define such hierarchies of real numbers, we will now study functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$ of several variables.

Definition 15.5.3. The class of *lower-elementary functions* is the smallest class of functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$

- containing the zero-function, the successor function $x \mapsto x + 1$ and the projection function $P_i : (x_1, ..., x_n) \mapsto x_i$,
- containing the addition $x + y$, the multiplication $x \cdot y$, and the modified subtraction $\max(x - y, 0)$,
- closed under composition, and
- closed under bounded summation.

The class of *elementary functions* is the smallest class which is also closed under bounded products.

Here, bounded summation (resp. product) is defined as

$$g(x, x_1, ..., x_n) = \sum_{a \leq x} f(a, x_1, ..., x_n) \text{ resp. } \prod_{a \leq x} f(a, x_1, ..., x_n) \ .$$

Elementary functions contain exponentials 2^n, whereas lower elementary function do not. The levels of the above hierarchy are strict [TZ].

The main result about periods proven in [Y, TZ] is:

Theorem 15.5.4. *Real periods are lower elementary real numbers.*

In fact, Yoshinaga proved that periods are elementary computable numbers, and Tent-Ziegler made the refinement that periods are even lower-elementary numbers. The proofs are based on Hironaka's theorem on semi-algebraic sets which we have used already in chapter 2. The main idea is to reduce periods to volumes of bounded semi-algebraic sets, and then use Riemann sums to approximate the volumes inside the class of lower elementary computable functions.

Corollary 15.5.5. *One has inclusions:*

$$\bar{\mathbb{Q}} \subsetneq \mathbb{P}^{\mathrm{eff}}(\mathbb{Q}) \subset \mathbb{C}_{\mathrm{low-elem}} \subsetneq \mathbb{C}_{\mathrm{elem}} \subsetneq \mathbb{C}_{\mathrm{comp}} \ .$$

Hence, in order to construct a non-period, one needs to exhibit a computable number which is not elementary computable. By Tent-Ziegler, it would also be enough to write down an elementary computable number which is not lower elementary.

Here is how Yoshinaga proceeds. First, using a result of Mazzanti [Maz], one can show that elementary functions are generated by composition from the following functions:

- The successor function $x \mapsto x+1$,
- the modified subtraction $\max(x-y, 0)$,
- the floor quotient $(x, y) \mapsto \lfloor \frac{x}{y+1} \rfloor$, and
- the exponential function $(x, y) \mapsto x^y$.

Using this, there is an explicit enumeration $(f_n)_{n\in\mathbb{N}_0}$ of all elementary functions $f : \mathbb{N}_0 \to \mathbb{N}_0$. Together with the standard enumeration of $\mathbb{Q}_{>0}$, we obtain an explicit enumeration $(g_n)_{n\in\mathbb{N}_0}$ of all elementary maps $g : \mathbb{N}_0 \to \mathbb{Q}_{>0}$. Using a trick, see [Y, pg. 9], one can "speed up" each function g_n, so that $g_n(m)$ is a Cauchy sequence (hence, convergent) in m for each n.

Following [Y], we therefore obtain

$$\mathbb{R}_{\mathrm{elem}} = \{\beta_0, \beta_1, ...\}, \text{ where } \beta_n = \lim_{m\to\infty} g_n(m) \ .$$

Finally, Yoshinaga defines

$$\alpha := \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sum_{i=1}^{n} \epsilon_i 3^{-i} \ ,$$

where $\epsilon_0 = 0$, and recursively

$$\epsilon_{n+1} := \begin{cases} 0, & \text{if } g_n(n) > \alpha_n + \frac{1}{2\cdot 3^n} \\ 1, & \text{if } g_n(n) \leq \alpha_n + \frac{1}{2\cdot 3^n} \end{cases} .$$

Now, it is quite easy to show that α does not occur in the list $\mathbb{R}_{\text{elem}} = \{\beta_0, \beta_1, ...\}$, see [Y, Prop. 17]. Note that the proof is essentially a version of Cantor's diagonal argument.