Slice filtration on motives and the Hodge conjecture (with an appendix by J. Ayoub*)

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We clarify the expected properties of the slice filtration on triangulated motives from the point of view of the generalized Hodge conjecture. In the appendix, J. Ayoub proves unconditionally that the slice filtration does not respect geometric motives.

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0 Introduction

The slice filtration on Voevodsky's triangulated category of motives is defined by effectivity conditions. It is constructed and studied in [10]. An analogous filtration on the homotopy category was introduced by Voevodsky.

For Artin–Tate motives, the weight filtration agrees with the slice filtration (see [10]). In the abelian (as opposed to derived) category of Hodge structures it is possible to reconstruct the weights from the slice filtration and its dual. So I had hoped to define the weight filtration on motives from the slice filtration. B. Kahn pointed out that my construction was assuming a number of nice (maybe too nice) properties of the slice filtration.

In this note we try to get a – conjectural – picture of these properties by systematic use of the realization functor to the derived category of Hodge structures. A key ingredient is Grothendieck's Generalized Hodge Conjecture about the analogous filtration on pure Grothendieck motives.

This approach is successful, even if the answers are contrary to what I had hoped for. (In fact, the application to the weight filtration does not work.) An old example of Griffiths's allows to deduce, using deep but standard conjectures, the following (see Proposition 5.3):

1. *The slice filtration does not respect the subcategory of geometrical motives.*
2. *The slice filtration does not commute with the weight filtration.*
3. *The induced slice filtration functors on the (conjectural) abelian category of mixed motives are left exact but not exact.*

Moreover, the induced filtration on the (conjectural) abelian category of pure motives agrees with the coniveau filtration. As a byproduct of our considerations, Grothendieck's Generalized Hodge Conjecture is generalized to triangulated motives. The generalization is implied by the same set of conjectures.

J. Ayoub communicated a non-conditional argument for property 1. to me. It is given in an appendix. This may be read as a confirmation for the conjectural picture we have of the theory of motives.

The note was written in context of the joint project with B. Kahn on the slice filtration and its properties, see [10]. I would like to thank him heartily for many interesting discussions. Several people helped me in my hunt for a good example. I am indebted to H. Esnault, B. Herzog, U. Jannsen, B. Moonen, A. Mukherjee and C. Voisin. It is a pleasure to thank them. I would also like to thank J. Ayoub for writing the Appendix and the referee for suggesting that it be written. B. Kahn pointed out a problem with the original proof of Proposition 3.6. I am

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1 Definition of the slice filtration

We review the construction of the slice filtration as constructed in [10]. For the purpose of the present article it suffices to work over the field of complex numbers $\mathbb{C}$. Most results extend to all fields of characteristic zero. We restrict to a $\mathbb{Q}$-rational theory.

Let $D\mathcal{M}_{\mathrm{gm}} = D\mathcal{M}_{\mathrm{gm}}(\mathbb{C})\otimes\mathbb{Q}$ be Voevodsky's category of geometrical motives ([14, Section 2.1]), $D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}} = D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}(\mathbb{C}) \otimes \mathbb{Q}$ the full subcategory of effective motives ([14, Definition 2.1.1]). Let $D\mathcal{M}^{\mathrm{eff}}_{-}$ be the category of bounded above complexes of Nisnevich sheaves with transfers which have homotopy invariant cohomology, i.e., Voevodsky's category of motivic complexes ([14, Section 3.1]). Let $D\mathcal{M}_{-}$ be the category obtained from $D\mathcal{M}^{\mathrm{eff}}_{-}$ by formally inverting the Tate object. Quasi-invertibility (e.g. [10, Prop. A.1]) can be used to show that there is a natural full embedding

$$\iota : D\mathcal{M}^{\mathrm{eff}}_{-} \longrightarrow D\mathcal{M}_{-}\ .$$

In all, there is a commutative diagram of full embeddings

$$\begin{array}{ccc} D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}} & \longrightarrow & D\mathcal{M}_{\mathrm{gm}} \\ \downarrow & & \downarrow \\ D\mathcal{M}^{\mathrm{eff}}_{-} & \xrightarrow{\iota} & D\mathcal{M}_{-} \end{array}$$

Lemma 1.1 ([10, Prop. 1.1]) *The functor ι has a right adjoint τ, i.e., $\tau : D\mathcal{M}_{-} \to D\mathcal{M}^{\mathrm{eff}}_{-}$ and a natural transformation $\tau \to \mathrm{id}$ such that*

$$\mathrm{Hom}_{D\mathcal{M}_{-}}(\iota N, M) = \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(N, \tau(M))$$

for all $N \in D\mathcal{M}^{\mathrm{eff}}_{-}$, $M \in D\mathcal{M}_{-}$.

Proof. Let M, N be as in the lemma. By definition, $M(m)$ is effective for m large enough. We put

$$\tau(M) := \lim_{m\to\infty} \underline{\mathrm{Hom}}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(\mathbb{Q}(m), M(m))$$

where $\underline{\mathrm{Hom}}$ is internal Hom in $D\mathcal{M}^{\mathrm{eff}}$ ([14, 3.2]). By quasi-invertibility of $\mathbb{Z}(1)$, the limit stabilizes: if $M = \widetilde{M}(-m)$ with $\widetilde{M} \in D\mathcal{M}^{\mathrm{eff}}_{-}$, $m \geq 0$, then

$$\tau(M) = \underline{\mathrm{Hom}}_{D\mathcal{M}^{\mathrm{eff}}_{-}}\big(\mathbb{Q}(m), \widetilde{M}\big).$$

The universal property is easy to check. □

For $n \in \mathbb{Z}$ let $D\mathcal{M}^{\geq n}_{-} = D\mathcal{M}^{\mathrm{eff}}_{-}(n)$. There is a sequence of functors

$$\nu^{\geq n} : D\mathcal{M}_{-} \longrightarrow D\mathcal{M}^{\geq n}_{-}$$

right adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(M) = \tau(M(-n))\,(n).$$

Definition 1.2 ([10, (1.1)]) The sequence of transformations

$$\nu^{\geq n} \longrightarrow \nu^{\geq n-1} \longrightarrow \cdots \longrightarrow \mathrm{id}$$

is called the *slice filtration*.

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The same type of filtration is also considered by Voevodsky in terms of homotopy theory of schemes.

Let $D\mathcal{M}^{\leq n}_{-}$ be the category of motives on which $\nu^{\geq n+1}$ vanishes. Quite formally one deduces from the adjunction properties of the slice filtration the existence of a sequence of functors

$$\nu_{\leq n} : D\mathcal{M} \longrightarrow D\mathcal{M}^{\leq n}_{-}$$

sitting in natural distinguished triangles

$$\nu^{\geq n} \longrightarrow \mathrm{id} \longrightarrow \nu_{\leq n-1} \longrightarrow \nu^{\geq n}[1].$$

Example 1.3 Let $M = \mathbb{Q}(n)$. Then

$$\tau(M) = \begin{cases} \underline{\mathrm{Hom}}(\mathbb{Q}, \mathbb{Q}(n)) = \mathbb{Q}(n), & n \geq 0, \\ \underline{\mathrm{Hom}}(\mathbb{Q}(-n), \mathbb{Q}) = 0, & n < 0. \end{cases}$$

Lemma 1.4 ([10, Prop. 1.7]) *Let $M = M^c(X)$ where X is a variety of dimension at most d. Then*

$$\nu^{\geq n} M = \begin{cases} M, & n \leq 0, \\ 0, & n > d. \end{cases}$$

P r o o f. The first part only says that M is effective. The second assertion follows from duality and the known facts on motivic cohomology with values in $\mathbb{Q}$. For details, see [10, Prop. 1.7]. □

This means that the slice filtration is finite, separated and exhaustive on geometrical motives. However, we do not know:

Question 1.5 *If $M \in D\mathcal{M}_{\mathrm{gm}}$, is it true that all $\nu^{\geq n} M$ are geometric?*

Contrary to my original hope, the answer is no in general, see Proposition 5.3 below for an argument relying on conjectures. A non-conditional argument by Ayoub is given in the appendix.

2 Slice filtration on mixed Hodge structures

In order to understand what the slice filtration means let us first consider the toy model of mixed Hodge structures. We denote $\mathcal{H}$ the category of mixed polarizable $\mathbb{Q}$-Hodge structures. A Hodge structure is called *effective* if its non-zero Hodge numbers are concentrated in the first quadrant. The category of effective Hodge structure is denoted $\mathcal{H}^{\mathrm{eff}}$. Note that this category is stable under subquotients and extensions. A mixed Hodge structure is effective if and only if its simple subquotients are effective.

Lemma 2.1 *The inclusion $\iota : \mathcal{H}^{\mathrm{eff}} \to \mathcal{H}$ has a left adjoint τ, i.e., $\tau : \mathcal{H} \to \mathcal{H}^{\mathrm{eff}}$ and natural transformation* $\mathrm{id} \to \tau$ *such that*

$$\mathrm{Hom}_{\mathcal{H}}(N, \iota M) = \mathrm{Hom}_{\mathcal{H}^{\mathrm{eff}}}(\tau N, M)$$

for all $N \in \mathcal{H}$, $M \in \mathcal{H}^{\mathrm{eff}}$.

P r o o f. This is just linear algebra. Let H be an object of $\mathcal{H}$. Then we define τH as the biggest quotient of H which is effective. We have to verify existence of this biggest quotient. Consider the set of all effective quotients of H ordered by the natural projections. This is an Artinian category. Suppose $p_1 : H \to H^1$ and $p_2 : H \to H^2$ are two effective quotients. Let K^i be the kernel of p_i and $K = K^1 \cap K^2$. We consider $H \to H/K$. It dominates H^1 and H^2. As subobject of $H^1 \oplus H^2$ the quotient H/K is effective. Hence the category of effective quotients has a unique maximal object. It is functorial. It is easy to check the universal property. □

Remark 2.2 We have switched from right adjoints in motives to left adjoints in Hodge structures. This corresponds to the fact that the Hodge realization functor is contravariant.

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Let $\mathcal{H}^{\geq n} = \mathcal{H}^{\mathrm{eff}}(-n)$. There is a sequence of functors

$$\nu^{\geq n} : \mathcal{H} \longrightarrow \mathcal{H}^{\geq n}$$

left adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(H) = \tau(H(n))\,(-n).$$

Definition 2.3 The sequence of transformations

$$\mathrm{id} \longrightarrow \cdots \longrightarrow \nu^{\geq n-1} \longrightarrow \nu^{\geq n}$$

is called the *slice cofiltration*.

If $\tau H = 0$, this does *not* mean that H has Hodge numbers only outside the first quadrant. It is easy to write down a simple, pure Hodge structure of weight 0 with Hodge type $\{(-1,1),(0,0),(1,-1)\}$. This Hodge structure has no effective quotient! This effect only occurs with $\mathbb{Q}$-Hodge structures as every simple $\mathbb{R}$-Hodge structure has Hodge type of the form $\{(p,q),(q,p)\}$ or $\{(p,p)\}$. Indeed, the slice functors become exact on $\mathbb{R}$-Hodge structures.

Lemma 2.4 *The functors $\nu^{\geq n}$ are right exact but not exact on $\mathcal{H}$.*

Proof. It suffices to consider τ. The functor τ is right exact because it is a left adjoint of an exact functor. Assume now that τ is exact. Let H be a simple polarizable Hodge structure of positive weight which is *not* effective and $H^\vee$ its dual. Note that $H^\vee$ is not effective either. Let E be a non-trivial extension

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow E \longrightarrow H \longrightarrow 0\,.$$

They are classified by

$$\mathrm{Ext}^1_{\mathcal{H}}(H, \mathbb{Q}(0)) = \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), H^\vee) = \mathrm{Coker}\left(H^\vee_{\mathbb{Q}} \oplus F^0 H^\vee_{\mathbb{C}} \longrightarrow H^\vee_{\mathbb{C}}\right) \neq 0\,.$$

In fact, this is an infinite dimensional $\mathbb{Q}$-vector space because $H^\vee$ is not effective. Hence E exists. We apply τ to the sequence and get

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \tau E \longrightarrow 0 \longrightarrow 0$$

because $\mathbb{Q}(0)$ is effective and H is not effective but simple. The isomorphism $\mathbb{Q}(0) \to \tau E$ together with the projection $E \to \tau E$ splits the original sequence, contradiction. □

3 Hodge conjecture

Recall ([8, 2.3.5], [9]) that there is a *Hodge realization functor*

$$\underline{R}_{\mathcal{H}} : D\mathcal{M}_{\mathrm{gm}} \longrightarrow D^b(\mathcal{H})\,.$$

We write $\underline{H}_{\mathcal{H}}(M) = \bigoplus H^i\big(\underline{R}_{\mathcal{H}}(M)\big) \in \mathcal{H}$.

If X is a smooth variety, then by construction $\underline{H}_{\mathcal{H}}(M(X))$ is a singular cohomology $H^*(X(\mathbb{C}),\mathbb{Q})$ of the complex manifold $X(\mathbb{C})$ with the Hodge structure defined by Deligne [5, Theorem 3.2.5 (iii)].

If M is effective, then $\underline{H}_{\mathcal{H}}(M)$ is also effective. What about the converse? This is the set-up of the generalized Hodge conjecture.

Let $\mathcal{M}$ be Grothendieck's category of pure motives up to homological equivalence, see e.g. [13, 1.4].

Conjecture 3.1 (Hodge) *The functor $\underline{H}_{\mathcal{H}} : \mathcal{M} \to \mathcal{H}$ is fully faithful.*

In more down to earth terms this says something about (p,p)-cycles. There was also a more general conjecture by Hodge for (p,q)-cycles. It was "false for trivial reasons" as Grothendieck pointed out. The corrected version is:

Conjecture 3.2 (GHC Grothendieck [7]) *The Hodge conjecture holds and a pure motive $M \in \mathcal{M}$ is effective if and only if $\underline{H}_{\mathcal{H}}(M)$ is effective.*

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This usually goes by the name of generalized Hodge conjecture. I propose to extend the conjecture to $D\mathcal{M}_{\mathrm{gm}}$.

Conjecture 3.3 (GHC for triangulated motives) *The Hodge conjecture holds and an object $M \in D\mathcal{M}_{\mathrm{gm}}$ is effective if and only if its Hodge realization is effective.*

Why should this be true?

Today's standard conjectures

- GHC for pure Grothendieck motives up to homological equivalence (3.2).
- $D\mathcal{M}_{\mathrm{gm}}$ admits a t-structure τ^{mot}. Its heart $\mathcal{MM}$ (mixed motives) contains $\mathcal{M}$ as full subcategory. For each object of $D\mathcal{M}_{\mathrm{gm}}$ the filtration induced by the truncation functors $\tau^{\mathrm{mot}}_{\leq n}$ is finite, separated and exhaustive.
- There are weight filtration functors $W_{\leq n}$ on $D\mathcal{M}_{\mathrm{gm}}$ which commute with the t-structure and such that the pure objects in $\mathcal{MM}$ are in $\mathcal{M}$. For each object of $D\mathcal{M}_{\mathrm{gm}}$ the filtration induced by the truncation functors $W_{\leq n}$ is finite, separated and exhaustive.
- The functor $\underline{H}_{\mathcal{H}}$ is compatible with t-structure and weights.

The cohomological functor of the motivic t-structure τ^{mot} is denoted H^i. Note that the Hodge realization is contravariant. This implies that $\underline{H}_{\mathcal{H}}(H^i(X)) = \underline{H}_{\mathcal{H}}^{-i}(X)$. We normalize the weight filtration such that

$$\underline{H}_{\mathcal{H}}(W_{\leq n}M) = \underline{H}_{\mathcal{H}}(M)/W_{-(n+1)}\underline{H}_{\mathcal{H}}(M)\ ,$$

i.e., a pure motive of weight n is mapped to a pure Hodge structure of weight $-n$. Note that this means $M(X)$ has cohomology in non-positive degrees and non-positive weights. If X is a smooth proper variety the conjectures imply that $H^i(X)$ is pure of weight i.

Proposition 3.4 *We assume the above conjectures. Then:*

1. *The functor $\underline{H}_{\mathcal{H}}$ is conservative on $D\mathcal{M}_{\mathrm{gm}}$, i.e., if $\underline{H}_{\mathcal{H}}(M) = 0$ then $M = 0$.*
2. *A pure Grothendieck motive is effective in $\mathcal{M}$ if and only if it is effective in $D\mathcal{M}_{\mathrm{gm}}$.*
3. *An object $M \in D\mathcal{M}_{\mathrm{gm}}$ is effective if and only if all $H^i(M)$ are effective in $\mathcal{MM}$ and if and only if all $\mathrm{Gr}_j H^i(M)$ are effective in $\mathcal{M}$.*
4. *GHC holds for triangulated motives, i.e., Conjecture 3.3 is true.*

Proof. We start with assertion 1. By today's standard conjectures, the H^i and Gr^W_j are conservative and commute with $\underline{H}_{\mathcal{H}}$. This reduces the question to pure Grothendieck motives. In this case it is the faithfulness part of the Hodge conjecture.

Now consider assertion 2. Suppose M is an effective object of $\mathcal{M}$. By the Hodge conjecture, $\mathcal{M}$ is a full subcategory of the semi-simple category of polarizable pure Hodge structures, hence semi-simple. Without restriction we may assume that M is pure of weight $-i$. By definition this means that it is a direct summand of $H^{-i}(X)$ for a smooth projective variety X. By the Hodge conjecture, the $H^{-i}(X)$ satisfy hard Lefschetz. By a general argument of Deligne (see [6]) this implies that in $D\mathcal{M}_{\mathrm{gm}}$

$$M(X) = \bigoplus H^{-i}(X)[i]\ .$$

Hence M is a direct summand of $M(X)[-i]$. As $D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}$ is pseudo-abelian, this implies that M is also effective viewed as object of $D\mathcal{M}_{\mathrm{gm}}$. Conversely, if M is in $\mathcal{M} \cap D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}$, then its Hodge realization is effective. By GHC for pure motives, M is an object of M^{eff}.

For property 3, note that $\underline{H}_{\mathcal{H}}\left(D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}\right) \subset \mathcal{H}^{\mathrm{eff}}$ and that $D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}$ is stable under triangles: if two vertices of a triangle are effective, then so is the third. Now let M be a triangulated motive such that $\underline{H}_{\mathcal{H}}(M) = \bigoplus \underline{H}_{\mathcal{H}}(H^{-i}(M))$ is effective. Hence all $H^{-i}\left(\mathrm{Gr}^W_j M\right)$ are effective in $\mathcal{M}$ by the GHC for $\mathcal{M}$. By the considerations above this implies that all $H^{-i}\left(\mathrm{Gr}^W_j M\right)$ are effective in $D\mathcal{M}_{\mathrm{gm}}$. The motive M is successive extension of effective objects, hence effective.

This implies the nontrivial part of GHC for triangulated motives. The remaining statements follows from GHC for $D\mathcal{M}_{\mathrm{gm}}$. □

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Wete need to understand the relation between the motivic t-structure and boundedness conditions for motivic complexes.

Lemma 3.5 *Assume today's standard conjectures. The following motivic complexes in $D\mathcal{M}_-^{\mathrm{eff}}$ are concentrated in degrees at most zero:*

1. *$M[m]$ for $M \in \mathcal{MM}^{\mathrm{eff}}$ of weight greater or equal than $-m$.*
2. *$M(c)[i+c]$ for M a subquotient in $\mathcal{MM}$ of $H^{-i}(X)$ for X smooth, $c \geq 0$.*
3. *$\tau_{>i}^{\mathrm{mot}} K$ for $K \in D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}$ a bounded complex of finite correspondences in degrees at most zero.*

Proof. Let M be pure of weight $-m$. Then M is direct summand of some $H^{-m}(X)$ for X smooth and projective. As in the proof of Proposition 3.4, $H^{-m}(X)[m]$ is a direct summand of $M(X)$. As direct summand of $M(X)$, the complex $M[m]$ is concentrated in non-positive degrees. Assertion (i) follows from this case by induction on the weight filtration.

Now we consider general subquotients M of $H^{-i}(X)$ for X smooth. If X is complete, then M is pure of weight $-i$ and the assertion holds. We have $M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$. Hence $\mathbb{Q}(1)[1]$ is concentrated in non-positive degrees. The property is stable under $\otimes$ in $D\mathcal{M}_-^{\mathrm{eff}}$. Hence, if the claim holds for some M and $c = 0$, it holds for M and all $c \geq 0$.

Let X be a smooth variety. There is a sequence of open embeddings of smooth varieties

$$X = U_N \subset U_{N-1} \subset U_{N-2} \subset \cdots \subset U_0$$

with U_0 complete and such that all strata $Z_i = U_i \setminus U_{i+1}$ are smooth of some codimension c_i. We argue by induction on i and induction on the dimension of X. The long exact cohomology sequence for the localization triangle reads

$$\cdots \to H^{-n+2c_i-1}(Z_i)(c_i) \to H^{-n}(U_{i+1}) \to H^{-n}(U_i) \to H^{-n+2c_i}(Z_i)(c_i) \to \cdots$$

If M is a subquotient of $H^{-n}(U_{i+1})$, it can be written as an extension

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathcal{MM}$ with M_1 a subquotient of $H^{-n+2c_i-1}(Z_i)(c_i)$ and M_2 a subquotient of $H^{-n}(U_i)$. By inductive hypothesis, M_1 is concentrated in degrees at most $n - 2c_i + 1 + c_i = n - c_i + 1 \leq n$ and M_2 in degrees at most n. Hence the assertion holds for M.

Finally consider $K \in D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}$ concentrated in degrees at most zero as a complex of finite correspondences. By induction on the length of the complex and (ii) all $H^{-j}(K)[j]$ are concentrated in degrees at most zero. The motivic t-structure is separated and exhaustive on geometric motives. Hence the statement on truncations follows from the statement on cohomology objects. □

Proposition 3.6 *Assume the motivic t-structure and the weight filtration exist on $D\mathcal{M}_{\mathrm{gm}}$ and satisfy today's standard conjectures. Then the weight filtration and the t-structure extend to $D\mathcal{M}_-$. Let $\mathcal{MM}_-$ be the heart of this t-structure. Then $\mathcal{MM}$ and $\mathcal{MM}_-^{\mathrm{eff}}$ are closed under extensions and subquotients in $\mathcal{MM}_-$.*

Proof. We consider the case of the motivic t-structure. The case of the weight filtration is similar but simpler (a t-structure with heart 0).

The essential property that we use is compactness of geometric objects, i.e., for $T \in D\mathcal{M}_{\mathrm{gm}}$ and $M = \bigoplus_{i \in I} M_i \in D\mathcal{M}_-$ we have

$$\mathrm{Hom}_{D\mathcal{M}_-}(T, M) = \bigoplus_{i \in I} \mathrm{Hom}_{D\mathcal{M}_-}(T, M_i).$$

We start with $D\mathcal{M}_-^{\mathrm{eff}}$. We follow Ayoub in in [1] section 2.1.3 with $G = \mathcal{M}^{\mathrm{eff}}$. (In our normalization "positive" corresponds to "negative" in Ayoub's.) Note that these objects are compact. The argument needs the existence of arbitrary direct sums. We embedd $D\mathcal{M}_-^{\mathrm{eff}}$ into the bigger category $D\mathcal{M}^{\mathrm{eff}}$ defined by Cisinski and

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Déglise, [2] Example 3.15. It is a localization of a category of unbounded complexes of sheaves and allows direct sums.

Put

$$\tau^{\rm mot}_{>0}D\mathcal{M}^{\rm eff} = \{M \in D\mathcal{M}^{\rm eff} \mid \mathrm{Hom}_{D\mathcal{M}^{\rm eff}}(T, M[-n]) = 0 \text{ for all } T \in \mathcal{M}^{\rm eff}, n \geq 0\},$$
$$\tau^{\rm mot}_{\leq 0}D\mathcal{M}^{\rm eff} = \{M \in D\mathcal{M}^{\rm eff} \mid \mathrm{Hom}_{D\mathcal{M}}(M, T) = 0 \text{ for all } T \in \tau_{>0}D\mathcal{M}^{\rm eff}_-\},$$

It is easy to check that $\tau^{\rm mot}_{>0}D\mathcal{M}^{\rm eff}_{\rm gm} \subset \tau^{\rm mot}_{>0}D\mathcal{M}^{\rm eff}$ and $\tau^{\rm mot}_{\leq 0}D\mathcal{M}^{\rm eff}_{\rm gm} \subset \tau^{\rm mot}_{\leq 0}D\mathcal{M}^{\rm eff}$. By [1] Proposition 2.1.70 this defines a t-structure extending the motivic t-structure on $D\mathcal{M}^{\rm eff}_{\rm gm}$.

Finally we check that the t-structure respects $D\mathcal{M}^{\rm eff}_-$. It suffices to consider $\tau^{\rm mot}_{>0}$. Let $K \in D\mathcal{M}^{\rm eff}_-$ be a complex concentrated in degrees at most N. K has a projective resolution, i.e., it can be written as a total complex of the form

$$\cdot \to \bigoplus_{i\in I_n} M(X_i) \to \bigoplus_{i\in I_{n+1}} M(X_i) \to \cdots \to \bigoplus_{i\in I_N} M(X_i) \to 0$$

with X_i smooth. The model structure used in the definition of $D\mathcal{M}^{\rm eff}$ is the injective one. Hence this presentation implies that K can be written as homotopy colimit of compact subobjects, more specifically as homotopy colimit of bounded complexes of finite correspondences concentrated in degrees at most N. By construction $\tau^{\rm mot}_{\geq 0}$ commutes with homotopy colimits. By Lemma 3.5 $\tau^{\rm mot}_{\geq 0}K$ is concentrated in degrees at most N for all these compact subcomplexes K. This finishes the proof of existence of the t-structure on $D\mathcal{M}^{\rm eff}_-$.

By [1] 2.1.70, $\tau_{\geq 0}D\mathcal{M}^{\rm eff}_-$ is the smallest suspended subcategory of $D\mathcal{M}^{\rm eff}_-$ closed under direct sums and containing $\mathcal{M}^{\rm eff}$. This implies

$$D\mathcal{M}^{\rm eff}_-(1) \cap \tau_{\geq 0}D\mathcal{M}^{\rm eff}_- = \tau_{\geq 0}D\mathcal{M}_-(1)$$

This means the t-structure extends to $D\mathcal{M}_-$.

Recall that $D\mathcal{M}_{\rm gm}$ is a full triangulated subcategory. Hence if two objects in a short exact sequence in $\mathcal{M}\mathcal{M}_-$ are in $\mathcal{M}\mathcal{M}$, then so is the third. It remains to show $\mathcal{M}\mathcal{M}$ that is is closed under subobjects. It suffices to consider subobjects of simple $M \in \mathcal{M}$. Let $U \subset M$ be a subobject in $\mathcal{M}\mathcal{M}_-$. Without loss of generality $U, M \in D\mathcal{M}^{\rm eff}_-$. By our proof of the existence of the t-structure, $U = \mathrm{hocolim}\, U_i$ with $U_i \in D\mathcal{M}^{\rm eff}_{\rm gm}$. As the truncation functors commute with homotopopy colimits, the U_i can be assumed in $\mathcal{M}\mathcal{M}^{\rm eff}_{\rm gm}$. It follows that $U = \lim_i U_i$ in $\mathcal{M}\mathcal{M}^{\rm eff}_-$. Without loss of generality the transition maps of the direct system are injective. Hence $U_i \subset M$. As M was assumed simple, this means $U_i = 0$ or $U_i = M$ for all i. Hence in the limit $U = 0$ or $U = M$.

$D\mathcal{M}^{\rm eff}_-$ is also a full triangulated subcategory of $D\mathcal{M}_-$. It remains to show that $\mathcal{M}\mathcal{M}^{\rm eff}_-$ is closed under subobjects. Let $U \subset M$ with $M \in \mathcal{M}\mathcal{M}^{\rm eff}_-$. As before, $U = \lim_i U_i$ with $U_i \in \mathcal{M}\mathcal{M}$ and $M = \lim_j M_j$ with $M_j \in \mathcal{M}\mathcal{M}^{\rm eff}$. Without loss of generality all transition maps are injective. By assumption $U_i \subset M$. As U_i is compact, this implies $U_i \subset M_j$ for some j. Subobjects of objects in $\mathcal{M}\mathcal{M}^{\rm eff}$ are effective by the trianguled GHC. As limit of effective objects U is effective. □

Proposition 3.7 *Assume again today's standard conjectures. An object $M \in \mathcal{M}\mathcal{M}$ is in $\mathcal{M}\mathcal{M}^{\rm eff}(n)$ if and only if $H^0\nu^{\geq n}M = M$. The functors $H^0\nu^{\geq n}$ and $H^0\nu_{\leq n}$ are left exact on $\mathcal{M}\mathcal{M}$. The functor $H^0\nu^{\geq n}$ respects $\mathcal{M}\mathcal{M}$ and is right adjoint to the inclusion $\iota : \mathcal{M}\mathcal{M}^{\rm eff}(n) \to \mathcal{M}\mathcal{M}$.*

One should think of $\nu^{\geq 0}$ as the derived functor of $H^0\nu^{\geq 0}$.

Proof. It suffices to consider $n = 0$. Let $M \in \mathcal{M}\mathcal{M}$. That the motive M is effective means $M = \nu^{\geq 0}M$, in particular $H^i\big(\nu^{\geq 0}M\big) = 0$ for $i \neq 0$. Conversely, assume $M = H^0\big(\nu^{\geq 0}M\big)$. Clearly $\nu^{\geq 0}M$ is effective and hence also its H^0.

For left exactness let $M \in \mathcal{M}\mathcal{M}$ be a mixed motive. Consider the distinguished triangle

$$\nu^{\geq 0}M \longrightarrow M \longrightarrow \nu_{<0}M\ .$$

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We first want to show that $H^i(\nu_{<0}M)$ is effective for $i \neq 0$. Consider the long exact sequence with respect to H^i. It yields isomorphisms

$$H^i\nu_{<0}M \longrightarrow H^{i+1}\nu^{\geq 0}M$$

for $i \neq -1, 0$. The object on the right is effective hence so is the object on the left. The same sequence yields

$$0 \longrightarrow H^{-1}\nu_{<0}M \longrightarrow H^0\nu^{\geq 0}M \longrightarrow M \longrightarrow H^0\nu_{<0}M \longrightarrow H^1\nu^{\geq 0}M \longrightarrow 0.$$

As subobject of an effective motive $H^{-1}\nu_{<0}M$ is effective.

Note that $\tau^{\mathrm{mot}}_{<0}M = 0$, hence $\tau^{\mathrm{mot}}_{<0}(\nu_{<0}(M[-1])) \cong \tau^{\mathrm{mot}}_{<0}\nu^{\geq 0}M$. Together with effectiveness of $H^{-1}\nu_{<0}M$ this means that $\tau^{\mathrm{mot}}_{<0}\nu_{<0}M$ is effective. By adjunction

$$\mathrm{Hom}\left(\tau^{\mathrm{mot}}_{<0}\nu_{<0}M, \nu_{<0}M\right) = \mathrm{Hom}\left(\tau^{\mathrm{mot}}_{<0}\nu_{<0}M, \nu_{\geq 0}\nu_{<0}M\right) = 0,$$

i.e., $\tau^{\mathrm{mot}}_{<0}M = 0$. In particular, $H^0\nu_{<0}$ is left exact. By the above isomorphisms this also implies that $H^0\nu^{\geq 0}$ is left exact. $H^0\nu^{\geq 0}M$ is a subobject of M, hence in $\mathcal{MM}$ itself.

Let $M \in \mathcal{MM}$, $N \in \mathcal{MM}^{\mathrm{eff}}$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{MM}}(N, M) &= \mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(N, M) &&= \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}\left(N, \nu^{\geq 0}M\right) \\ &= \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}\left(N, H^0\nu^{\geq 0}M\right) &&= \mathrm{Hom}_{\mathcal{MM}}\left(N, H^0\nu^{\geq 0}M\right). \end{aligned}$$

The crucial third equality holds because N and $\nu^{\geq 0}M$ have cohomology concentrated in degree 0 and in non-negative degrees respectively. □

Question 3.8 *Does the slice filtration commute with the weight filtration?*

I think that the answer is no, see Proposition 5.3 below for an argument relying on conjectures.

4 Coniveau filtration

In this section we concentrate on pure motives. As a left exact functor, $H^0\nu^{\geq n}$ respects the category of pure motives $\mathcal{M}$. In fact, on a simple pure motive it is either the zero or the full object. We are going to review Grothendieck's coniveau filtration. Note that we have to reverse all arrows because we use covariant motives whereas his setting was contravariant.

Definition 4.1 (Compare [7].) Let X be a smooth proper variety. The *coniveau filtration* on $M = H^{-i}(X)$ is defined as

$$F'^pM = M/\operatorname{Im}\left(\bigoplus H^{-i}(U)\right)$$

where the sum runs over all open subvarieties $U \subset X$ such that $T = X \smallsetminus U$ has codimension at least p.

Alternatively, F'^pM can be described as the smallest quotient of M such that all Gysin morphisms $H^{-i}(X) \to H^{-i+2q}\big(\widetilde{T}\big)(q)$ for all $\widetilde{T} \to X$ with $\widetilde{T}$ smooth, projective, $\dim X - \dim \widetilde{T} = q \geq p$ factor through F'^pM.

Proposition 4.2 *Assume today's standard conjectures. Let X be smooth and proper, $M = H^{-i}(X)$. Then the composition*

$$H^0\nu^{\geq p}M \longrightarrow M \longrightarrow F'^pM$$

is an isomorphism, hence the slice filtration provides a splitting of the coniveau filtration.

Proof. The key observation is that the $H^{-i+2q}\big(\widetilde{T}\big)(q)$ of the alternative description are in $D\mathcal{M}^{\geq q} \subset D\mathcal{M}^{\geq p}$. Hence a simple constituent of M which is not in $D\mathcal{M}^{\geq p}$ is also mapped to zero in F'^pM. Conversely, let $M' \subset M$ be a simple direct summand which is in $D\mathcal{M}^{\geq p}$. It is direct summand of some $H^{-i+2p}(Y)(p)$ with Y smooth and projective. The projection $M \to M'$ is induced by a morphism of motives $\phi : H^{-i}(X) \to H^{-i+2p}(Y)(p)$. We assume that pure motives in $\mathcal{MM}$ are Grothendieck motives, hence this morphism is represented by a cycle T in $X \times Y$ with $\dim X - \dim T = p$. Let $\widetilde{T}$ be a desingularization of T. Then ϕ is the composition $H^{-i}(X) \to H^{-i+2p}\big(\widetilde{T}\big)(p) \to H^{-i+2p}(Y)(p)$. As M' is a direct summand of $H^{-i+2p}(Y)(p)$, it is also a direct summand of $H^{-i+2p}\big(\widetilde{T}\big)(p)$. This implies that M' does not vanish in F'^pM either. □

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GHC for pure motives can be formulated as saying that $H^0\nu^{\geq p}$ commutes with the Hodge realization functor. We ask if this can be extended to the triangulated case.

Question 4.3 *Does $\nu^{\geq n}$ on $D\mathcal{M}_-$ commute with the Hodge realization?*

In order for this question to make sense, we first have to extend the Hodge realization to a functor on $D\mathcal{M}_-$. It will have values in $D^+(\mathrm{Pro}{-}\mathcal{H})$ where $\mathrm{Pro}{-}\mathcal{H}$ is the pro-category of Hodge structures. The question can be reduced to the case of $M \in \mathcal{M}$. However, I do not have a guess for the answer.

5 The counterexample

Let X be a smooth projective variety and Z a cohomologically trivial cycle of codimension 2. By the Abel–Jacobi map it induces an extension of mixed Hodge structures

$$0 \longrightarrow \underline{H}^3_{\mathcal{H}}(X) \longrightarrow H_Z \longrightarrow \mathbb{Q}(-2) \longrightarrow 0.$$

Lemma 5.1 *Let X be a generic quintic in $\mathbb{P}^4$ and let $H = \underline{H}^3_{\mathcal{H}}(X)$. Then H is a simple Hodge structure of weight 3 with $H^{3,0} \neq 0$. The image of the Abel–Jacobi map in $\mathrm{Ext}^1\left(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X)\right)$ is not finite dimensional.*

P r o o f. Quintics in $\mathbb{P}^4$ are simply connected Calabi–Yau threefolds and very well studied. In particular, their H^3 is primitive and has Hodge type

$$(3,0), \quad (2,1), \quad (1,2), \quad (0,3).$$

By [12, Corollary 18] it is simple for a generic X. The Abel–Jacobi map on homologically trivial cycles was studied by Griffiths and Clemens in this example. By [3, Theorem 0.2] or [4, Theorem 6] its image in $\mathrm{Ext}^1\left(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X)\right)$ is not finite dimensional. $\square$

I would like to thank C. Voisin and B. Moonen for pointing these arguments out to me.

Corollary 5.2 *Assume today's standard conjectures. Let X be as in the lemma, $M = H^{-3}(X)^\vee(2)$. Then $H^0\nu^{\geq 0}M = 0$ and $\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), M)$ is infinite dimensional.*

P r o o f. $M = H^3(X)^\vee(2)$ is simple because its Hodge realization is simple. Moreover, its Hodge realizations is of type $(-1,2), (0,1), (1,0), (2,-1)$, i.e., M is not effective. Hence it does not have any effective subobjects. By duality and functoriality

$$\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), M) \cong \mathrm{Ext}^1_{\mathcal{MM}}\left(H^{-3}(X), \mathbb{Q}(2)\right) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}\left(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X)\right).$$

The Abel–Jacobi map factors through this map. By the lemma the dimension of the Ext-group has to be infinite. $\square$

Proposition 5.3 (See Questions 1.5 and 3.8.) *Assume today's standard conjectures. Then $H^0\nu^{\geq n}$ is not exact, the functors $\nu^{\geq n}$ do not respect geometrical motives and do not commute with the weight filtration.*

P r o o f. It suffices to consider $n = 0$. Let M be as in the corollary. Consider a nontrivial extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathbb{Q}(0) \longrightarrow 0.$$

The long exact sequence for $H^i\nu^{\geq 0}$ starts

$$0 \longrightarrow 0 \longrightarrow H^0\nu^{\geq 0}E \longrightarrow \mathbb{Q}(0) \longrightarrow H^1\nu^{\geq 0}M \longrightarrow .$$

If $H^0\nu^{\geq 0}$ was exact, then $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. If $\nu^{\geq 0}$ commuted with the weight filtration, then $H^1\nu^{\geq 0}M$ would be pure of weight 1. The boundary map would vanish and again $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. In both cases this isomorphism together with the inclusion $H^0\nu^{\geq 0}E \to E$ would split the sequence and we would have a contradiction.

Now assume that $\nu^{\geq 0}M$ is geometric. We have

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}_{-}^{\mathrm{eff}}}\left(\mathbb{Q}, \nu^{\geq 0}M[1]\right).$$

As M is simple and non-effective, $H^0\nu^{\geq 0}M = 0$. By assumption $H^1\nu^{\geq 0}M$ is geometric, hence

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}}\left(\mathbb{Q}, H^1\nu^{\geq 0}M\right).$$

For pure motives, we have

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, N) = \mathrm{Hom}_{\mathcal{H}}\left(\underline{H}_{\mathcal{H}}(N), \mathbb{Q}\right)$$

by the Hodge conjecture, in particular this is a finite dimensional vector space. Hence this is also true for mixed motives. This contradicts the infinite dimensionality established in Corollary 5.2. □

Remark 5.4 Our results are over the complex numbers. The situation may be different over a number field, where the group of cohomologically trivial cycles is expected to be *finite-dimensional*. In this special case, the slice filtration might still respect geometric motives. The other two assertions of the proposition depend on the existence of just some nontrivial element in this group.

A Appendix (by Joseph Ayoub). The slice filtration on $\mathrm{DM}(k)$ does not preserve geometric motives

In this appendix we give an unconditional argument for the following (un)-property of the slice filtration on $\mathbf{DM}(k)$:

Proposition A.1 *The slice filtration on* $\mathbf{DM}(k)$ *does not preserve geometric motives.*

Recall (Definition 1.2) that the slice filtration is a sequence of transformations:

$$\nu^{\geq n} \longrightarrow \nu^{\geq n-1} \longrightarrow \ldots \longrightarrow \mathrm{id}$$

where $\nu^{\geq n}(M) = \tau(M(-n))(n)$, with $\tau : \mathbf{DM}(k) \to \mathbf{DM}_{\mathrm{eff}}(k)$, is the right adjoint to the full embedding $\mathbf{DM}_{\mathrm{eff}}(k) \subset \mathbf{DM}(k)$. When M is effective (e.g. the motive $\mathrm{M}(X)$ of a smooth projective variety X) we have by the proof of Lemma 1.1 that $\tau(M(-n)) = \underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(n), M)$ where $\underline{\mathsf{Hom}}_{\mathrm{eff}}$ stands for the internal hom in $\mathbf{DM}_{\mathrm{eff}}(k)$. We will prove the following:

Proposition A.2 *Assume that k is big enough. There exists a smooth projective k-variety X such that* $\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1), \mathrm{M}(X))$ *is not a geometric motive.*

We will implicitly assume k of characteristic zero and algebraically closed. We also work with rational coefficients for simplicity.

A.1 Compacity in $\mathrm{DM}(k)$

Recall the following classical notions (see [11]):

Definition A.3 Let $\mathcal{T}$ be a triangulated category with arbitrary infinite sums. An object $U \in \mathcal{T}$ is called *compact* if the functor $\hom_{\mathcal{T}}(U, -) : \mathcal{T} \to \mathcal{A}b$ commutes with sums. The category $\mathcal{T}$ is *compactly generated*, if there exists a set $\underline{G}$ of compact objects in $\mathcal{T}$ such that the family of triangulated functors $\hom_{\mathcal{T}}(U[n], -)$, where $U \in \underline{G}$ and $n \in \mathbb{Z}$, is conservative (that is, detects isomorphisms).

If $\mathcal{T}$ is compactly generated by $\underline{G}$ then the subcategory $\mathcal{T}_{\mathrm{comp}}$ of compact objects is the pseudo-abelian envelop of the triangulated sub-category of $\mathcal{T}$ generated by $\underline{G}$.

Let $(A_n)_{n\in\mathbb{N}}$ be an inductive system in $\mathcal{T}$. Its *homotopy colimit* is the cône of:

$$(\mathrm{id} - s) : \oplus_{n\in\mathbb{N}} A_n \longrightarrow \oplus_{n\in\mathbb{N}} A_n$$

where s is the composition $A_{n_0} \to A_{n_0+1} \to \oplus_{n\in\mathbb{N}} A_n$ on the factor A_{n_0}. It is denoted by $hocolim_{n\in\mathbb{N}} A_n$. We have the following lemma:

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Lemma A.4 *If $U \in \mathcal{T}$ is compact, then $\hom_{\mathcal{T}}(U, -)$ commutes with $\mathbb{N}$-indexed homotopy colimits.*

The following proposition is well-known. It follows immediately from the commutation of Nisnevic hypercohomology with infinite sums of complexes:

Proposition A.5 *The category $\mathbf{DM}_{\text{eff}}(k)$ is compactly generated by the set of $\mathrm{M}(X)$ with X in a set representing isomorphism classes of smooth k-varieties. Moreover the sub-category $\mathbf{DM}^{\text{gm}}_{\text{eff}}(k)$ is the sub-category of compact objects of $\mathbf{DM}_{\text{eff}}(k)$.*

A.2 Finite generation in $\mathbf{HI}(k)$

Recall that $\mathbf{DM}_{\text{eff}}(k)$ admits a natural t-structure whose heart $\mathbf{HI}(k)$ is the category of homotopy invariant Nisnevic sheaves with transfers. For an object $M \in \mathbf{DM}_{\text{eff}}(k)$ we denote $h_i(M)$ the truncation with respect to this t-structure. Recall that $h_i(M)$ is simply the i-th homology sheaf of the complex M. We will also write $h_i(X)$ for $h_i(\mathrm{M}(X))$ when X is a smooth k-variety. We make the following definition:

Definition A.6 A sheaf $F \in \mathbf{HI}(k)$ is called *finitely generated* if there exists a smooth variety X and a surjection $h_0(X) \twoheadrightarrow F$.

It is clear that the property of being finitely generated is stable by quotients. It is also stable by extensions. Indeed, let $F \subset G$ in $\mathbf{HI}(k)$ be such that F and G/F are finitely generated and chose surjections $a: \ h_0(X) \twoheadrightarrow F$ and $b: \ h_0(Y) \twoheadrightarrow G/F$. There exists a Nisnevic cover $U \to Y$ such that $b|_U$ lifts to $b': \ h_0(U) \longrightarrow G$. We get in this way a surjection $a \coprod b': \ h_0(X \coprod U) \twoheadrightarrow G$.

Assuming that k is countable we say that a sheaf F is countable if for any smooth k-variety X the set $F(X)$ is countable. Note the following technical lemma:

Lemma A.7 *Let F be a sheaf in $\mathbf{HI}(k)$ which is countable. There exists a chain $(S_n)_{n\in\mathbb{N}}$ of finitely generated sub-sheaves of F such that $F = \bigcup_{n\in\mathbb{N}} S_n$.*

Proof. Consider the set $\mathcal{S}$ whose elements are the finitely generated sub-sheaves of F. This set is countable as every finitely generated sub-sheaf of F is the image of a map $a : h_0(X) \to F$ with X a smooth k-variety and $a \in F(X)$. Fix a bijection $b : \mathbb{N} \xrightarrow{\sim} \mathcal{S}$ and denote $S_n = \sum_{i=0}^{n} b(i)$. We clearly have that $F = \bigcup_{n\in\mathbb{N}} S_n$. □

As a corollary we have the following:

Proposition A.8 *Let F be a countable sheaf in $\mathbf{HI}(k)$. Suppose that $\hom_{\mathbf{HI}(k)}(F, -)$ commutes with $\mathbb{N}$-indexed colimits. Then F is finitely generated.*

Proof. By Lemma A.7 we can write $F = \mathrm{colim}_{n\in\mathbb{N}}(S_n)$ with S_n finitely generated sub-sheaves of F. Using $\hom(F, F) = \mathrm{colim}\hom(F, S_n)$ one can find $n_0 \in \mathbb{N}$ such that the identity of F factors trough the inclusion $S_{n_0} \subset F$. This implies that $F = S_{n_0}$. □

Remark A.9 By working a little bit more, one shows under the hypothesis of A.8 that F is finitely presented in the sense that there exists an exact sequence:

$$h_0(X_2) \longrightarrow h_0(X_1) \longrightarrow F \longrightarrow 0$$

with X_1 and X_2 two smooth k-varieties.

A.3 Conclusion

Using Propositions A.5 and A.8 we can prove the following:

Theorem A.10 *Let M be a geometric motive in $\mathbf{DM}_{\text{eff}}(k)$. Suppose that $h_i(M) = 0$ for $i < 0$. Then $h_0(M)$ is finitely generated*[1].

[1] In fact $h_0(M)$ is even finitely presented (see A.9).

Proof. The motive M being geometric, it is defined over a finitely generated field (in particular a countable one). Hence, we may assume our base field k countable. It follows that the sheaves $h_i(M)$ are countable. This can be proved by reducing to the case $M = \mathrm{M}(X)$ with X a smooth k-variety and using Voevodsky's identification $\mathrm{M}(X) = C_*\mathbb{Z}_{tr}(X)$ with C_* the Suslin–Voevodsky complex.

By A.8 we need only to check that $\hom_{\mathbf{HI}(k)}(h_0(M), -)$ commutes with $\mathbb{N}$-colimits. Let $(A_n)_{n\in\mathbb{N}}$ be an inductive system and denote A its colimit. First, remark that A is also the homotopy colimit of $(A_n)_{n\in\mathbb{N}}$ in $\mathbf{DM}_{\mathrm{eff}}(k)$. Indeed, one has an exact triangle:

$$\oplus A_n \xrightarrow{\mathrm{id}-s} \oplus A_n \longrightarrow hocolim A_n \longrightarrow$$

It is easy to see that the morphism of sheaves $\mathrm{id} - s$ is injective. It follows that $hocolim_{n\in\mathbb{N}} A_n$ is the co-kernel of $\mathrm{id} - s$ which is canonically isomorphic to A.

Having this in mind, we can write:

$$\begin{aligned}
\hom_{\mathbf{HI}(k)}(h_0(M), colim A_n) &\overset{1}{=} \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(h_0(M), hocolim A_n)\\
&\overset{2}{=} \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(M, hocolim A_n)\\
&\overset{3}{=} colim \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(M, A_n)\\
&\overset{4}{=} colim \hom_{\mathbf{HI}(k)}(h_0(M), A_n).
\end{aligned}$$

Equality (1) follows from the above discussion. Equalities (2) and (4) follow from the condition $h_i(M) = 0$ for $i < 0$. Equality (3) is the compactness of M. This proves the theorem. □

Let X be a smooth projective variety of dimension d. Using [14, Theorem 4.2.2 and Proposition 4.2.3] we have:

- the sheaf $h_i\big(\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1)[2], M(X))\big)$ is zero for $i < 0$,
- the sheaf $h_0\big(\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1)[2], M(X))\big)$ is canonically isomorphic to the Nisnevic sheaf $\mathrm{CH}^{d-1}_{/X}$ associated to the pre-sheaf: $U \rightsquigarrow \mathrm{CH}^{d-1}(U \times_k X)$.

To prove A.2 it suffices by A.10 to find a smooth projective variety X of dimension $d = 3$ such that $\mathrm{CH}^{d-1}_{/X}$ is not finitely generated. To do this, we will construct a quotient of $\mathrm{CH}^{d-1}_{/X}$ which is constant but not finitely generated.

Definition A.11 Let U be a smooth k-scheme. A cycle $[Z] \in \mathrm{CH}^{d-1}(U \times_k X)$ is said to be *U-algebraically equivalent to zero* if there exist a smooth and connected U-scheme $V \to U$, a finite correspondence of degree zero $\sum_i n_i[T_i] \in Cor(V/U)$ (i.e., $n_i \in \mathbb{Z}$ and $t_i : T_i \to U$ are finite and surjective) and a cycle $[W] \in \mathrm{CH}^{d-1}(V \times_k X)$ such that $[Z]$ is rationally equivalent to $\sum_i n_i(t_i \times id_X)_*[W \cup (T_i \times X)]$.

We denote $\mathrm{NS}^{d-1}(U \times_k X)_U$ the quotient of $\mathrm{CH}^{d-1}(U \times_k X)$ with respect to the U-algebraic equivalence. We let also $\mathrm{NS}^{d-1}_{/X}$ be the Nisnevic sheaf associated to the pre-sheaf $U \rightsquigarrow \mathrm{NS}^{d-1}(U \times_k X)_U$.

We have clearly a surjective morphism $\mathrm{CH}^{d-1}_{/X} \to \mathrm{NS}^{d-1}_{/X}$. The latter sheaf is constant (because our base field k is algebraically closed). Indeed, for any finitely generated extension $k \subset K$ we have $\mathrm{NS}^{d-1}_{/X}(K) = \mathrm{NS}^{d-1}(X \otimes_k K)$. It is a well-known fact that the Neron-Severi group is invariant by extensions of an algebraically closed field.

Now, it is easy to see that a constant sheaf is finitely generated if and only if its group of sections over k is a finite dimensional $\mathbb{Q}$-vector space (using that a map from $h_0(X)$ to a constant sheaf factors trough $\mathbb{Q}_{tr}(\pi_0(X))$ with $\pi_0(X)$ the set of connected components of the variety X). We are done since $\mathrm{NS}^2(X)$ is not finite dimensional for a generic quintic in $\mathbb{P}^4$ (see [3, Theorem 0.2]).

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