POLYLOGARITHM FOR FAMILIES OF COMMUTATIVE GROUP SCHEMES

ANNETTE HUBER AND GUIDO KINGS

ABSTRACT. We generalize the definition of the polylogarithm classes to the case of commutative group schemes, both in the sheaf theoretic and the motivic setting. This generalizes and simplifies the existing cases.

CONTENTS

1. INTRODUCTION

Since its invention by Deligne, the importance of the cyclotomic polylogarithm and its elliptic analogue increased with each new aspect discovered about it. The main reason for this is the fact that the polylogarithm remains the only systematic way to construct interesting classes in motivic cohomology and that its realizations are related to important functions like Euler's polylogarithm or real analytic Eisenstein series. Many results about special values of L-functions rely on the motivic classes of the polylogarithm and we just mention the Tamagawa number conjecture for abelian number fields ([HuKi03] and [BuG03]), for CM elliptic curves ([Ki01]) and modular forms ([G06]), or Kato's work on the conjecture of Birch and Swinnerton-Dyer ([Ka04]).

It was already a vision of Beilinson and Levin (unpublished) that it should be possible to define the polylogarithm for general $K(\pi, 1)$-spaces, a program realized to a large extent by Wildeshaus in [Wi97]. There the polylogarithm was defined for extensions of abelian schemes by tori, a restriction which is unfortunate when dealing with degenerations, and the motivic construction of the polylogarithm was lacking.

In this paper we propose a new definition of the polylogarithm which works for arbitrary smooth commutative group schemes with connected fibres. This is not quite a generalization of Wildeshaus' definition (it agrees with it in some special cases, e.g. for abelian schemes), but the better functoriality properties of our definition make this look like the right construction. What is more, and highly important for applications, we can construct a class in motivic cohomology for our polylogarithm building on the techniques and results developed in [AHP] and [Ki99].

To explain the novel features in our construction, let us briefly review the definition of the polylogarithm (as we propose it) in the sheaf theoretic setting. Let S be noetherian finite dimensional scheme, $\pi : G \to S$ a smooth commutative group scheme with connected fibres of dimension d. Let

$$\mathscr{H} := \mathscr{H}_G := R^{2d-1}\pi_! \mathbf{Q}(d) = R^{-1}\pi_! \pi^! \mathbf{Q}$$

be the first homology of the group scheme. This is the sheaf of the Tate-modules of the fibres. The main player is the universal Kummer extension

$$0 \to \pi^* \mathscr{H} \to \mathcal{L}og^{(1)} \to \mathbf{Q} \to 0$$

on G. Taking symmetric powers $\mathcal{L}og^{(n)} := \mathrm{Sym}^n \mathcal{L}og^{(1)}$ one gets a projective system of sheaves $\mathcal{L}og$. The $\mathcal{L}og^{(n)}$ have obviously a filtration whose associated graded are just the $\mathrm{Sym}^n \mathscr{H}$. Moreover, $\mathcal{L}og$ has the important property that for torsion

sections $t : S \to G$ one has

$$t^* \mathcal{L}og = \prod_{n=0}^{\infty} \mathrm{Sym}^n \mathscr{H}$$

(as pro-objects) which is called the splitting principle. This applies in particular to the unit section e. The pro-object $\mathcal{L}og$ together with the splitting is characterized by a universal property, which we are able to verify in the sheaf theoretic setting (Theorem 3.3.2) and under some more restrictive assumptions also in the motivic setting (see Theorem 4.5.2).

We then turn to the construction of the polylogarithm. Let $j : U := G \backslash e(S) \to G$ be the open immersion of the complement of the unit section. The polylogarithm is a class

$$\mathrm{pol} \in \mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_* j^* \mathcal{L}og(d))$$

whose image under the residue map

$$\mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_* j^* \mathcal{L}og(d)) \xrightarrow{\mathrm{res}} \mathrm{Hom}_S(\mathscr{H}, e^* \mathcal{L}og)$$

is given by the natural inclusion $\mathscr{H} \to \prod_{n=0}^{\infty} \mathrm{Sym}^n \mathscr{H}$. The difference of our definition to the existing ones in the literature is the use of $R\pi_!$. In fact it is one of our main insights that everything becomes much more natural using cohomology with compact support.

In the sheaf theoretic setting the existence and uniqueness of pol follows from the vanishing of the higher direct images of $R\pi_! \mathcal{L}og$. In the motivic setting, we cannot make the same computation. However, analyzing the operation of multiplication by $a \in \mathbb{Z}$ we get a decomposition of $R\pi_! \mathcal{L}og$ into generalized eigenspaces. We get existence and a unique characterization of pol when asking it in addition to be in the right eigenspace. By either approach, the classes can easily be seen to be natural with respect to both S and G. By construction, the realization functors map the motivic classes to the sheaf theoretic ones.

We would also like to advocate a slight variant of the above definition, which appears already in [BeLe91] but not so much in other literature on the polylog. For each $\mathbb{Q}$-valued function α of degree 0 on a finite subscheme D of torsion points one can define

$$\mathrm{pol}_\alpha \in \mathrm{Ext}^{2d-1}(\mathbf{Q}, R\pi_! j_{D*} j_D^* \mathcal{L}og(d)).$$

This class has the advantage of having very good norm compatibility properties, which are useful in Iwasawa theoretic applications (see [Ki15]).

How can we have a more general motivic construction and still a simpler one? The main reason is that by the work of Ayoub and Cisinski-Deglise the theory of triangulated motives over a general base has now been developed to a point that makes calculations possible. One such is the computation of motives of commutative groups schemes in [AEH]. The original constructions could only use motivic cohomology with coefficients in $\mathbf{Q}(j)$. All the interesting non-constant nature of $\mathcal{L}og$ had to be encoded in complicated geometric objects. In the case of the classical polylog, the basic object $\mathcal{L}og^{(1)}$ had to be defined using relative cohomology - forcing the use of simplicial schemes in [HuWi98]. We are still missing the motivic t-structure on triangulated motives, but in our case $[a]$-eigenspace arguments as in [Ki99], which generalize [BeLe91], can be used as a replacement. Indeed, also this

part of the argument is clarified by applying it to objects rather than cohomology groups. For a complete list of earlier results, see the discussion in Section 6.3.

What is missing in contrast to the cases already in the literature is an explicit description of the monodromy matrices of $\mathcal{L}og$ and pol or the computation of other realizations.

Organization of the paper. The paper starts with a section on notation; fixing the geometric situation and also explaining the various settings we are going to work in.

Section 3 gives the sheaf theoretic construction of $\mathcal{L}og$, including the formulation of the universal property. Section 4 mimicks the construction in the motivic setting.

From this point on, we work in parallel in the sheaf theoretic and motivic setting. Section 5 explains the polylogarithm extension and its properties. In Section 6 we relate the present construction to the ones in the literature. The particularly important case of the cyclotomic polylog is discussed in more detail. Finally, Section 7 provides a couple of longer, technical proofs on properties of $\mathcal{L}og$, which had been delayed for reasons of readability.

An appendix discusses the decomposition into generalized eigenspaces in general $\mathbb{Q}$-linear triangulated categories.

Acknowledgements. It should be already clear from the introduction how much we are influenced by the ideas and constructions of Beilinson-Levin and Deligne-Beilinson. It is pleasure to thank F. Ivorra and S. Pepin-Lehalleur for discussions.

2. Setting and preliminaries

2.1. **Geometric situation.** We fix the following notation. Let S be a base scheme, subject to further conditions depending on the setting. Let

$$\pi : G \to S$$

be a smooth commutative group scheme with connected fibres of relative dimension d and unit section $e : S \to G$ and multiplication $\mu : G \times_S G \to G$. Let $j : U \to G$ be the open complement of $e(S)$.

Let $\iota_D : D \to G$ be a closed subscheme with structural map $\pi_D : D \to S$. Most of the time we will assume π_D étale and D contained in the N-torsion of G for some $N \geq 1$. Let $j_D : U_D = G \setminus D \to G$ be the open complement of D. This basic set up is summarized in the diagram

$$U_D := G \setminus D \xrightarrow{j_D} G \xleftarrow{\iota_D} D$$

We will also consider morphisms $\varphi : G_1 \to G_2$ of S-group schemes as above. In this case we decorate all notation with an index 1 or 2, e.g., d_1 for the relative dimension of G_1/S.

2.2. ℓ-**adic setting.** Let S be of finite type over a regular scheme of dimension 0 or 1. Let ℓ be a prime invertible on S, $X \to S$ separated and of finite type. We work in the category of constructible $\mathbb{Q}_\ell$-sheaves on X in the sense of [SGA 5, Exposé V] and its "derived" category in the sense of Ekedahl [Eke90]. They are triangulated

categories with a t-structure whose heart is the category of constructible $\mathbb{Q}_\ell$-sheaves. By loc. cit. Theorem 6.3 there is a full 6 functor formalism on these categories.

2.3. **Analytic sheaves.** Let S be separated and of finite type over the complex numbers. For $X \to S$ separated and of finite type, we denote X^{an} the set $X(\mathbb{C})$ equipped with the analytic topology. We work in the category of constructible sheaves of $\mathbb{Q}$-vector spaces on X^{an} and its derived category. There is a full 6 functor formalism on these categories, see e.g. [Di04].

2.4. **Hodge theoretic setting.** Let again S be separated and of finite type over the complex numbers. Let $X \to S$ be separated and of finite type. We work in the derived category of Hodge modules on X of Saito, e.g. [Sai88]. It has a natural forgetful functor into the derived category of constructible sheaves on X^{an}. By [Sai90, Section 4.6 Remarks 2. page 328-329] it also carries a t-structure whose heart maps to the abelian category of constructible sheaves via the forgetful functor. Note that this *not* the better known t-structure whose heart maps to perverse sheaves.

2.5. **Motivic setting.** Let S be noetherian and finite dimensional. Let $X \to S$ be separated and of finite type.

We denote $\mathbf{DA}(S)$ the triangulated category of étale motives without transfers with *rational coefficients*.

This is the same notation as in [AHP], our main reference in the sequel. The category is denoted $\mathbf{DA}^{\mathrm{et}}(S, \mathbb{Q})$ in the work of Ayoub [Ay07a], [Ay7b], [Ay14]. In the work of Cisinski and Déglise (see [CD09, 16.2.17]) it is the category $D_{\mathbb{A}^1,\mathrm{et}}(\mathrm{Sm}/S, \mathbb{Q})$.

There is a full 6 functor formalism for these categories. In particular, for $f : X \to S$ smooth of fibre dimension d, there is a natural object $M_S(X) \in \mathbf{DA}(S)$. In formulas:

$$M_S(X) = f_\# \mathbb{Q}_X = Rf_! \mathbb{Q}_X(d)[2d] = Rf_! f^! \mathbb{Q}_S.$$

Beside the formal properties of $\mathbf{DA}(S)$, we also are going to use the existence of a convenient *abelian* category mapping to it. Let $\mathrm{Sh}_{\mathrm{et}}(\mathrm{Sm})$ be the category étale sheaves of $\mathbb{Q}$-vector spaces on the category of smooth S-schemes of finite type. Then there is a tensor functor

$$C^b(\mathrm{Sh}_{\mathrm{et}}(\mathrm{Sm})) \to \mathbf{DA}(S)$$

which maps short exact sequences to exact triangles.

Remark 2.5.1. There are a number of different triangulated categories of motives over S. With integral or torsion coefficients, the differences between them are subtle; and comparison results like the Bloch-Kato conjecture are the deepest results in the theory. However, the situation is much more straightforward with rational coefficients. For example, we get the same categories when working with the Nisnevich or the étale topology. Under weak assumptions on S (e.g., S excellent and regular is more then enough) all definitions agree. In these cases, $\mathbf{DA}(S)$ is equivalent to the categories of motives for the qfh-topoly or for the h-topology, to triangulated motives with transfers, and to the category of Beilinson motives of Cisinski and Déglise

2.6. **Realizations.** Let $\mathbf{DA}_c(S)$ be the full subcategory of compact objects. If ℓ is invertible on S, then by [Ay14, Section 9] there is a covariant étale realization functor

$$R_\ell : \mathbf{DA}_c(S) \to D_c(S, \mathbb{Q}_\ell)$$

where $\mathbf{DA}_c(S)$ is the full subcategory of compact motives and $D_c(S, \mathbb{Q}_\ell)$ is the triangulated category of the ℓ-adic setting. The functors R_ℓ are compatible with the six functor formalism on both sides and map the Tate motive $\mathbb{Q}(j)$ to $\mathbb{Q}_\ell(j)$.

If S is of finite type over $\mathbb{C}$, then by [Ay10] there is a covariant Betti realization functor

$$R_B : \mathbf{DA}_c(S) \to D_c(S^{\mathrm{an}}, \mathbb{Q}).$$

It is compatible with the six functor formalism on both sides and maps the Tate motive $\mathbb{Q}(j)$ to $\mathbb{Q}$.

At the time of writing this paper, the situation for the Hodge theoretic realization is not yet as satisfactory. By work of Drew ([Dre13a], [Dre13b]) there is realization compatible with the 6 functor formalism into categories which are of Hodge theoretic flavour but a priori bigger than the derived category of Hodge modules. By work of Ivorra [Ivo14], there is realization into Hodge modules for compact motives over a smooth base of finite type over $\mathbb{C}$, but without knowledge about the 6 functors.

2.7. **Notation.** The bulk of our computations will be valid in the various settings without any changes. We are going to refer to the ℓ-adic, analytic or Hodge theoretic setting by the shorthand *sheaf theoretic setting*. By *triangulated setting* we are going to refer to computations on the level of derived categories in the ℓ-adic, analytic or Hodge theoretic setting as well as in the motivic setting. We denote them uniformly by $\mathbf{D}(X)$.

In any of the above sheaf theories we denote by $\mathbf{Q}$ the structure sheaf, i.e., $\mathbb{Q}_\ell$, $\mathbb{R}(0)$. In the motivic setting we denote $\mathbf{Q}$ the motive of S. It is defined by the image of the constant étale sheaf $\mathbb{Q}$.

To avoid confusion, we write $Rf_*, Rf_!$ etc. for the triangulated functors instead of f_* or $f_!$, which is sometimes used, in particular in [AHP]. The notation f_*, $f_!$ etc. is reserved for the functors between abelian categories of sheaves.

2.8. **Unipotent sheaves.** Let S be the base scheme and $\pi : X \to S$ separated and of finite type.

Recall that a sheaf $\mathscr{F}$ on X is *unipotent of length n*, if it has a filtration $0 = \mathscr{F}^{n+1} \subset \mathscr{F}^n \subset \ldots \subset \mathscr{F}^0 = \mathscr{F}$ such that $\mathscr{F}^i/\mathscr{F}^{i+1} \cong \pi^*\mathscr{G}^i$ for a sheaf $\mathscr{G}^i$ on S.

In any of the triangulated settings above, we call an object $M \in \mathbf{D}(X)$ *unipotent* if there is a finite sequence of objects $M_1 \to M_2 \to \ldots M_n = M$ and exact triangles

$$M_{i-1} \to M_i \to \pi_2^* N_i.$$

Lemma 2.8.1. *Let* $\pi_1 : X_1 \to S$ *and* $\pi_2 : X_2 \to S$ *be smooth of constant fibre dimension* d_1 *and* d_2. *Let* $f : X_1 \to X_2$ *be an* S-*morphism. Let* $M \in D(X_2)$ *be unipotent. Then*

$$f^! M = f^* M(d_1 - d_2)[2d_1 - 2d_2].$$

Proof. Put $c = d_1 - d_2$ the relative dimension of f. We start with the case $M = \pi_2^* N$. In this case

$$f^! M = f^! \pi_2^* N = f^! \pi_2^! N(-d_2)[-2d_2] = \pi_1^! N(-d_2)[-2d_2]$$
$$= \pi_1^* N(c)[2c] = f^* \pi_2^* N(c)[2c] = f^* M \otimes \mathbf{Q}(c)[2c].$$

In particular, $f^!\mathbf{Q} = \mathbf{Q}(c)[2c]$ and we may rewrite the formula as

$$f^*M \otimes f^!\mathbf{Q} = f^!(M \otimes \mathbf{Q}).$$

There is always a map from the left to right via adjunction from the projection formula

$$Rf_!(f^*M \otimes f^!\mathbf{Q}) = M \otimes Rf_!f^!\mathbb{Q} \to M \otimes \mathbf{Q}.$$

Hence we can argue on the unipotent length of M and it suffices to consider the case $M = \pi^*N$. This case was settled above. $\qquad\square$

Let $X \to S$ be a smooth scheme with connected fibres and $e : S \to X$ a section. Homomorphisms of unipotent sheaves are completely determined by their restriction to S via e^*:

Lemma 2.8.2. *We work in the sheaf theoretic setting. Let $\pi : X \to S$ be smooth with connected fibres and $e : S \to X$ a section of π and $\mathscr{F}$ a unipotent sheaf on X. Then*

$$e^* : \mathrm{Hom}_X(\mathbf{Q}, \mathscr{F}) \to \mathrm{Hom}_S(e^*\mathbf{Q}, e^*\mathscr{F})$$

is injective.

Proof. Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be a short exact sequence of unipotent sheaves on X. By exactness of e^* and left-exactness of Hom we get a commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_X(\mathbf{Q}, \mathscr{F}_1) & \longrightarrow & \mathrm{Hom}_X(\mathbf{Q}, \mathscr{F}_2) & \longrightarrow & \mathrm{Hom}_X(\mathbf{Q}, \mathscr{F}_3) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_S(\mathbf{Q}, e^*\mathscr{F}_1) & \longrightarrow & \mathrm{Hom}_S(\mathbf{Q}, e^*\mathscr{F}_2) & \longrightarrow & \mathrm{Hom}_S(\mathbf{Q}, e^*\mathscr{F}_3)
\end{array}
$$

If injectivity holds for $\mathscr{F}_1$ and $\mathscr{F}_3$, then by a small diagram chase it also holds for $\mathscr{F}_2$. Hence by induction on the unipotent length it suffices to consider the case $\mathscr{F} = \pi^*\mathscr{G}$. We claim that we even have an isomorphism in this case. It reads

$$\mathrm{Hom}_X(\pi^*\mathbf{Q}, \pi^*\mathscr{G}) \to \mathrm{Hom}_S(\mathbf{Q}, e^*\pi^*\mathscr{G}) = \mathrm{Hom}_S(\mathbf{Q}, \mathscr{G}).$$

As π is smooth, the left hand side is

$$\mathrm{Hom}_X(\pi^!\mathbf{Q}, \pi^!\mathscr{G}) = \mathrm{Hom}_S(R\pi_!\pi^!\mathbf{Q}, \mathscr{G}).$$

Recall that $H^0 R\pi_!\pi^!\mathbf{Q}$ is fibrewise 0-th homology of X. As we assume that π has connected fibres, this is isomorphic to $\mathbf{Q}$. Hence

$$\mathrm{Hom}_S(H^0 R\pi_!\pi^!\mathbf{Q}, \mathscr{G}) = \mathrm{Hom}_S(\mathbf{Q}, \mathscr{G}).$$

This proves the claim. $\qquad\square$

3. The logarithm sheaf

We work in one of the sheaf theoretic settings described in Section 2 and in the geometric situation described there. In particular, $\pi : G \to S$ is a smooth commutative group scheme with connected fibres of dimension d.

3.1. Definition of the logarithm sheaf.

Definition 3.1.1. For the group scheme $\pi : G \to S$ we let
$$\mathscr{H} := \mathscr{H}_G := R^{2d-1}\pi_! \mathbf{Q}(d) = R^{-1}\pi_! \pi^! \mathbf{Q}.$$

The formation of $\mathscr{H}_G$ is covariant functorial for S-group homomorphisms $\varphi : G_1 \to G_2$. The adjunction $\varphi_! \varphi^! \mathbf{Q} \to \mathbf{Q}$ induces by applying $R\pi_{2!}$ a map of sheaves

(1) $$\varphi_! : \mathscr{H}_{G_1} \to \mathscr{H}_{G_2}.$$

Using the "Leray spectral sequence" for $R\pi_! \pi^! \mathbf{Q}$ (i.e., the spectral sequence for the canonical filtration) we get

$$0 \to \mathrm{Ext}^1_S(\mathbf{Q}, \mathscr{H}) \xrightarrow{\pi^!} \mathrm{Ext}^1_G(\pi^! \mathbf{Q}, \pi^! \mathscr{H}) \to \mathrm{Hom}_S(\mathscr{H}, \mathscr{H}) \to$$
$$\to \mathrm{Ext}^2_S(\mathbf{Q}, \mathscr{H}) \xrightarrow{\pi^!} \mathrm{Ext}^2_G(\pi^! \mathbf{Q}, \pi^! \mathscr{H})$$

and the maps $\pi^!$ are injective because they admit the splitting $e^!$ induced by the unit section e. This gives

(2) $$0 \to \mathrm{Ext}^1_S(\mathbf{Q}, \mathscr{H}) \xrightarrow{\pi^!} \mathrm{Ext}^1_G(\pi^! \mathbf{Q}, \pi^! \mathscr{H}) \to \mathrm{Hom}_S(\mathscr{H}, \mathscr{H}) \to 0.$$

Note that $\mathrm{Ext}^1_G(\pi^! \mathbf{Q}, \pi^! \mathscr{H}) \cong \mathrm{Ext}^1_G(\mathbf{Q}, \pi^* \mathscr{H})$.

Definition 3.1.2. The *first logarithm sheaf* $(\mathcal{L}og^{(1)}, \mathbf{1}^{(1)})$ on G consists of an extension class
$$0 \to \pi^* \mathscr{H} \to \mathcal{L}og^{(1)} \to \mathbf{Q} \to 0$$
such that its image in $\mathrm{Hom}_S(\mathscr{H}, \mathscr{H})$ is the identity together with a fixed splitting $\mathbf{1}^{(1)} : e^* \mathbf{Q} \to e^* \mathcal{L}og^{(1)}$.

We define
$$\mathcal{L}og^{(n)} := \mathrm{Sym}^n \mathcal{L}og^{(1)}$$
and denote by $\mathbf{1}^{(n)}$ the induced splitting $\mathrm{Sym}^n(\mathbf{1}^{(1)}) : \mathbf{Q} \to \mathcal{L}og^{(n)}$.

The existence and uniqueness of $(\mathcal{L}og^{(1)}_G, \mathbf{1}^{(1)})$ follow directly from (2). The automorphisms of $\mathcal{L}og^{(1)}$ form a torsor under $\mathrm{Hom}_G(\mathbf{Q}, \pi^* \mathscr{H})$. In particular, the pair $(\mathcal{L}og^{(1)}, \mathbf{1}^{(1)})$ admits no automorphisms except the identity.

Consider $\mathcal{L}og^{(1)} \to \mathcal{L}og^{(1)} \oplus \mathbf{Q}$ induced by the identity and the natural projection $\mathcal{L}og^{(1)} \to \mathbf{Q}$. We define transition maps
$$\mathcal{L}og^{(n+1)} \cong \mathrm{Sym}^{n+1} \mathcal{L}og^{(1)} \to \mathrm{Sym}^{n+1}(\mathcal{L}og^{(1)} \oplus \mathbf{Q}) \to \mathrm{Sym}^n \mathcal{L}og^{(1)} \otimes \mathrm{Sym}^1 \mathbf{Q} \cong \mathcal{L}og^{(n)},$$
induced by the canonical projection. Under these transition maps $\mathbf{1}^{(n+1)}$ is mapped to $\mathbf{1}^{(n)}$ and one has an exact sequence
$$0 \to \pi^* \mathrm{Sym}^n \mathscr{H} \to \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n-1)} \to 0.$$

This implies that the sheaf $\mathcal{L}og^{(n)}$ is unipotent of length n with associated graded $\bigoplus_{k=0}^n \pi^* \mathrm{Sym}^k \mathscr{H}$. The section $\mathbf{1}^{(n)}$ induces an isomorphism

(3) $$e^* \mathcal{L}og^{(n)} \cong \prod_{k=0}^n \mathrm{Sym}^k \mathscr{H}.$$

Definition 3.1.3. The *logarithm sheaf* $(\mathcal{L}og, \mathbf{1})$ is the pro-system of $(\mathcal{L}og^{(n)}, \mathbf{1}^{(n)})$ with the above transition maps. The unipotent filtration is given by the kernels of the augmentation maps
$$\mathcal{L}og \to \mathcal{L}og^{(n)}.$$

For later reference, we also explain an explicit construction of $\mathcal{L}og^{(1)}$ as the universal Kummer extension. It is this point of view that will be used in the motivic case.

Note that the unit section induces an isomorphism $\mathbf{Q} \to R^0\pi_!\pi^!\mathbf{Q}$ and a splitting

$$(4) \qquad R\pi_!\pi^!\mathbf{Q} \cong \mathbf{Q} \oplus \tau_{\leq -1}R\pi_!\pi^!\mathbf{Q}.$$

We apply this to the G-group scheme $\tilde{G} = G \times G$ with structure map $\tilde{\pi} = \pi \times \mathrm{id}$. Its unit section is $\tilde{e} = e \times \mathrm{id}$. The diagonal $\Delta : G \to G \times G$ is a morphism of G-schemes, hence Δ induces a natural morphism of functors

$$\mathrm{id} = R\tilde{\pi}_!R\Delta_!\Delta^!\tilde{\pi}^! \to R\tilde{\pi}_!\tilde{\pi}^!,$$

which we apply to $\mathbf{Q}$. Together this yields a natural map in $\mathbf{D}(G)$

$$(5) \qquad \mathbf{Q} \to R\tilde{\pi}_!\tilde{\pi}^!\mathbf{Q} \to \tau_{\leq -1}R\tilde{\pi}_!\tilde{\pi}^!\mathbf{Q} \to R^{-1}\tilde{\pi}_!\tilde{\pi}^!\mathbf{Q}[1] = \pi^*\mathcal{H}[1].$$

Lemma 3.1.4. *The above composition (5) of morphisms in $\mathbf{D}(G)$ agrees with $\mathcal{L}og^{(1)}$ as element of $\mathrm{Ext}^1_G(\mathbf{Q}, \pi^*\mathcal{H})$.*

Proof. Let $\mathcal{L}$ be extension class in the Lemma. By Definition 3.1.2 we have to check that

(1) $e^*(\mathcal{L}) = 0$ (the 1-extension is split),
(2) the image of $\mathcal{L}$ in $\mathrm{Hom}_S(\mathcal{H}, \mathcal{H})$ under the map induced from the Leray spectral sequence is the identity map $\mathcal{H} \to \mathcal{H}$.

The first statement is true by construction because the restriction of Δ and $\tilde{e}$ to the unit section is the unit section e. The splitting of $e^*\mathcal{L}$ is the one induced from $\tilde{e}$.

We turn to the second statement and review the construction of the map to $\mathrm{Hom}_S(\mathcal{H}, \mathcal{H})$. We view $[\mathcal{L}]$ in $\mathrm{Hom}_G(\pi^!\mathbf{Q}, \pi^!\mathcal{H}[1])$. Using the adjunction between $\pi^!$ and $R\pi_!$ amounts to the composition

$$R\pi_!\pi^!\mathbf{Q} \xrightarrow{R\pi_!\mathcal{L}} R\pi_!\pi^!\mathcal{H}[1] \to \mathcal{H}[1].$$

The map "given by the Leray spectral sequence" is the one obtained by precomposing with

$$\tau_{\leq -1}R\pi_!\pi^!\mathbf{Q} \to R\pi_!\pi^!\mathbf{Q}.$$

The result naturally factors via

$$\mathcal{H}[1] \to \mathcal{H}[1]$$

for degree reasons. The map $R\pi_!\mathcal{L}$ is induced from

$$R\pi_!\mathbf{Q} \xrightarrow{\Delta_!} R(\pi \times \pi)_!\mathbf{Q} = R\pi_!\mathbf{Q} \otimes R\pi_!\mathbf{Q} \to R\pi_!\mathbf{Q} \otimes \mathcal{H}[1].$$

We compose with $R\pi_!\pi^!\mathbf{Q} \to \mathbf{Q}$ in the first factor. This agrees with projection to the second factor of $G \times G$, i.e., to the map induced by the identity. $\square$

3.2. Functoriality and splitting principle.

We collect some fundamental properties of the logarithm sheaf.

The first important property is the functoriality. Let

$$\varphi : G_1 \to G_2$$

be a homomorphism of group schemes of relative dimension d_1, d_2, respectively, and $\varphi_! : \mathcal{H}_{G_1} \to \mathcal{H}_{G_2}$ be the associated morphism of the homology.

Theorem 3.2.1 (Functoriality). *Let $c := d_1 - d_2$ be the relative dimension of the homomorphism $\varphi : G_1 \to G_2$. Then there is a unique homomorphism of sheaves*

$$\varphi_\# : \mathcal{L}og_{G_1} \to \varphi^* \mathcal{L}og_{G_2} \cong \varphi^! \mathcal{L}og_{G''} (-c)[-2c]$$

such that $\mathbf{1}_{G_1}$ maps to $\mathbf{1}_{G_2}$ and which respects the canonical filtrations on both sides. The induced map on the associated graded

$$\operatorname{gr} \varphi_\# : \bigoplus_{n \geq 0} \pi_1^* \operatorname{Sym}^n \mathscr{H}_{G_1} \to \bigoplus_{n \geq 0} \pi_1^* \operatorname{Sym}^n \mathscr{H}_{G_2}$$

coincides with $\operatorname{Sym}^{\cdot} \varphi_!$. If φ is an isogeny one has $\varphi_\# : \mathcal{L}og_{G_1} \to \varphi^! \mathcal{L}og_{G_2}$.

Proof. We are going to define a homomorphism

$$\mathcal{L}og_{G_1}^{(n)} \to \varphi^* \mathcal{L}og_{G_2}^{(n)}.$$

Assuming this, the right hand side agrees with $\varphi^! \mathcal{L}og_{G_2}^{(n)}(-c)[-2c]$ where $c = d_1 - d_2$ by Lemma 2.8.1.

As φ^* is compatible with tensor products, it suffices to prove the statement for $\mathcal{L}og^{(1)}$. The sheaf $\varphi^* \mathcal{L}og_{G_2}^{(1)}$ defines an extension class in $\operatorname{Ext}^1_{G_1}(\mathbf{Q}, \pi_1^* \mathscr{H}_{G_2})$. The push-out of $\mathcal{L}og_{G_1}^{(1)}$ by $\varphi_! : \mathscr{H}_{G_1} \to \mathscr{H}_{G_2}$ defines also a class in this Ext-group and from the definition one sees that these classes agree. Hence, one has a map of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1^* \mathscr{H}_{G_1} & \longrightarrow & \mathcal{L}og_{G_1}^{(1)} & \longrightarrow & \mathbf{Q} & \longrightarrow & 0 \\
 & & \varphi_! \downarrow & & h \downarrow & & \| & & \\
0 & \longrightarrow & \pi_1^* \mathscr{H}_{G_2} & \longrightarrow & \varphi^* \mathcal{L}og_{G_2}^{(1)} & \longrightarrow & \mathbf{Q} & \longrightarrow & 0.
\end{array}
$$

Taking the pull-back by e_1^* and using purity one gets a splitting

$$e_1^*(h) \circ \mathbf{1}_{G_1}^{(1)} : \mathbf{Q} \to e_1^* \mathcal{L}og_{G_1}^{(1)} \to e_2^* \mathcal{L}og_{G_2}^{(1)}.$$

By uniqueness there is a unique isomorphism of the pair $(\mathcal{L}og_{G_2}^{(1)}, e_1^*(h) \circ \mathbf{1}_{G_1}^{(1)})$ with $(\mathcal{L}og_{G_2}^{(1)}, \mathbf{1}_{G_2}^{(1)})$. The composition of this with h gives the desired map.

The difference of any two maps $h, h' : \mathcal{L}og_{G_1}^{(1)} \to \varphi^* \mathcal{L}og_{G_2}^{(1)}$ induces a homomorphism $h - h' : \mathbf{Q} \to \pi_1^* \mathscr{H}$, which by Lemma 2.8.2 is uniquely determined by its pull-back $e_1^*(h - h') : \mathbf{Q} \to e_2^* \mathcal{L}og_{G_2}^{(1)}$. If h and h' are compatible with the splittings the map $e_1^*(h - h')$ has to be zero, so that $h = h'$. $\square$

Corollary 3.2.2 (Splitting principle). *Let $\varphi : G_1 \to G_2$ be an isogeny, then*

$$\varphi_\# : \mathcal{L}og_{G_1} \to \varphi^! \mathcal{L}og_{G_2}$$

is an isomorphism. In particular, if $t : S \to G_1$ is in the kernel of φ, then

$$t^* \mathcal{L}og_{G_1} \cong \prod_{n \geq 0} \operatorname{Sym}^n \mathscr{H}_{G_2}.$$

Proof. By Corollary 3.2.1 the map $\operatorname{gr} \varphi_\#$ is an isomorphism as $\varphi_! : \mathscr{H}_{G_1} \to \mathscr{H}_{G_2}$ is already an isomorphism (recall that we have $\mathbf{Q}$-coefficients). From this one sees that $\varphi_\# : \mathcal{L}og_{G_1}^{(n)} \to \varphi^! \mathcal{L}og_{G_2}^{(n)}$ is an isomorphism. Applying $t^!$ gives, as $\varphi \circ t = e_2$, the isomorphism $t^! \mathcal{L}og_{G_1} \cong t^! \varphi^! \mathcal{L}og_{G_2} \cong (e_2)^! \mathcal{L}og_{G_2}$. By purity or more precisely Lemma 2.8.1 we get $t^* \mathcal{L}og_{G_1} \cong (e_2)^* \mathcal{L}og_{G_2} \cong \prod_{n \geq 0} \operatorname{Sym}^n \mathscr{H}_{G_2}$. $\square$

3.3. **Vanishing of cohomology.** The second property of the logarithm sheaf concerns the cohomology, which is important for the proof of all other properties and the definition of the polylogarithm.

Theorem 3.3.1 (Vanishing of cohomology). *One has*

$$R^i \pi_! \mathcal{L}og \cong \begin{cases} \mathbf{Q}(-d) & i = 2d \\ 0 & i \neq 2d. \end{cases}$$

Let G be an extension of an abelian scheme of relative dimension g by a torus or rank r. Then $\mathscr{H}$ is a locally constant $\mathbf{Q}$-sheaf of dimension $h := \dim_{\mathbf{Q}} \mathscr{H} = 2g + r$, and one also has

$$R^i \pi_* \mathcal{L}og \cong \begin{cases} \bigwedge^h \mathscr{H}^\vee & i = h \\ 0 & i \neq h \end{cases}$$

where $\mathscr{H}^\vee = \underline{\mathrm{Hom}}_S(\mathscr{H}, \mathbf{Q})$ is the dual of $\mathscr{H}$.

The proof of this theorem will be given in Section 7, see Corollary 7.1.3 and Corollary 7.1.6.

The sheaf $\mathcal{L}og$ can also be characterized by a universal property. Let $\mathscr{F}$ be a unipotent sheaf of some finite length n on G. Consider the homomorphism

(6) $$\pi_* \underline{\mathrm{Hom}}_G(\mathcal{L}og, \mathscr{F}) \to e^* \mathscr{F}$$

defined as the composition of

$$\pi_* \underline{\mathrm{Hom}}_G(\mathcal{L}og, \mathscr{F}) \to \pi_* e_* e^* \underline{\mathrm{Hom}}_G(\mathcal{L}og, \mathscr{F}) \to \underline{\mathrm{Hom}}_S(e^* \mathcal{L}og, e^* \mathscr{F})$$

with

$$\underline{\mathrm{Hom}}_S(e^* \mathcal{L}og, e^* \mathscr{F}) \xrightarrow{(1)^*} \underline{\mathrm{Hom}}_S(\mathbf{Q}, e^* \mathscr{F}) \cong e^* \mathscr{F}.$$

The same composition on the derived level defines a morphism

(7) $$R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, \mathscr{F}) \to e^* \mathscr{F}$$

Theorem 3.3.2 (Universal property). *Let $\mathscr{F}$ be a unipotent sheaf, then the map (6) induces an isomorphism*

$$\pi_* \underline{\mathrm{Hom}}(\mathcal{L}og, \mathscr{F}) \cong e^* \mathscr{F}.$$

Let M be a unipotent object in the derived category of sheaves $\mathbf{D}(G)$. Then the morphism (7) is an isomorphism

$$R\pi_* R\underline{\mathrm{Hom}}(\mathcal{L}og, M) \cong e^* M.$$

As a consequence the functor $\mathscr{F} \to \Gamma(S, e^* \mathscr{F})$ is pro-represented by $\mathcal{L}og$.

Proof. It suffices to treat the triangulated version. Indeed, if $M = \mathscr{F}$ is a sheaf, then $e^* \mathscr{F}$ is concentrated in degree 0, and hence

$$R\pi_* R\underline{\mathrm{Hom}}(\mathcal{L}og, \mathscr{F}) = \pi_* \underline{\mathrm{Hom}}(\mathcal{L}og, \mathscr{F}).$$

We will show the theorem by induction on the length n of the unipotent object M. We start in the case $n = 0$, $M = \pi^* N$. We claim that the natural map is an isomorphism

$$R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, \pi^* N) \cong N$$

Writing $\pi^* N \cong \pi^! N(-d)[-2d]$ then one has by adjunction and because $R\pi_! \mathcal{L}og \cong \mathbf{Q}(-d)[-2d]$

$$R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, \pi^* N) \cong R\underline{\mathrm{Hom}}_S(R\pi_! \mathcal{L}og, N(-d)[-2d]) \cong R\underline{\mathrm{Hom}}_S(\mathbf{Q}, N)$$

As $\underline{\mathrm{Hom}}_S(\mathbf{Q}, N) \cong N$ is the identity functor, the claim follows.

Now assume that the theorem is proven for unipotent objects of length $n-1$ and let M be unipotent of length n. Then we have an exact triangle

$$M' \to M \to M''$$

with M' and M'' unipotent of length less than n. We get a morphism of exact triangles

$$
\begin{array}{ccc}
R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, M') \longrightarrow & R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, M) \longrightarrow & R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, M'') \\
\Big\downarrow{\cong} & \Big\downarrow & \Big\downarrow \\
e^* M' \xrightarrow{\hspace{3cm}} & e^* M \xrightarrow{\hspace{3cm}} & e^*(M'').
\end{array}
$$

By induction the outer vertical morphisms are isomorphisms, hence the same is true in the middle. $\qquad\square$

4. Motivic Logarithm

We work in the motivic setting described in Section 2 and the geometric situation described there. In particular, let S be noetherian and finite dimensional. Let $X \to S$ be separated and of finite type. Recall that we work in the category $\mathbf{DA}(X)$ the triangulated category of étale motives without transfers with *rational coefficients*, see Section 2.5.

4.1. **Motives of commutative group schemes.** Let G/S be a smooth commutative group scheme with connected fibres of relative dimension d. The group G defines two natural étale sheaves of $\mathbb{Q}$-vector spaces on the category of smooth S-schemes:

- on the one hand $T \mapsto \mathbb{Q}[G(T)]$; its image in $\mathbf{DA}(S)$ is the motive $M_S(G)$.
- on the other hand $T \mapsto G(T) \otimes \mathbb{Q}$. Following [AHP, Definition 2.1, 2.3] we write $\underline{G}_{\mathbb{Q}}$ for the étale sheaf and $M_1(G)$ for its image in $\mathbf{DA}(S)$.

The summation map $\mathbb{Q}[G] \to \underline{G}_{\mathbb{Q}}$ induces a natural map $M_S(G) \to M_1(G)$.

Let $\mathrm{kd}(G)$ be the Kimura dimension of G (see [AHP, Definition 1.3]). It is at most $2d$. The main result of [AHP] (see loc.cit. Theorem 3.3) is the existence of a decomposition

$$(8) \qquad\qquad M_S(G) = \bigoplus_{i=0}^{\mathrm{kd}(G)} M_n(G),$$

which is natural in G and S. Moreover, we have

$$M_n(G) = \mathrm{Sym}^n M_1(G)$$

and the isomorphism in (8) is an isomorphism of Hopf objects. The motive $M_n(G)$ is uniquely determined by naturality.

By [AHP, Section 5.2] the image of $M_1(G)$ under the (covariant) ℓ-adic realization is $\mathcal{H}_\ell[1]$ where $\mathcal{H}_\ell$ is the relative Tate-module of Definition 3.1.1. Its image under the Betti-realization is the relative first homology $R^{-1}\pi_! \pi^! \mathbb{Q}[1]$. This motivates the following definition:

Definition 4.1.1. Let G/S be a smooth commutative group scheme with connected fibres. Let $\mathcal{H} := \mathcal{H}_{G/S} \in \mathbf{DA}(S)$ be defined as $M_1(G)[-1]$.

4.2. Kummer motives.

Definition 4.2.1. Let G/S be a smooth commutative group scheme with connected fibres. Let $s : S \to G$ be a section. The *Kummer motive* $\mathcal{K}(s)$ given by s is the image of the complex of étale sheaves

$$[\mathbb{Q}_S \xrightarrow{s} \underline{G}_{\mathbb{Q}}]$$

(with $\mathbb{Q}_S$ in degree 0) in the category $\mathbf{DA}(S)$. The *Kummer extension* of s is the natural triangle

$$\mathscr{H}_{G/S} \to \mathcal{K}(s) \to \mathbb{Q}_S \xrightarrow{s} \mathscr{H}_{G/S}[1] \ .$$

This defines a natural group homomorphism (the *motivic Kummer map*)

$$G(S) \to \mathrm{Hom}_{\mathbf{DA}(S)}(\mathbb{Q}_S, \mathscr{H}_{G/S}[1]) \ .$$

It maps the unit section to the trivial extension. More precisely, $\mathcal{K}(e)$ is the image of the complex of étale sheaves $[\mathbb{Q}_S \xrightarrow{0} \underline{G}_{\mathbb{Q}}]$, hence the natural inclusion $[\mathbb{Q}_S \to 0] \to [\mathbb{Q}_S \xrightarrow{0} \underline{G}_{\mathbb{Q}}]$ induces a distinguished splitting

$$\mathcal{K}(e) = \mathbb{Q}_S \oplus \mathscr{H}_{G/S}[-1]$$

Remark 4.2.2. It may seem strange at first glance that the motivic extension $\mathcal{L}og^{(1)}$ has a distinguished splitting, whereas the $\mathcal{L}og^{(1)}$ sheaf has not. In fact, there is a unique splitting of the sheaf theoretic version of $\mathcal{L}og^{(1)}$, which is compatible with all isogenies (see [BKL14, Section 1.5.] for an elaboration). This splitting coincides with the motivic splitting under the realizations.

Lemma 4.2.3. *The Kummer extension is given by the projection*

$$M_S(S) \xrightarrow{M_S(s)} M_S(G) = \bigoplus_i M_i(G) \to M_1(G) = \mathscr{H}_{G/S}[1]$$

under the decomposition of [AHP].

Proof. By construction in loc.cit. the map $M_S(G) \to M_1(G)$ is induced from the morphism of étale sheaves $\mathbb{Q}[G] \to \underline{G}_{\mathbb{Q}}$. Also by construction $s : M_S(S) \to M_S(G)$ is induced from $s : \mathbb{Q}_S = \mathbb{Q}[S] \to \mathbb{Q}[G]$. Hence the composition is induced from $\mathbb{Q}_S \to \underline{G}_{\mathbb{Q}}$. $\qquad\square$

Remark 4.2.4. Let ℓ be a prime invertible on S. Then the realization of the Kummer extension is the ℓ-adic Kummer extension

$$0 \to \mathscr{H} \to \mathcal{K}(s)_l \to \mathbb{Q}_l \to 0$$

in $\mathrm{Ext}^1_S(\mathbb{Q}_l, \mathscr{H})$. We do not go into details because we will not need this fact.

4.3. Logarithm sheaves.
Let G/S be smooth commutative group scheme with connected fibres.

Definition 4.3.1. Consider $G \times_S G \to G$ via the first projection. Let $\Delta : G \to G \times G$ be the diagonal. We put

$$\mathcal{L}og^{(1)} = \mathcal{K}(\Delta) \in \mathbf{DA}(G)$$

together with the splitting $\mathbf{1}^{(1)} : \mathbf{Q} \to e^* \mathcal{L}og^{(1)}$ given by $e^* \mathcal{K}(\Delta) = \mathcal{K}(e)$ as before.
    We define

$$\mathcal{L}og^{(n)} = \mathrm{Sym}^n \mathcal{L}og^{(1)}$$

and denote by $\mathbf{1}^{(n)}$ the induced splitting $\mathrm{Sym}^n(\mathbf{1}^{(1)}) : \mathbb{Q}_S \to \mathcal{L}og^{(n)}$.

We first establish the basic properties analogous to the sheaf theoretic case.

Lemma 4.3.2. *The section $\mathbf{1}^{(n)}$ induces isomorphisms $e^*\mathcal{L}og^{(n)} \to \bigoplus_{i=0}^n \mathrm{Sym}^i\mathcal{H}$ and $e^!\mathcal{L}og^{(n)} \to \bigoplus_{i=0}^n \mathrm{Sym}^i\mathcal{H}(-d)[-2d]$.*

Proof. The case $n = 1$ was discussed above. Passing to symmetric powers, we get

$$\mathrm{Sym}^n\mathcal{L}og^{(1)} \cong \bigoplus_{i=0}^n \mathrm{Sym}^i\mathbb{Q}_S \otimes \mathrm{Sym}^{n-i}\mathcal{H}$$

as claimed. The statement on $e^!\mathcal{L}og^{(n)}$ follows by Lemma 2.8.1. $\square$

Proposition 4.3.3. *For $n \geq 1$ there is a system of exact triangles in $\mathbf{DA}(G)$:*

$$\mathrm{Sym}^n\pi^*\mathcal{H}_{G/S} \to \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n-1)} .$$

Proof. Consider first the case $n = 1$. By definition, we have a distinguished triangle

$$\mathcal{H}_{G\times G/G} \to \mathcal{L}og^{(1)} \to \mathbb{Q}_S .$$

By compatibility of $M_1(G)$ with pull-back (see [AHP, Proposition 2.7]) we have

$$\pi^*M_1(G/S) = M_1(G \times G/G) .$$

This finishes the proof in this case. We abbreviate $\mathcal{H}$ for both $\mathcal{H}_{G/S}$ and $\pi^*\mathcal{H}_{G\times G/G}$.

Recall that $\mathcal{L}og^{(n)}$ is the image of a complex $\underline{\mathcal{L}og}^{(n)}$ of étale sheaves on G. The complex $\underline{\mathcal{L}og}^{(1)}$ has a filtration

$$0 \to \pi^*\mathcal{H} \to \underline{\mathcal{L}og}^1 \to \mathbb{Q}_G \to 0$$

in the abelian category of complexes of étale sheaves. Hence the symmetric powers also have a natural filtration (for full details see [AEH] Appendix C). Its associated gradeds are

$$\mathrm{Sym}^i(\mathcal{H}) \otimes \mathrm{Sym}^j\mathbb{Q}_G = \mathrm{Sym}^i\mathcal{H} .$$

In the same way as in the ℓ-adic case, see the discussion before Definition 3.1.3, we get short exact sequences of complexes of sheaves

$$0 \to \mathrm{Sym}^n\mathcal{H} \to \underline{\mathcal{L}og}^{(n)} \to \underline{\mathcal{L}og}^{(n-1)} \to 0 .$$

We view them as triangles in $\mathbf{DA}(G)$. $\square$

4.4. Functoriality.

Theorem 4.4.1. *Let $\varphi : G_1 \to G_2$ be morphism of smooth group schemes with connected fibres over S. Let $c = d_1 - d_2$ be the relative fibre dimension. Then there is a natural map*

$$\varphi_\# : \mathcal{L}og_{G_1}^{(n)} \to \varphi^*\mathcal{L}og_{G_2}^{(n)} = \varphi^!\mathcal{L}og_{G_2}^{(n)}(-c)[-2c].$$

Proof. We construct the map to $\varphi^*\mathcal{L}og_{G_2}^{(n)}$. By Lemma 2.8.1 one has $\varphi^*\mathcal{L}og_{G_2}^{(n)} = \varphi^!\mathcal{L}og_{G_2}^{(n)}(-c)[-2c]$. As φ^* commutes with tensor product, it suffices to treat the case $n = 1$. We have the commutative diagram

$$\begin{array}{ccc}
G_1 & \xrightarrow{\;\Delta\;} & G_1 \times G_1 \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\
G_2 & \xrightarrow{\;\Delta\;} & G_2 \times G_2
\end{array}$$

i.e., $\Delta_{G_1} \in G_1 \times G_1(G_1)$ is mapped to $\Delta_{G_2} \in G_2 \times G_2$. This implies that the diagram of sheaves on G commutes

$$
\begin{array}{ccc}
\mathbb{Q}_{G_1} & \longrightarrow & \underline{G_1 \times G_1}_{\mathbb{Q}} = \pi_{G_1}^* \underline{G_1}_{\mathbb{Q}} \\
{\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi_{G_1}^* \varphi} \\
\varphi^* \mathbb{Q}_{G_2} & \longrightarrow & \varphi^* \underline{G_2 \times G_2}_{\mathbb{Q}} = \pi_{G_2}^* \underline{G_2}_{\mathbb{Q}}
\end{array}
$$

We take the image of this diagram in $\mathbf{DA}(G_1)$. The statement follows because $\varphi^* M_1(G_2) = M_1(\varphi^* G_2) = M_1(G_1)$ by [AHP, Proposition 2.7]. $\qquad \square$

Corollary 4.4.2 (Splitting principle)**.** *Let $\varphi : G_1 \to G_2$ be an isogeny, then*

$$
\varphi_\# : \mathcal{L}og_{G_1}^{(n)} \to \varphi^! \mathcal{L}og_{G_2}^{(n)}
$$

is an isomorphism. In particular, if $t : S \to G_1$ is in the kernel of φ, then

$$
t^* \mathcal{L}og_{G_1} \cong \prod_{n \geq 0} \mathrm{Sym}^n \mathcal{H}_{G_2}.
$$

Proof. As φ^* is compatible with tensor product and exact triangles, it suffices to show $\varphi^* \mathcal{H}_{G_2} = \mathcal{H}_{G_1}$ or equivalently $\mathcal{H}_{G_2} = \mathcal{H}_{G_1}$ as motives on S. This holds by construction because $\underline{G_2}_{\mathbb{Q}} = \underline{G_1}_{\mathbb{Q}}$. The rest of the argument is the same as in the sheaf theoretic case, see Corollary 3.2.2. $\qquad \square$

4.5. Vanishing of cohomology.
The second property of the logarithm sheaf concerns the vanishing of the cohomology, which is important for the proof of all other properties and the definition of the polylogarithm.

Theorem 4.5.1 (Vanishing of cohomology)**.** *Assume that S is a scheme of characteristic 0 or that G/S is affine. One has*

$$
R\pi_! \mathcal{L}og \cong \mathbf{Q}(-d)[-2d]
$$

The proof of this theorem will be given in Section 7.2.

As in the sheaf theoretic case, this implies a universal property of the motivic logarithm. Let M be a unipotent sheaf of length n on G. In the same way as in the case of sheaves (see equation (6)) one has a map

$$
(9) \qquad\qquad R\pi_* R\underline{\mathrm{Hom}}_G(\mathcal{L}og, M) \to e^* M.
$$

Theorem 4.5.2 (Universal property)**.** *Let S be a scheme of characteristic 0 or assume that G/S is affine. Let M be a unipotent motive on G, then the map (9) induces an isomorphism*

$$
R\pi_* R\underline{\mathrm{Hom}}(\mathcal{L}og, M) \cong e^* M.
$$

Proof. The argument is the same as in the sheaf theoretic case, with Theorem 4.5.1 replacing Theorem 3.3.1. $\qquad \square$

4.6. Realizations.

Proposition 4.6.1.   (1) *Assume the prime ℓ is invertible on S and S of finite over a regular scheme of dimension 0 or 1. Then the ℓ-adic realization R_ℓ maps the motivic $\mathcal{L}og_G^{(n)}$ to the ℓ-adic $\mathcal{L}og_G^{(n)}$ as defined in Section 3.1 .*
    (2) *Assume S is of finite type over $\mathbb{C}$. Then the Betti realization R_B maps the motivic $\mathcal{L}og_G^{(n)}$ to the constructible $\mathcal{L}og_G^{(n)}$ in Section 3.1*

Proof. The argument is the same in both cases. By construction it suffices to consider the case $n = 1$. We use the description of the Kummer extension for Δ given in Lemma 4.2.3. After applying the realization functor (which commutes with all 6 functors), we obtain the same class as constructed in Equation (5). By Lemma 3.1.4 this is $\mathcal{L}og^{(1)}$ in the realization. $\qquad\square$

Remark 4.6.2. The same argument will also apply in the Hodge theoretic setting once we have a realization functor compatible with the 6 functor formalism. See the discussion in Section 2.5 on the state of the art.

5. THE POLYLOGARITHM SHEAF/MOTIVE

Unless stated otherwise, we work in the sheaf theoretic and in the motivic setting in parallel. The pro-sheaf $\mathcal{L}og = (\mathcal{L}og^{(n)})_{n \geq 0}$ is the one of Definition 3.1.3 and Definition 4.3.1, respectively.

5.1. **Residue sequences.** As before let $\iota_D : D \to G$ be a closed subscheme which is étale over S and contained in some scheme of torsion points $G[N]$. Of particular interest is the case $D = e(S)$. Recall the localization triangle attached to $j_D : U_D \to X \leftarrow D : \iota_D$. For any $\mathscr{F}$ it defines a connecting morphism

$$R\pi_! Rj_{D*} j_D^* \mathscr{F}[-1] \to R\pi_! \iota_{D!} \iota_D^! \mathscr{F} = \pi_{D!} \iota_D^! \mathscr{F} \ .$$

We apply this to $\mathscr{F} = \mathcal{L}og^{(n)}(d)[2d]$. This is unipotent, so by Lemma 2.8.1, we may replace $\iota_D^!$ by ι_D^*. Moreover, recall the sheaf theoretic and motivic splitting principles 3.2.2 and Lemma 4.4.2, respectively. Together we have a canonical identification

$$\pi_{D!} \iota_D^! \mathcal{L}og^{(n)}(d)[2d] \cong \bigoplus_{i=0}^{n} \pi_{D!} \mathrm{Sym}^i \pi_D^* \mathscr{H}.$$

Definition 5.1.1. The composition of the above morphisms

$$R\pi_! Rj_{D*} j_D^* \mathcal{L}og^{(n)}(d)[2d-1] \to \pi_{D!} \iota_D^! \mathcal{L}og^{(n)}(d)[2d] = \bigoplus_{i=0}^{n} \pi_{D!} \mathrm{Sym}^i \pi_D^* \mathscr{H}$$

is called *residue map at D*.

The residue triangle also induces a connecting homomorphism, also called residue map,

$$\mathrm{Ext}_S^{2d-1}(\mathscr{F}, R\pi_! Rj_{D*} j_D^* \mathcal{L}og^{(n)}(d)) \to \mathrm{Hom}_S(\mathscr{F}, \bigoplus_{i=0}^{n} \pi_{D!} \mathrm{Sym}^i \pi_D^* \mathscr{H}).$$

Lemma 5.1.2 (Functoriality)**.** *The residue map is functorial. More precisely, let* $\varphi : G_1 \to G_2$ *be a morphism of smooth group schemes with connected fibres over* S. *Let* $D_1 \subset G_1$ *and* $D_2 \subset G_2$ *be closed subschemes étale over* S *such that* $\varphi(D_1) \subset D_2$. *Then the morphism*

$$\varphi_\# : \mathcal{L}og_{G_1}^{(n)}(d_1)[2d_1] \to \varphi^! \mathcal{L}og_{G_2}^{(n)}(d_2)[2d_2]$$

of Theorem 3.2.1 and Lemma 4.4.1, respectively, induces a morphism of exact triangles

$$\begin{array}{ccccc}
R\varphi_! \mathcal{L}og_{G_1}^{(n)}(d_1)[2d_1-1] & \longrightarrow & R\varphi_! Rj_{D_1*}j_{D_1}^* \mathcal{L}og_{G_1}^{(n)}(d_1)[2d_1-1] & \longrightarrow & R\varphi_! \iota_{D_1!} \bigoplus_{i=0}^n \mathrm{Sym}^i \pi_{D_1}^* \mathscr{H}_{G_1} \\
\downarrow & & \varphi_\# \downarrow & & \downarrow \\
\mathcal{L}og_{G_2}^{(n)}(d_2)[2d_2-1] & \longrightarrow & Rj_{D_2*}j_{D_2}^* \mathcal{L}og_{G_2}^{(n)}(d_2)[2d_2-1] & \longrightarrow & \iota_{D_2!} \bigoplus_{i=0}^n \mathrm{Sym}^i \pi_{D_2}^* \mathscr{H}_{G_2}
\end{array}$$

Proof. Let c be the relative dimension of G_1 over G_2 and denote by U_{D_i} the complement of D_i and by $U_{\varphi^{-1}D_2} \subset U_{D_1}$ the complement of $\varphi^{-1}D_2$. We apply $j_{D_1*}j_{D_1}^*$ to $\varphi_\#$ and restrict to $U_{\varphi^{-1}D_2}$ and obtain

$$j_{D_1*}j_{D_1}^* \mathcal{L}og^{(n)} \to j_{D_1*}j_{D_1}^* \varphi^! \mathcal{L}og^{(n)}(c)[2c] \to j_{\varphi^{-1}D_2*}j_{\varphi^{-1}D_2}^* \varphi^! \mathcal{L}og^{(n)}(c)[2c].$$

We have a cartesian square

$$\begin{array}{ccc}
U_{\varphi^{-1}D_2} & \xrightarrow{\ j_{\varphi^{-1}D_2}\ } & G_1 \\
\varphi \downarrow & & \downarrow \varphi \\
U_{D_2} & \xrightarrow{\ j_{D_2}\ } & G_2
\end{array}$$

which implies $j_{D_2}^* \varphi^! = \varphi^! j_{\varphi^{-1}D_2}^*$. Together with the base change $Rj_{\varphi^{-1}D_2*}\varphi^! = \varphi^! Rj_{D_2*}$ this gives a map

$$Rj_{D_1*}j_{D_1}^* \mathcal{L}og^{(n)} \to \varphi^! Rj_{D_2*}j_{D_2}^* \mathcal{L}og^{(n)}(c)[2c]$$

or equivalently

$$R\varphi_! Rj_{D_1*}j_{D_1}^* \mathcal{L}og^{(n)}(d_1)[2d_1-1] \to Rj_{D_2*}j_{D_2}^* \mathcal{L}og^{(n)}(d_2)[2d_2-1]$$

The analogous argument for $\iota_{D_1!}\iota_{D_1}^!$ gives

$$R\varphi_! \iota_{D_1!}\iota_{D_1}^! \mathcal{L}og^{(n)}(d_1)[2d_1] \to \iota_{D_2!}j\iota_{D_1}^! \mathcal{L}og^{(n)}(d_2)[2d_2].$$

This defines a morphism of exact triangles. We now apply the identification via the splitting principle on D_1 and D_2.   □

5.2. The main result.
We formulate all results on polylog in two big statements. We keep the notation and the setting of Section 2.

Theorem 5.2.1 (Polylog with respect to the unit section)**.** *Let S be a base scheme satisfying the assumptions of the respective setting, see Section 2. Let G/S be a smooth commutative S-group scheme with connected fibres of dimension d.*

(1) *There is a unique system of classes*

$$\mathrm{pol}^{(n)} \in \mathrm{Ext}_S^{2d-1}(\mathscr{H}_G, R\pi_! Rj_* j^* \mathcal{L}og_G^{(n)}(d))$$

   such that
   (a) *their residue in $e^! \mathcal{L}og_G^{(n)}(d)[2d] \cong \bigoplus_{i=0}^n \mathrm{Sym}^i \mathscr{H}_G$ is the natural inclusion of $\mathscr{H}_G$;*
   (b) *they are compatible under the transition maps $\mathcal{L}og_G^{(n+1)} \to \mathcal{L}og_G^{(n)}$;*

(c) *they are functorial with respect to homomorphisms of groups schemes*
$\varphi : G_1 \to G_2$, *i.e., the diagrams*

$$
\begin{array}{ccc}
\mathscr{H}_{G_1} & \xrightarrow{\mathrm{pol}_{G_1}^{(n)}} & R\pi_{1!}Rj_{1*}j_1^*\mathcal{L}og_{G_1}^{(n)}(d_1)[2d_1-1] \\
{\scriptstyle \varphi_!}\downarrow & & \downarrow{\scriptstyle \varphi_{\#} \ (5.1.2)} \\
\mathscr{H}_{G_2} & \xrightarrow{\mathrm{pol}_{G_2}^{(n)}} & R\pi_{2!}Rj_{2*}j_2^*\mathcal{L}og_{G_2}^{(n)}(d_2)[2d_2-1]
\end{array}
$$

commute.

(2) *The classes* $\mathrm{pol}^{(n)}$ *are contravariantly functorial under morphisms* $S' \to S$.

(3) *If* ℓ *is invertible on* S *which is of finite type over a regular scheme of dimension* 0 *or* 1, *then the motivic class is mapped to the* ℓ-*adic class by the* ℓ-*adic realization functor* R_ℓ.

(4) *If* S *is of finite type over* $\mathbb{C}$, *then the motivic class is mapped to the analytic class by the Betti-realization functor* R_B.

Let $D \subset G$ be a closed subscheme which is étale over S and contained in $G[N]$ for some N.

Definition 5.2.2. Let
$$
\mathbf{Q}[D]^0 := \ker\left(H^0(S, \pi_{D!}\mathbf{Q}) \to H^0(S, \mathbf{Q})\right),
$$
where $\pi_{D!}\mathbf{Q} \to \mathbf{Q}$ is the trace map.

This should be thought of as $\mathbf{Q}$-valued functions f on D with $\sum_{d \in D} f(d) = 0$, which is literally true in the case where D is a disjoint set of sections.

Note that by the isomorphism $\pi_{D!}\iota_D^!\mathcal{L}og_G^{(n)}(d)[2d] \cong \pi_{D!}\bigoplus_{i=0}^n \mathrm{Sym}^i\mathscr{H}_G$ induced by the splitting principle, one has an inclusion
$$
\mathbf{Q}[D]^0 \subset \ker\left(H^0(S, \pi_{D!}\iota_D^!\mathcal{L}og_G^{(n)}) \to H^0(S, \mathbf{Q})\right).
$$

Let $\varphi : G_1 \to G_2$ is a homomorphism of smooth group schemes with connected fibres, $D_1 \subset G_1$ and $D_2 \subset G_2$ as above such that $\varphi(D_1) \subset D_2$. Then the trace map also induces
$$
\varphi_! : \mathbf{Q}[D_1]^0 \to \mathbf{Q}[D_2]^0.
$$

Theorem 5.2.3 (Polylog with respect to a subscheme)**.** *Let* S *be a base scheme satisfying the assumptions of the respective setting, see Section 2. Let* G/S *be a smooth* S-*group scheme with connected fibres of dimension* d. *Let* $D \subset G$ *be a closed subscheme which is étale over* S *and contained in* $G[N]$ *for some* N *and étale. Let* $\alpha \in \mathbf{Q}[D]^0$.

(1) *There is a unique system of classes*
$$
\mathrm{pol}_\alpha^{(n)} \in \mathrm{Ext}^{2d-1}(\mathbf{Q}, R\pi_! j_{D*}j_D^*\mathcal{L}og^{(n)}(d))
$$
such that

(a) *their residue in* $\ker\left(H^0(S, \pi_{D!}\iota_D^!\mathcal{L}og_G^{(n)}) \to H^0(S, \mathbf{Q})\right)$ *is given by* α;

(b) *they are compatible under the transition maps* $\mathcal{L}og_G^{(n+1)} \to \mathcal{L}og_G^{(n)}$;

(c) *they are functorial with respect to homomorphism of group schemes* $\varphi : G_1 \to G_2$ *mapping* $D_1 \subset G_1$ *into* D_2, *i.e., the class* $\mathrm{pol}_\alpha^{(n)}$ *is mapped to* $\mathrm{pol}_{\varphi_!\alpha}^{(n)}$ *under the map*

$$
\varphi_{\#} : \mathrm{Ext}_S^{2d_1-1}(\mathbf{Q}, R\pi_{1!}j_{D*}j_D^*\mathcal{L}og_{G_1}^{(n)}(d_1)) \to \mathrm{Ext}_S^{2d_2-1}(\mathbf{Q}, R\pi_{2!}j_{\varphi D*}j_{\varphi(D)}^*\mathcal{L}og_{G_2}^{(n)}(d_2))
$$

induced from Lemma 5.1.2.

(2) *The classes* $\mathrm{pol}_\alpha^{(n)}$ *are contravariantly functorial under morphisms* $S' \to S$.

(3) *If* ℓ *is invertible on* S *which is of finite type over a regular scheme of dimension* 0 *or* 1, *then the motivic class is mapped to the* ℓ-*adic class by the* ℓ-*adic realization functor* R_ℓ.

(4) *If* S *is of finite typer over* $\mathbb{C}$, *then the motivic class is mapped to the analytic class by the Betti-realization functor* R_B.

Remark 5.2.4. The proof of the theorems are nearly identical and will be given together. We are going to give two different arguments:

- The first proof uses the cohomological vanishing of Theorem 3.3.1. It has the advantage of being quick and direct. The argument is valid in the sheaf theoretic setting and relies on the fact that the polylogarithm classes for G are uniquely determined by their residues and compatibility with respect to n. It also applies in the motivic setting under the more restrictive assumptions of Theorem 4.5.1.
- The second proof is valid in any setting and relies on the fact that the polylogarithm classes for G are uniquely determined by their residues and uses the functoriality with respect to multiplication $[a] : G \to G$ for a single $a \in \mathbb{Z}, a \neq 0, \pm 1$ (satisfying $[a]^*D \subset D$ in the case of polylog with respect to a divisor). Indeed, they are going to be characterized as the unique preimages of their residues on which $[a]$ operates by multiplication by a^1 and a^0, respectively.

Remark 5.2.5. The argument for compatibility with realizations will also apply in Hodge theoretic setting once a Hodge realization functor compatible with the six functor formalism is constructed. This is not yet the case, see the discussion at the end of Section 2.6 for the state of the art.

Remark 5.2.6. In the simplest case $G = \mathbb{G}_m$, the above class is not the same as the one in the literature, but rather maps to it. See Section 6 for the precise relation.

5.3. **First proof.** We work in the sheaf theoretic setting. The same arguments also apply in the motivic setting if the characteristic is 0 or if G/S is affine.

Recall that by Theorem 3.3.1 and Theorem 4.5.1, respectively, we have

$$R\pi_! \mathcal{L}og(d)[2d] = \mathbf{Q}.$$

Proposition 5.3.1. *We work either in the sheaf theoretic setting or the motivic setting with* S *of characteristic* 0 *or* G/S *affine. Let* $\mathscr{F} = \mathscr{H}$ *or* $\mathscr{F} = \mathbf{Q}$. *There is an exact sequence*

$$0 \to \mathrm{Ext}_S^{2d-1}(\mathscr{F}, R\pi_! Rj_{D*}j_D^* \mathcal{L}og(d)) \xrightarrow{\mathrm{res}} \mathrm{Hom}_S(\mathscr{F}, \pi_{D!}\iota_D^* \mathcal{L}og) \to \mathrm{Hom}_S(\mathscr{F}, \mathbf{Q}).$$

where the last map is the composition of the augmentation $\pi_{D!}\iota_D^* \mathcal{L}og \to \pi_{D!}\iota_D^* \mathbf{Q}$ *and the the trace map* $\pi_{D!}\iota_D^* \mathbf{Q} \to \mathbf{Q}$.

Proof. We apply $R\pi_!$ and $\mathrm{Hom}_S(\mathscr{F}, -)$ to the localization triangle and using the computation of $R\pi_! \mathcal{L}og(d)[2d]$.

It remains to show that $\mathrm{Hom}_S(\mathscr{F}, \mathbf{Q})$ vanishes for $\mathscr{F} = \mathscr{H}$ and $\mathscr{F} = \mathbf{Q}$. This is clear in the sheaf theoretic setting because negative Ext-groups vanish.

We now turn to the motivic setting. If $\mathscr{F} = \mathbf{Q}$, the vanishing of $\mathrm{Hom}_S(\mathbf{Q}, \mathbf{Q}[-1])$ is [Ay14, Proposition 11.1]. If $\mathscr{F} = \mathscr{H}$, then

$$\mathrm{Hom}_S(\mathscr{H}[1], \mathbf{Q}) = \mathrm{Hom}_S(M_1(G), \mathbf{Q}) \subset \mathrm{Hom}_S(M_S(G), \mathbf{Q}) = \mathrm{Hom}_G(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q}$$

again by [Ay14, Proposition 11.1]. The morphism $\mathrm{Hom}_S(\mathbf{Q}, \mathbf{Q}) \to \mathrm{Hom}_G(\mathbf{Q}, \mathbf{Q})$ is an isomorphism, hence the direct summand $\mathrm{Hom}_S(M_1(G), \mathbf{Q})$ vanishes. $\qquad\square$

Proof of Theorem 5.2.1 and Theorem 5.2.3. We first apply Proposition 5.3.1 with $\mathscr{F} = \mathscr{H}$ and $D = e(S)$. We obtain the exact sequence

$$0 \to \mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_* j^* \mathcal{L}og(d)) \to \mathrm{Hom}_S(\mathscr{H}, \prod_{i=0}^{\infty} \mathrm{Sym}^i \mathscr{H}) \to \mathrm{Hom}_S(\mathscr{H}, \mathbf{Q}).$$

with the last map induced from the natural projection. We define

$$\mathrm{pol} \in \mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_* j^* \mathcal{L}og(d))$$

as the preimage of the natural inclusion of the $\mathscr{H}$ into $\prod_{i=0}^{\infty} \mathrm{Sym}^i \mathscr{H}$. This means we have defined a system of elements

$$\mathrm{pol}^{(n)} \in \mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_* j^* \mathcal{L}og^{(n)}(d))$$

compatible under transition maps. It is uniquely determined by these properties.

We now turn to functoriality under $\varphi : G_1 \to G_2$. By functoriality, pol_{G_1} and pol_{G_2} both define elements in $\mathrm{Ext}_S^{2d_2-1}(\mathscr{H}_{G_1}, R\pi_{2!} Rj_{2*} j_2^* \mathcal{L}og_{G_2}(d_2))$ with the same residue in $\mathrm{Hom}_S(\mathscr{H}_{G_1}, \prod_{i=0}^{\infty} \mathrm{Sym}^i \mathscr{H}_{G_2})$. By Proposition 5.3.1 this implies that they agree.

The behaviour under realizations follows from these properties for $\mathcal{L}og$ (see Proposition 4.6.1) and uniqueness.

In the case of pol_α, we obtain the sequence

$$0 \to \mathrm{Ext}_S^{2d-1}(\mathbf{Q}, R\pi_! Rj_{D*} j_D^* \mathcal{L}og(d)) \to \mathrm{Hom}_S(\mathbf{Q}, \prod_{i=0}^{\infty} \pi_{D!} \mathrm{Sym}^i \pi_D^* \mathscr{H}) \to \mathrm{Hom}_S(\mathbf{Q}, \mathbf{Q}).$$

By assumption α is in the kernel of the last map. We define pol_α as its preimage. All other argument are the same as in the case of pol with respect to the unit section. $\qquad\square$

5.4. **Second proof.** We work in the sheaf theoretic and in the motivic setting in parallel. The argument relies on analysing the eigenspace decomposition under the operation of multiplication by $a \in \mathbb{Z}$ on G. Let $[a] : G \to G$ be the morphism on G.

Recall that an $[a]$-linear operation on an object $X \in \mathbf{D}(G)$ is the datum of a morphism $X \to [a]^! X$ or equivalently $f_a : [a]_! X \to X$. By naturality it induces a map $\pi_! X = \pi_! [a]_! X \xrightarrow{\pi_! f_a} \pi_! X$.

Such an $[a]$-linear operation on $\mathcal{L}og^{(n)}$ was defined in Theorem 3.2.1 and Theorem 4.4.1, respectively.

Recall also from Appendix A the notion of a finite decomposition into generalized $[a]$-eigenspaces in a $\mathbb{Q}$-linear triangulated category.

Proposition 5.4.1. *Let $a \in \mathbb{Z}$.*

(1) *Then $R\pi_!\mathbf{Q}$ has a finite decomposition into a-eigenspaces*

$$R\pi_!\mathbf{Q} = \bigoplus_{i=0}^{\mathrm{kd}(G)} \mathrm{Sym}^i \mathscr{H}(-d)[i-2d]$$

with a operating on $\mathrm{Sym}^i \mathscr{H}$ by multiplication by a^i.

(2) *Let $n \geq 0$. Under the operation of $[a]$ on the associated graded of $\mathcal{L}og^{(n)}$, the object $R\pi_!\pi^*\mathrm{Sym}^n\mathscr{H}$ on S has a finite decomposition into $[a]$-eigenspaces with eigenvalues $a^n, \ldots, a^{n+\mathrm{kd}(G)}$.*

(3) *The object $R\pi_!\mathcal{L}og^{(n)}$ on S has a finite decomposition into generalized $[a]$-eigenspaces with eigenvalues $a^0, \ldots, a^{n+\mathrm{kd}(G)}$.*

(4) *For $n \geq 1$ the map $R\pi_!\mathcal{L}og^{(n)} \to R\pi_!\mathcal{L}og^{(n-1)}$ induces an isomorphism on a^0-eigenspaces. In particular, this eigenspace is isomorphic to $\mathbb{Q}_S(-d)[-2d]$.*

(5) *For $n \geq 1$, the a^1-eigenspace of $R\pi_!\mathcal{L}og^{(n)}$ vanishes.*

The decompositions are independent of the choice of a.

Proof. We have the formula

$$R\pi_!\mathbf{Q} = R\pi_!\pi^!\mathbf{Q}(-d)[-2d] = \bigoplus_{i=0}^{\mathrm{kd}(G)} \mathrm{Sym}^i \mathscr{H}(-d)[i-2d]$$

hence it suffices to show that $[a]$ operates as multiplication by a on $\mathscr{H}$. The motivic case is established in [AHP, Theorem 3.3.] (it follows directly from the description of $M_1(G)$ as the motive induced by $\underline{G}_\mathbb{Q}$). The sheaf theoretic case is classical. It also follows immediately from the motivic case and compatibility under realizations. This finishes the proof of the first claim.

By Theorem 3.2.1 and Theorem 4.4.1, the operation of $[a]$ on $\pi^*\mathrm{Sym}^n\mathscr{H}$ under the functoriality of $\mathcal{L}og^{(n)}$ is given by $\mathrm{Sym}^n[a]_! = a^n$. By the projection formula

$$R\pi_!\pi^*\mathrm{Sym}^n\mathscr{H} = (R\pi_!\mathbf{Q}) \otimes \mathrm{Sym}^n\mathscr{H}.$$

Hence the second statement follows from the first.

For the third assertion, consider the exact triangle

$$R\pi_!\mathrm{Sym}^n\mathscr{H} \to R\pi_!\mathcal{L}og^{(n)} \to R\pi_!\mathcal{L}og^{(n-1)} \ .$$

By induction and Proposition A.0.6 , we get a decomposition for $R\pi_!\mathcal{L}og^{(n)}$ with eigenvalues as stated. Passing to the a^0-eigenspace preserves exact triangles by the same Proposition A.0.6. There is no contribution from $R\pi_!\mathrm{Sym}^n\mathscr{H}$ for $n \geq 1$. In the case $n = 0$, the contribution is the component $i = 0$ in assertion (1).

We now consider the generalized eigenspace for the eigenvalue a^1. There is no contribution from $R\pi_!\mathrm{Sym}^n\mathscr{H}$ for $n \geq 2$. Hence it suffices to show the vanishing for $n = 1$. We pass to the a^1-eigenspace in the triangle for $n = 1$ and have

$$\mathscr{H} \otimes \mathbf{Q}(-d)[-2d] \to \ ? \ \to \mathscr{H}(G)(-d)[1-2d].$$

It remains to show that the connecting morphism is the identity. In the sheaf theoretic case, this is true by definition of $\mathcal{L}og^{(1)}$, see Definition 3.1.2. In the motivic case, this was checked during the proof of Proposition 4.6.1 on compatibility of the motivic logarithm with realizations.

Let $a \neq b$ be integers. Note that $[a]$ and $[b]$ commute. By Lemma A.0.7, the object $\mathcal{L}og^{(n)}$ has a simultaneous decomposition into generalized eigenspaces with respect to both. We show inductively that the generalized eigenspaces for a^i and b^i agree from the same statement for $\mathrm{Sym}^i(\mathscr{H})$. $\square$

Consider $e : S \to G$. Recall from Lemma 5.1.2 (with $\varphi = [a]$, $D_1 = D_2 = e(S)$) that there is an $[a]$-linear operation on the residue sequence

$$e_! e^! \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n)} \to Rj_* j^* \mathcal{L}og^{(n)}$$

compatible with the operation on $\mathcal{L}og^{(n)}$.

Proposition 5.4.2.   (1) *We have*

$$R\pi_! Re_! e^! \mathcal{L}og^{(n)} = e^! \mathcal{L}og^{(n)} = \bigoplus_{i=0}^{n} \mathrm{Sym}^n \mathscr{H}(-d)[-2d]$$

   and $[a]$ operates on the i-th summand by multiplication by a^i.
   (2) *The object $R\pi_! Rj_* j^* \mathcal{L}og^{(n)}(-d)$ has a finite decomposition into generalized eigenspaces for the operation of $[a]$ with $a \in \mathbb{Z}$. The eigenvalues are a^i for $1 \le i \le n + \mathrm{kd}(G)$.*
   (3) *For $a \ne \pm 1$, the generalized $[a]$-eigenspace of $R\pi_! Rj_* j^* \mathcal{L}og^{(n)}(-d)$ for the eigenvalue a^i is given by $\mathrm{Sym}^i \mathscr{H}(-d)[-2d+1]$ via the residue map. It is actually an eigenspace, i.e., $[a]$ operates by multiplication by a^i.*
   The decomposition is independent of the choice of a.

Proof. The formula for $e^! \mathcal{L}og^{(n)}$ is given in Lemma 4.3.2. The operation of $[a]$ is the same as on the associated gradeds of $\mathcal{L}og^{(n)}$. By Theorem 3.2.1 and Theorem 4.4.1, respectively, it has the shape claimed in the Proposition.

Consider the triangle on G

$$e_! e^! \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n)} \to Rj_* j^* \mathcal{L}og^{(n)} \ .$$

It induces an exact triangle on S

$$\bigoplus_{i=0}^{n} \mathrm{Sym}^i \mathscr{H}(-d)[-2d] \to R\pi_! \mathcal{L}og^{(n)} \to R\pi_! Rj_* j^* \mathcal{L}og^{(n)}.$$

By the first assertion and Proposition 5.4.1, the first two objects have a finite decomposition into generalized $[a]$-eigenvalues with eigenvalues as stated. Hence by Proposition A.0.6 the object on the right also has a finite decomposition into generalized eigenspaces. We pass to the generalized eigenspace for the eigenvalue a^1 and get

$$\mathscr{H}(-d)[-2d] \to 0 \to?$$

This proves the last assertion.

The decompositions are independent of a by Lemma A.0.7 because the different $[a]$ commute and the assertion is true for $\mathrm{Sym}^i \mathscr{H}$. $\square$

As before let $\iota_D : D \to G$ be the inclusion of a closed subscheme which is étale over S and contained in $G[N]$ for some N. Let $a \in \mathbb{Z}$ such that $[a]^{-1}D \subset D$. Recall from Lemma 5.1.2 (with $\varphi = [a]$, $D_1 = D_2 = D$) that there is an $[a]$-linear operation on the residue sequence

$$R\pi_{D!} \pi_D^! \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n)} \to Rj_{D*} j_D^* \mathcal{L}og^{(n)}$$

compatible with the operation on $\mathcal{L}og^{(n)}$.

Proposition 5.4.3. *Let $\iota_D : D \to G$ be as before. Let $a \in \mathbb{Z}$ such that $D \subset [a]^{-1}D$.*
   Then the object $R\pi_! Rj_{D*} j_D^* \mathcal{L}og^{(n)}(d)$ has a finite decomposition into generalized eigenspaces for the operation of $[a]$.

For $a \neq \pm 1, 0$, the generalized $[a]$-eigenspace for the eigenvalue a^0 sits in a distinguished triangle

$$\left(R\pi_! R j_{D*} j_D^* \mathcal{L}og^{(n)}(-d)\right)^{a^0} \to R\pi_{D!}\mathbb{Q}[-2d+1] \to \mathbb{Q}[-2d+1]$$

via the residue map.

If $a, b \in \mathbb{Z}$ are integers with $D \subset [a]^{-1}D, [b^-]D \subset [ab]^{-1}D$, then they the decompositions with respect to a and b agree.

Remark 5.4.4. The assumptions on a are satisfied if $D \subset G[N]$ and $a \equiv 1 \mod N$.

Proof. The arguments are the same as in the proof of Proposition 5.4.2. It remains to compute explicitly for the eigenvalue a^0. We apply $R\pi_!$ to the localization triangle and pass to the generalized $[a]$-eigenspace for the eigenvalue a^0. The eigenspace for $R\pi_! \mathcal{L}og^{(n)}$ was computed in Proposition 5.4.1 (3). The eigenspace for

$$R\pi_! \iota_{D!} \iota_D^! \mathcal{L}og^{(n)}(d) = R\pi_{D!} \iota_D^* \mathcal{L}og^{(n)}[-2d] = \pi_{D!} \bigoplus_{i=0}^{n} \mathrm{Sym}^i \pi_D^* \mathcal{H}[-2d]$$

is given by the summand for $i = 0$.

Under the compatibility assumption on a and b, it is easy to check along the lines of the proof of Lemma 5.1.2 that the induced operations commute. Hence the decompositions agree by Lemma A.0.7. $\square$

Second Proof of Theorem 5.2.1 and Theorem 5.2.3. We want to construct an element in $\mathrm{Ext}_S^{2d-1}(\mathcal{H}, \pi_! j_* \mathcal{L}og^{(n)}|_U(d))$ Choose $a \in \mathbb{Z}$, $a \neq \pm 1, 0$. We define

$$\mathrm{pol}^{(n)} \in \mathrm{Ext}_S^{2d-1}(\mathcal{H}, \pi_! j_* \mathcal{L}og^{(n)}|_U(d))$$

be the unique preimage of $\mathrm{id} \in \mathrm{Hom}(\mathcal{H}, \bigoplus_{i=0}^n \mathrm{Sym}^n \mathcal{H})$ under the residue map of Definition 5.1.1 such that $\mathrm{pol}^{(n)}$ maps to the generalized $[a]$-eigenspace of $\pi_! j_* \mathcal{L}og^{(n)}$ with eigenvalue a^1.

By construction it is compatible under restriction and with the realization functors. By uniqueness, it is also functorial with respect to group homomorphisms $\varphi : G_1 \to G_2$. In particular, $\mathrm{pol}^{(n)}$ is independent of the choice of a.

Now let $\alpha \in \mathbb{Q}[D]^0$. We choose $a \in \mathbb{Z}$ with $a \neq \pm 1, 0$ such that $[a]^{-1}D \subset D$, e.g., $a \equiv 1 \mod N$ with $D \subset G[N]$. We define

$$\mathrm{pol}_\alpha^{(n)} \in \mathrm{Ext}^{2d-1}(\mathbf{Q}, \pi_! j_{D*} \mathcal{L}og^{(n)}(d))$$

as be the unique preimage of α under the residue map of Definition 5.1.1 which maps to the generalized $[a]$-eigenspace of $\pi_! j_{N!} \mathcal{L}og^{(n)}$ for the eigenvalue a^0. By construction, it is compatible under restriction and with realization functors. By uniqueness, it is also functorial with respect to group homomorphisms $\varphi : G_1 \to G_2$ such that $\varphi^{-1}D_2 \subset D_1$. In particular, it is independent of the choice of a. $\square$

6. Comparison with other definitions of the polylog

We work in the sheaf theoretic and in the motivic setting in parallel.

In order to relate our constructions to the existing literature, we also need a version of polylog with respect to $R\pi_*$.

6.1. **Comparing $R\pi_!$ and $R\pi_*$.** Recall that there is always a natural map of functors $R\pi_! \to R\pi_*$.

If $D \subset G$ is finite étale over S, then there is a commutative diagram

$$(10) \quad R\pi_! Rj_{D*} j_D^* \mathcal{L}og(d))[2d] \xrightarrow{\quad\text{comp}\quad} R\pi_* Rj_{D*} j_D^* \mathcal{L}og(d)[2d]$$

$$\pi_{D*}\iota_D^* \mathcal{L}og)[1].$$

Let $D \subset G$ be finite étale over S and contained in $G[N]$ for some N. By applying $R\pi_*$ instead of $R\pi_!$, we obtain another variant of the residue triangle:

$$R\pi_* \iota_{D!}\iota_D^! \mathcal{L}og^{(n)}(d)[2d] \to R\pi_* \mathcal{L}og^{(n)}(d)[2d] \to R\pi_* Rj_{D*} j_D^* \mathcal{L}og^{(n)}(d)[2d].$$

Again under the identification of Definition 3.1.3 and Definition 4.3.1 and because ι_D is proper, we have

$$R\pi_* \iota_{D!}\iota_D^! \mathcal{L}og^{(n)}(d)[2d] = R\pi_{D*} \bigoplus_{i=0}^{n} \pi_{D!}\operatorname{Sym}^i \pi_D^* \mathcal{H}.$$

Hence the connecting morphism induces by adjunction another map, again called residue map,

$$\operatorname{Ext}_{U_D}^{2d-1}(\mathscr{F}, j_D^* \mathcal{L}og^{(n)}(d)) \to \operatorname{Hom}_S(\mathscr{F}, \bigoplus_{i=0}^{n} \pi_{D*}\operatorname{Sym}^i \pi_D^* \mathcal{H}).$$

Lemma 6.1.1. *Let $\mathscr{F}$ be an object of $\mathbf{D}(S)$. There is an exact sequence*

$$\operatorname{Ext}_{U_D}^{2d-1}(j_D^* \mathscr{F}, j_D^* \mathcal{L}og(d)) \xrightarrow{\text{res}} \operatorname{Hom}_S(\mathscr{F}, \pi_{D*}\iota_D^* \mathcal{L}og) \to \operatorname{Hom}_S(\mathscr{F}, \mathbf{Q}).$$

In the sheaf theoretic setting, let $\mathscr{F}$ be a sheaf on S. Then the residue map is injective.

Proof. Same argument as for $R\pi_!$, see Lemma 5.3.1. $\qquad\qquad\square$

6.2. **Polylog with $R\pi_*$.** The map comp from (10) induces maps

$$(11) \quad \operatorname{Ext}_S^{2d-1}(\mathcal{H}, R\pi_! Rj_* j^* \mathcal{L}og^{(n)}(d)) \longrightarrow \operatorname{Ext}_S^{2d-1}(\mathcal{H}, R\pi_* Rj_* j^* \mathcal{L}og^{(n)}(d))$$

$$\operatorname{Ext}_U^{2d-1}(\pi_U^* \mathcal{H}, j^* \mathcal{L}og^{(n)}(d))$$

and similarly

$$(12) \quad \operatorname{Ext}_S^{2d-1}(\mathbf{Q}, R\pi_! Rj_{D*} j_D^* \mathcal{L}og^{(n)}(d)) \to \operatorname{Ext}_{U_D}^{2d-1}(\mathbf{Q}, j_D^* \mathcal{L}og^{(n)}(d)).$$

We define the polylog with respect to $R\pi_*$ as the image of the polylog under these maps.

Definition 6.2.1. We denote by

$$\overline{\operatorname{pol}}^{(n)} \in Ext_U^{2d-1}(\pi_U^* \mathcal{H}, j^* \mathcal{L}og^{(n)}(d))$$

the image of $\operatorname{pol}^{(n)}$ under the map (11) and for $\alpha \in \mathbb{Q}[D]^0$, we denote by

$$\overline{\operatorname{pol}}_\alpha^{(n)} \in \operatorname{Ext}_{U_D}^{2d-1}(\mathbf{Q}, j_D^* \mathcal{L}og^{(n)}(d))$$

the image of $\operatorname{pol}_D^{(n)}$ under the map (12).

These classes have the advantage of having an interpretation on U and U_D, respectively. They have the disadvantage of having a more restrictive functoriality.

Proposition 6.2.2.   (1) $\overline{\mathrm{pol}}^{(n)}$ and $\overline{\mathrm{pol}}_\alpha^{(n)}$ are compatible under the transition maps $\mathcal{L}og^{(n)} \to \mathcal{L}og^{(n-1)}$. We write $\overline{\mathrm{pol}} \in \mathrm{Ext}_U^{2d-1}(\pi_U^* \mathscr{H}, j^* \mathcal{L}og(d))$ and $\overline{\mathrm{pol}}_\alpha \in \mathrm{Ext}_{U_D}^{2d-1}(\mathbf{Q}, j_D^* \mathcal{L}og(d))$ for the resulting classes.
  (2) $\overline{\mathrm{pol}}^{(n)}$ and $\overline{\mathrm{pol}}_\alpha^{(n)}$ are contravariantly functorial in the base scheme S.
  (3) The image of $\overline{\mathrm{pol}}^{(n)}$ under the residue map is given by the natural inclusion of $\mathscr{H}$ into $\bigoplus_{n=0}^n \mathrm{Sym}^i \mathscr{H}$.
  (4) The image of $\overline{\mathrm{pol}}_\alpha^{(n)}$ under the residue map is given by α.
  (5) Let $\varphi : G_1 \to G_2$ be a proper morphism of S-group schemes.
    (a) The diagram

$$
\begin{array}{ccc}
\varphi_* \mathscr{H}_{G_1} & \xrightarrow{\mathrm{pol}_{G_1}^{(n)}} & \varphi_* j_1^* \mathcal{L}og_{G_1}^{(n)}(d_1)[2d_1 - 1] \\
\varphi \downarrow & & \downarrow \varphi_\# \\
\mathscr{H}_{G_2} & \xrightarrow{\mathrm{pol}_{G_2}^{(n)}} & j_2^* \mathcal{L}og_{G_2}^{(n)}(d_2)[2d_2 - 1]
\end{array}
$$

    commutes.
    (b) the class $\overline{\mathrm{pol}}^{(n)} \alpha$ is mapped to $\overline{\mathrm{pol}}_{\varphi_! \alpha}^{(n)}$ under

$$
\varphi_\# : \mathrm{Ext}_{G_1}^{2d_1 - 1}(\mathbf{Q}, j_D^* \mathcal{L}og_{G_1}^{(n)}(d)) \to \mathrm{Ext}_{G_2}^{2d_2 - 1}(\mathbf{Q}, j_{\varphi(D)}^* \mathcal{L}og_{G_2}^{(n)}(d)).
$$

Proof. The argument as the same as in the proof of Theorem 5.2.1. The main ingredient is the functoriality of $\mathcal{L}og^{(n)}$ in Theorem 3.2.1.   □

Functoriality is of particular interest in the case where φ is an isogeny, e.g., multiplication by N with N invertible on S.

Remark 6.2.3. It is not clear in general if $\overline{\mathrm{pol}}^{(n)}$ and $\overline{\mathrm{pol}}_\alpha^{(n)}$ are uniquely determined by their residues. In a more special geometric situation, which covers the cases in the existing literature, uniqueness is at least true in the sheaf theoretic setting.

Proposition 6.2.4. *In the sheaf theoretic setting, the map*

$$
\mathrm{comp} : \mathrm{Ext}_S^{2d-1}(\mathscr{H}, R\pi_! Rj_{D*} j_D^* \mathcal{L}og(d)) \to \mathrm{Ext}_{G \backslash D}^{2d-1}(\mathscr{H}, \mathcal{L}og(d))
$$

is an isomorphism, if either

  (1) G *is an abelian scheme,*
  (2) G *is an extension of an abelian scheme A/S of dimension g by a torus T/S of dimension r, and the considered sheaf theory admits weights.*

In these cases $\overline{\mathrm{pol}}$ is uniquely determined by its compatibility under the restriction maps or by functoriality for some $a \in \mathbb{Z}$, $a \neq 0, \pm 1$.

Note that in the second case $\mathscr{H}$ is a lisse of rank $h = 2g + r$.

Proof. If G is an abelian scheme, the map comp is just the natural adjunction, hence an isomorphism and there is nothing to show.

Now let G be an extension of A/S by T/S as in the statement. Let $h := \dim_{\mathbf{Q}} \mathscr{H} = 2g + r$, then by Theorem 3.3.1 one has

$$\operatorname{Ext}_G^j(\pi^*\mathscr{H}, \mathcal{L}og(d)) = \operatorname{Ext}_S^j(\mathscr{H}, R\pi_*\mathcal{L}og(d)) \cong \operatorname{Ext}_S^{j-h}(\mathscr{H}, \mathbf{Q}).$$

The weights of $\mathscr{F} = \mathscr{H}$ are ≤ -1 and the Ext-groups $\operatorname{Ext}_S^{j-h}(\mathscr{F}, \mathbf{Q})$ vanish. Then the localization sequence gives rise, with the arguments from Proposition 5.3.1, to an isomorphism

$$\operatorname{Ext}_{G\backslash D}^{2d-1}(\pi^*\mathscr{H}, \mathcal{L}og(d)) \cong \operatorname{Hom}_S(\mathscr{H}, \pi_{D*}\iota_D^*\mathcal{L}og)$$

because $\operatorname{Hom}_S(\mathscr{H}, \mathbf{Q}) = 0$. Together with 5.3.1 this shows that comp is an isomorphism. $\square$

6.3. **Special cases.** We review the existing literature and how the present paper fits. In all cases, it is $\overline{\operatorname{pol}}^{(n)}$ and $\overline{\operatorname{pol}}_\alpha^{(n)}$ defined in Definition 6.2.1 that appears. Recall that for abelian schemes one has $\operatorname{pol}^{(n)} = \overline{\operatorname{pol}}^{(n)}$. By Proposition 6.2.4, the class $\overline{\operatorname{pol}}^{(n)}$ is not identical, but has the same information as $\operatorname{pol}^{(n)}$, at least in the sheaf theoretic setting.

(1) If $G = \mathbb{G}_m$, then we are in the situation of the classical polylog on the projective line minus three points. Its sheaf theoretic construction by Deligne in [Del89] was the starting point of the whole field. The motivic construction over $S = \mathbb{Z}$ (that is enough by functoriality) is due to Beilinson and Deligne. Full details can be found in [HuWi98] by Huber and Wildeshaus. We are going to explain this case in more detail below.

(2) If $G = E$ is an elliptic curve, it agrees with the sheaf theoretic polylog for elliptic curves as defined by Beilinson and Levin [BeLe91]. They also constructed the motivic elliptic polylog. Their treatment served as the role model for all later definitions of the polylogarithm.

(3) If $G = A$ is abelian and S is regular, the motivic polylog constructed in the present paper agrees with the one constructed by the second author in [Ki99]. In this paper the decomposition under the $[a]$-operation, as used by Beilinson and Levin, was amplified and made into a flexible tool, which motivated the approach in the present paper.

(4) If the considered sheaf theory admits weights and G is an extension of an abelian scheme by a torus, then the polylogarithm class $\overline{\operatorname{pol}}^{(n)}$ of Definition 6.2.1

(13) $$\overline{\operatorname{pol}} \in \operatorname{Ext}_{G\backslash\{e\}}^{2d-1}(\pi^*\mathscr{H}, \mathcal{L}og(d))$$

agrees with the polylogarithm defined by Wildeshaus in [Wi97, page 161]. In particular, we achieve the construction of the motivic classes inducing his sheaf theoretic polylogarithm.

6.4. **Classical polylog.** As the case $G = \mathbb{G}_m$ is of particular interest, and our approach is a considerable technical simplification of the existing motivic construction in [HuWi98], we spell out the details. It suffices to consider $S = \operatorname{Spec}\mathbb{Z}$. We work in the motivic and sheaf theoretic setting in parallel.

Lemma 6.4.1. *For $G = \mathbb{G}_m$ we have*

$$M_1(G) = \mathbf{Q}(1)[1], \quad \mathscr{H}_G = \mathbf{Q}(1), \ \text{and} \ \operatorname{Sym}^k\mathscr{H} = \mathbf{Q}(k).$$

Moreover,

$$R\pi_!\mathbf{Q}(k) = \mathbf{Q}(k) \oplus \mathbf{Q}(k+1)[1].$$

with the splitting induced by the unit section.

Proof. The first statement is a classical computation of Voevodsky: $\mathbb{Z}(1)[1]$ is represented by the sheaf $\mathcal{O}^* = \underline{\mathbb{G}_m}$. [Voe00, Theorem 3.4.2]. All the others follow. $\square$

This means that $\mathcal{L}og^{(n)}$ is an iterated extension of Tate motives/sheaves on $\mathbb{G}_m$.

Definition 6.4.2. Let S be finite dimensional and noetherian. The *triangulated category* $D_{\mathbf{MT}}(S)$ *of mixed Tate motives* on S is defined as the full triangulated subcategory of $\mathbf{DA}(S)$ generated by $\mathbf{Q}(k)$ for $k \in \mathbb{Z}$.

Note that this category is closed under tensor products and duality.
We say that Tate motives on S satisfy the Beilinson-Soulé vanishing conjectures if

$$\operatorname{Hom}_{\mathbf{DA}(S)}(\mathbf{Q}(i), \mathbf{Q}(j)[N]) = 0$$

for all $N < 0$. This implies the existence of a t-structure on $D_{\mathbf{MT}}(\operatorname{Spec}\mathbb{Z})$ such that the Betti- or ℓ-adic realizations are t-exact and conservative.

Definition 6.4.3. Let $\mathbf{MT}(S)$ be the *abelian category of mixed Tate motives* on S be defined as the heart of the motivic t-structure on $D_{\mathbf{MT}}(\operatorname{Spec}\mathbb{Z})$.

Lemma 6.4.4. *Tate motives on* $\operatorname{Spec}\mathbb{Z}$, $\mathbb{G}_m$ *and* U *satisfy the Beilinson-Soulé vanishing conjectures.*

Proof. Borel's computation of higher algebraic K-theory of $\mathbb{Z}$ implies the case of $S = \operatorname{Spec}\mathbb{Z}$.
For $S = \mathbb{G}_m$ we consider

$$
\begin{aligned}
\operatorname{Hom}_{\mathbb{G}_m}(\mathbf{Q}(i), \mathbf{Q}(j)[N]) &= \operatorname{Hom}_{\operatorname{Spec}\mathbb{Z}}(R\pi_!\pi^!\mathbf{Q}(i), \mathbf{Q}(j)[N]) \\
&= \operatorname{Hom}_{\operatorname{Spec}\mathbb{Z}}(\mathbf{Q}(i) \oplus \mathbf{Q}(i+1)[1], \mathbf{Q}(j)[N]) \\
&= \operatorname{Hom}_{\operatorname{Spec}\mathbb{Z}}(\mathbf{Q}(i), \mathbf{Q}(j)[N]) \oplus \operatorname{Hom}_{\operatorname{Spec}\mathbb{Z}}(\mathbf{Q}(i+1), \mathbf{Q}(j)[N-1]).
\end{aligned}
$$

Both summands vanish for $N < 0$.
For $S = U$ consider the localizing triangle

$$R\pi_{U!}\pi_U^!\mathbf{Q}(i) \to R\pi_!\pi^!\mathbf{Q}(i) \to e_*e^*\mathbf{Q}(i+1)[2]$$

and the long exact sequence for $\operatorname{Hom}_{\operatorname{Spec}\mathbb{Z}}(\cdot, \mathbf{Q}(j)[N])$ to get the same vanishing. $\square$

Corollary 6.4.5. *The motives* $\mathcal{L}og^{(n)}$ *and* $j^*\mathcal{L}og^{(n)}$ *are objects of* $\mathbf{MT}(\mathbb{G}_m)$ *and* $\mathbf{MT}(U)$, *respectively.*
The motives $R\pi_!\mathcal{L}og^{(n)}$ *and* $R\pi_!j_*j^*\mathcal{L}og^{(n)}$ *are objects of the triangulated category of mixed Tate motives on* $\operatorname{Spec}\mathbb{Z}$.

Proof. Immediate from the triangle

$$\mathbf{Q}(n) \to \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n-1)}$$

the computation of $R\pi_!\pi^!\mathbf{Q}$. $\square$

Hence the spectral sequence computation of Section 7 and its conclusion in Theorem 3.3.1 are also true in the motivic setting. Note that the argument simplifies considerably in this special case, see [HuKi99, Appendix A] for the cohomological case. The homological case agrees with this up to a shift because $\mathbf{Q}(i)^\vee = \mathbf{Q}(-i)$.

Corollary 6.4.6. *The localization sequence with respect to the unit section* $e :$ $\mathrm{Spec}\mathbb{Z} \to \mathbb{G}_m$ *induces a long exact sequence*

$$\mathbf{Q}(-1) \to R^1\pi_! j_* j^* \mathcal{L}og^{(n)}(1) \to \bigoplus_{k=0}^{n} \mathbf{Q}(k) \to 0$$

of mixed Tate motives.

Moreover, the proof of Proposition 6.2.4 also applies in the motivic setting because the theory of mixed Tate motives has weights.

Definition 6.4.7. Let $\mathrm{pol}^{(n)} \in \mathrm{Ext}^1_{\mathrm{Spec}\mathbb{Z}}(\mathbf{Q}(1), R\pi_! j_* j^* \mathcal{L}og^{(n)})$ be the unique element with residue the natural inclusion $\mathbf{Q}(1) \to \bigoplus_{k=0}^{n} \mathbf{Q}(k)$.

Let $\overline{\mathrm{pol}}^{(n)} \in \mathrm{Ext}^1_{\mathbb{G}_m}(\mathbf{Q}(1), j^* \mathcal{L}og^{(n)})$ be the unique element with residue the natural inclusion $\mathbf{Q}(1) \to \bigoplus_{k=0}^{n} \mathbf{Q}(k)$.

Remark 6.4.8.   (1) The analogous discussion can also be carried out for $\mathrm{pol}^{(n)}_\alpha$. It involves Artin-Tate motives because $R\pi_{D!}\pi^!_D \mathbf{Q}$ is Artin-Tate. Borel's result on motivic cohomology is still available. We omit the precise formulation.
    (2) The same arguments are also valid for all tori over a base S where Tate motives satisfy the Beilinson-Soulé vanishing conjectures.

7. PROOF OF THE VANISHING THEOREM

7.1. **Proof of Theorem 3.3.1.** We work in the sheaf theoretic setting.

Before we give the proof we start with some general remarks concerning $R\pi_! \mathbf{Q}$ and the definition of $\mathcal{L}og^{(1)}$. First note that the group multiplication $\mu : G \times_S G \to G$ induces a product

$$\mu : R^i\pi_! \mathbf{Q}(d) \otimes R^j\pi_! \mathbf{Q}(d) \to R^{i+j-2d}\pi_! \mathbf{Q}(d)$$

and the diagonal $\Delta : G \to G \times_S G$ a coproduct

$$\Delta : R^i\pi_! \mathbf{Q}(d) \to \bigoplus_j R^j\pi_! \mathbf{Q}(d) \otimes R^{2d+i-j}\pi_! \mathbf{Q}(d).$$

In particular, $\bigoplus_i R^i\pi_! \mathbf{Q}(d)$ is a Hopf algebra and a direct computation shows that $R^{2d-1}\pi_! \mathbf{Q}(d) = \mathcal{H}$ are the primitive elements. As usual we get an isomorphism

$$R^i\pi_! \mathbf{Q}(d) \cong \bigwedge^{2d-i} \mathcal{H}.$$

Recall that we have given a description of $\mathcal{L}og^{(1)}$ in terms of the comultiplication in Lemma 3.1.4.

We want to compute $R\pi_! \mathcal{L}og$ by using the spectral sequence arising from the unipotent filtration on $\mathcal{L}og$. For this we need to identify the connecting homomorphisms.

Lemma 7.1.1. *The connecting homomorphism*

$$R^i\pi_! \mathbf{Q} \to R^{i+1}\pi_! \pi^* \mathcal{H} \cong R^{i+1}\pi_! \mathbf{Q} \otimes \mathcal{H}$$

of the long exact cohomology sequence of

$$0 \to \pi^* \mathcal{H} \to \mathcal{L}og^{(1)} \to \mathbf{Q} \to 0$$

is given (up to sign) by the composition of the comultiplication

$$\Delta : R^i\pi_!\mathbf{Q} \to \bigoplus_j R^j\pi_!\mathbf{Q} \otimes R^{2d+i-j}\pi_!\mathbf{Q}(d)$$

with the projection onto $R^{i+1}\pi_!\mathbf{Q} \otimes R^{2d-1}\pi_!\mathbf{Q}(d)$.

Proof. This is completely formal. The comultiplication is obtained by applying $R(\pi \times \pi)_!$ to

$$\Delta_!\Delta^!(\pi \times \pi)^!\mathbf{Q} \to (\pi \times \pi)^!\mathbf{Q}.$$

We factor $R(\pi \times \pi)_! = R(\mathrm{id} \times \pi)_! \circ R(\pi \times \mathrm{id})_!$ and get that the comultiplication is given by applying $R\pi_!$ to the map $\pi^!\mathbf{Q} \to \pi^! R\pi_!\pi^!\mathbf{Q}$. On the other hand, the connecting homomorphism is obtained by applying $R\pi_!$ to the composition $\pi^!\mathbf{Q} \to \pi^! R\pi_!\pi^!\mathbf{Q} \to \pi^!\mathscr{H}[1]$ from (5), which by the above lemma describes the extension $\mathcal{L}og^{(1)}$. $\qquad\square$

To compute the higher direct images of $\mathcal{L}og^{(n)}$ we need the exact Koszul complex (see [Ill71, 4.3.1.7])

$$(14) \quad 0 \to \overset{m}{\bigwedge}\mathscr{H} \xrightarrow{d_m^0} \dots \xrightarrow{d_m^{i-1}} \overset{m-i}{\bigwedge}\mathscr{H} \otimes \mathrm{Sym}^i\mathscr{H} \xrightarrow{d_m^i} \dots \xrightarrow{d_m^{m-1}} \mathrm{Sym}^m\mathscr{H} \to 0.$$

Recall that the differentials $d_m^i : \bigwedge^{m-i}\mathscr{H} \otimes \mathrm{Sym}^i\mathscr{H} \to \bigwedge^{m-i-1}\mathscr{H} \otimes \mathrm{Sym}^{i+1}\mathscr{H}$ are induced by the comultiplication $\bigwedge^{m-i}\mathscr{H} \to \bigwedge^{m-i-1}\mathscr{H} \otimes \mathscr{H}$ of the exterior algebra composed with the multiplication of the symmetric algebra.

Proposition 7.1.2. *The spectral sequence associated to the filtration of $\mathcal{L}og^{(n)}$ by unipotence length*

$$E_1^{p,q} = R^{p+q}\pi_!\pi^*\mathrm{Sym}^p\mathscr{H}(d) \Rightarrow R^{p+q}\pi_!\mathcal{L}og^{(n)}(d).$$

has $E_1^{p,q} \cong \bigwedge^{2d-p-q}\mathscr{H} \otimes \mathrm{Sym}^p\mathscr{H}$ for $0 \le p \le n$ and $p+q \ge 0$ and E_1-differential given by the Koszul differential. It degenerates at E_2 with

$$R^i\pi_!\mathcal{L}og^{(n)}(d) \cong \begin{cases} \mathbf{Q} & i = 2d \\ \mathrm{coker}\ d_{2d-i+n}^{n-1} & 0 < i < 2d \\ R^0\pi_!\mathbf{Q}(d) \otimes \mathrm{Sym}^n\mathscr{H} & i = 0. \end{cases}$$

where $d_{2d-i+n}^{n-1} : \bigwedge^{2d-i+1}\mathscr{H} \otimes \mathrm{Sym}^{n-1}\mathscr{H} \to \bigwedge^{2d-i}\mathscr{H} \otimes \mathrm{Sym}^n\mathscr{H}$ is the Koszul differential from (14).

Proof. The sheaf $\mathcal{L}og^{(n)}$ has a filtration $F^{\cdot}\mathcal{L}og^{(n)}$ such that the associated graded pieces are the $\pi^*\mathrm{Sym}^k\mathscr{H}$ for $0 \le k \le n$. We consider the associated spectral sequence

$$E_1^{p,q} = R^{p+q}\pi_!\pi^*\mathrm{Sym}^p\mathscr{H}(d) \Rightarrow R^{p+q}\pi_!\mathcal{L}og^{(n)}(d).$$

If we identify $R^{p+q}\pi_!\mathbf{Q}(d) \cong \bigwedge^{2d-p-q}\mathscr{H}$ we get

$$E_1^{\cdot,q} : 0 \to \overset{2d-q}{\bigwedge}\mathscr{H} \xrightarrow{d_1^{0,q}} \overset{2d-q-1}{\bigwedge}\mathscr{H} \otimes \mathscr{H} \xrightarrow{d_1^{1,q}} \dots \xrightarrow{d_1^{n-1,q}} \overset{2d-q-n}{\bigwedge}\mathscr{H} \otimes \mathrm{Sym}^n\mathscr{H},$$

where the first term is $E_1^{0,q}$ etc. We assume by induction on n that the differentials $d_1^{p,q}$ in the spectral sequence for $\mathcal{L}og^{(n-1)}$ are the Koszul differentials d_{2d-q}^p (up to sign). The case $n = 1$ is Lemma 7.1.1. Then the $d_1^{p,q}$ for $p \le n-2$ in the spectral sequence for $\mathcal{L}og^{(n)}$ coincide also with the Koszul differentials d_{2d-q}^p (up

to sign). We claim that this is also true for $d_1^{n-1,q}$. The differentials $d_1^{n-1,q}$ in the spectral sequence are the connecting homomorphisms for $R^{n-1+q}\pi_!$ of the short exact sequence

$$0 \to \pi^*\mathrm{Sym}^n \mathscr{H} \to F^{n-1}\mathcal{L}og^{(n)}/F^{n+1}\mathcal{L}og^{(n)} \to \pi^*\mathrm{Sym}^{n-1}\mathscr{H} \to 0.$$

By construction of $\mathcal{L}og^{(n)}$ this short exact sequence is isomorphic to the push-out of

$$0 \to \pi^*\mathrm{Sym}^{n-1}\mathscr{H} \otimes \pi^*\mathscr{H} \to \pi^*\mathrm{Sym}^n \mathscr{H} \otimes \mathcal{L}og^{(1)} \to \pi^*\mathrm{Sym}^{n-1}\mathscr{H} \to 0$$

by the multiplication map $\pi^*\mathrm{Sym}^{n-1}\mathscr{H} \otimes \pi^*\mathscr{H} \to \pi^*\mathrm{Sym}^n \mathscr{H}$. In particular, the connecting homomorphisms are the ones of $\mathcal{L}og^{(1)}$ tensored with $\pi^*\mathrm{Sym}^{n-1}\mathscr{H}$. If one unravels the definitions one gets that the $d_1^{n-1,q}$ are the Koszul differentials.

It follows that $E_1^{\cdot,q}$ is the truncated Koszul complex and hence the only non-zero E_2-terms are $E_2^{0,2d} = \mathbf{Q}$, $E_2^{n,q} = \mathrm{coker}\ d_1^{n-1,q}$ for $-n < q < 2d - n$ and $E_2^{n,-n} = R^0\pi_!\mathbf{Q}(d) \otimes \mathrm{Sym}^n \mathscr{H}$. For the higher direct images we get accordingly

$$R^i\pi_!\mathcal{L}og^{(n)}(d) = \begin{cases} \mathbf{Q} & i = 2d \\ \mathrm{coker}\ d_1^{n-1,i-n} & 0 < i < 2d \\ R^0\pi_!\mathbf{Q}(d) \otimes \mathrm{Sym}^n \mathscr{H} & i = 0, \end{cases}$$

which is the desired result. □

As a corollary we get the statement of Theorem 3.3.1:

Corollary 7.1.3. *One has*

$$R^i\pi_!\mathcal{L}og \cong \begin{cases} \mathbf{Q}(-d) & i = 2d \\ 0 & i \neq 2d. \end{cases}$$

Proof. From the computation of $R^{2d}\pi_!\mathcal{L}og^{(n)}$ it follows that the transition maps $R^{2d}\pi_!\mathcal{L}og^{(n)} \cong R^{2d}\pi_!\mathcal{L}og^{(n-1)}$ are all isomorphisms. In particular, $R^{2d}\pi_!\mathcal{L}og \cong \mathbf{Q}(-d)$.

It remains to show that $R^i\pi_!\mathcal{L}og = 0$ for $i \neq 2d$ and for this it is enough to show that $R^i\pi_!\mathcal{L}og^{(n)} \to R^i\pi_!\mathcal{L}og^{(n-1)}$ is the zero map. Consider the long exact cohomology sequence of

$$0 \to \pi^*\mathrm{Sym}^n \mathscr{H} \to \mathcal{L}og^{(n)} \to \mathcal{L}og^{(n-1)} \to 0.$$

By the computation of $R^i\pi_!\mathcal{L}og^{(n)}$ in Proposition 7.1.2 the map

$$R^i\pi_!\mathrm{Sym}^n \mathscr{H} \cong \bigwedge^{2d-i} \mathscr{H} \otimes \mathrm{Sym}^n \mathscr{H} \to R^i\pi_!\mathcal{L}og^{(n)}$$

is surjective, hence $R^i\pi_!\mathcal{L}og^{(n)} \to R^i\pi_!\mathcal{L}og^{(n-1)}$ is the zero map. □

We now turn to the case where G is an extension of an abelian scheme by a torus and hence $\mathscr{H}$ locally constant. We discuss the necessary modifications of this proof to get the statement for the higher direct images $R^i\pi_*\mathcal{L}og$. First note that one has by Poincaré duality a perfect pairing

$$R^i\pi_*\mathbf{Q} \otimes R^{2d-i}\pi_!\mathbf{Q} \to \mathbf{Q}(-d),$$

which shows $\bigoplus_i R^i\pi_*\mathbf{Q} \cong \bigwedge^i \mathscr{H}^\vee$. The dual of the quasi-isomorphism in (4) gives the decomposition

(15) $$R\pi_*\pi^*\mathbf{Q} \cong \mathbf{Q} \oplus \tau_{>0}R\pi_*\pi^*\mathbf{Q}.$$

To identify the extension class of $\mathcal{L}og^{(1)} \in \mathrm{Ext}_G^1(\mathbf{Q}, \pi^*\mathscr{H})$ consider the evaluation map $\mathrm{ev} : \mathscr{H} \otimes \mathscr{H}^\vee \to \mathbf{Q}$ and its dual

$$\mathrm{ev}^\vee : \mathbf{Q} \to \mathscr{H}^\vee \otimes \mathscr{H}.$$

Note further that by duality one has

$$R\pi_*\pi^*\mathscr{H} \cong (R\pi_!\pi^!\mathscr{H}^\vee)^\vee \cong (R\pi_!\pi^!\mathbf{Q} \otimes \mathscr{H}^\vee)^\vee \cong R\pi_*\pi^*\mathbf{Q} \otimes \mathscr{H}.$$

Lemma 7.1.4. *The class of $\mathcal{L}og^{(1)} \in \mathrm{Ext}_S^1(\mathbf{Q}, R\pi_*\pi^*\mathscr{H}) = \mathrm{Hom}_S(\mathbf{Q}, R\pi_*\pi^*\mathscr{H}[1])$ is given by the composition*

$$\mathbf{Q} \xrightarrow{\mathrm{ev}^\vee} \mathscr{H}^\vee \otimes \mathscr{H} \to \tau_{>0}R\pi_*\pi^*\mathbf{Q} \otimes \mathscr{H}[1] \xrightarrow{(15)} R\pi_*\pi^*\mathbf{Q} \otimes \mathscr{H}[1]$$

where the arrow in the middle is induced by the map $\mathscr{H}^\vee = R^1\pi_*\pi^*\mathbf{Q} \to \tau_{>0}R\pi_*\pi^*\mathbf{Q}[1]$.

Proof. By definition the extension class $\mathcal{L}og^{(1)} \in \mathrm{Hom}_S(R\pi_!\pi^!\mathbf{Q}, \mathscr{H}[1])$ is given by the map in (5), which induces

$$\mathbf{Q} \to (\tau_{\leq -1}R\pi_!\pi^!\mathbf{Q})^\vee \otimes \mathscr{H}[1] \cong \tau_{>0}R^1\pi_*\pi^*\mathbf{Q} \otimes \mathscr{H}[1].$$

If one unravels the definition of the map in (5) one gets the map in the lemma.   □

Let $h := \dim_\mathbb{Q} \mathscr{H}$ be the dimension of the local system $\mathscr{H}$, then the pairing $\bigwedge^i \mathscr{H}^\vee \otimes \bigwedge^{h-i} \mathscr{H}^\vee \to \bigwedge^h \mathscr{H}^\vee$ induces an isomorphism

$$R^i\pi_*\pi^*\mathbf{Q} \cong \bigwedge^i \mathscr{H}^\vee \cong \bigwedge^{h-i} \mathscr{H} \otimes \bigwedge^h \mathscr{H}^\vee.$$

The computation of $R^i\pi_*\mathcal{L}og^{(n)}(d)$ is exactly the same as before, once we have identified the connecting homomorphisms

$$R^i\pi_*\mathbf{Q} \to R^{i+1}\pi_*\pi^*\mathscr{H}$$

of the extension

$$0 \to \pi^*\mathscr{H} \to \mathcal{L}og^{(1)} \to \mathbf{Q} \to 0.$$

Lemma 7.1.5. *Using the above identification $R^i\pi_*\pi^*\mathbf{Q} \cong \bigwedge^{h-i} \mathscr{H} \otimes \bigwedge^h \mathscr{H}^\vee$ the connecting homomorphism*

$$\bigwedge^{h-i} \mathscr{H} \otimes \bigwedge^h \mathscr{H}^\vee \to \bigwedge^{h-i-1} \mathscr{H} \otimes \mathscr{H} \otimes \bigwedge^h \mathscr{H}^\vee$$

is induced by the comultiplication in $\bigwedge^\cdot \mathscr{H}$.

Proof. The connecting homomorphism is the map

$$R^i\pi_*\mathbf{Q} \to R^i\pi_*\mathbf{Q} \otimes R^1\pi_*\mathbf{Q} \otimes \mathscr{H} \to R^{i+1}\pi_*\mathbf{Q} \otimes \mathscr{H}$$

induced by the cup-product. If we make the identifications explicit, we get the desired formula.   □

Exactly as in the proof of Proposition 7.1.2 we get

$$R^i\pi_*\mathcal{L}og^{(n)} \cong \begin{cases} \bigwedge^h \mathscr{H}^\vee & i = h \\ \bigwedge^h \mathscr{H}^\vee \otimes \mathrm{coker}\ d_{2h-i+n}^{n-1} & 0 < i < h \\ \bigwedge^h \mathscr{H}^\vee \otimes \mathrm{Sym}^n \mathscr{H} & i = 0. \end{cases}$$

Corollary 7.1.6. *Let G be an extension of an abelian scheme by a torus. Then*

$$R^i\pi_*\mathcal{L}og \cong \begin{cases} \bigwedge^h \mathscr{H}^\vee & i = h \\ 0 & i \neq h. \end{cases}$$

Proof. This follows by the same argument as in Corollary 7.1.3. □

7.2. **Proof of Theorem 4.5.1.** We turn to the motivic setting with either S of characteristic 0 or G affine. As always, G/S is a smooth commutative group scheme with connected fibres.

Lemma 7.2.1. *Let S be a scheme of characteristic 0 or G affine. Let $a \in \mathbb{Z}$, $a \neq 0, \pm 1$. Then the generalized a^0-eigenspace for the operation of $[a]$ on $R\pi_!\mathcal{L}og^{(n)}$ is isomorphic to $\mathbb{Q}(-d)[-2d]$. The generalized a^j-eigenspace vanishes for $j > n + \mathrm{kd}(G)$ and for $0 < j < n$.*

This is a refined version of the vanishing in Proposition 5.4.1 (3). Its proof relies on much deeper input from the theory of motives.

Proof. The computation of the generalized a^0 eigenspace was carried out in Proposition 5.4.1 (3). The vanishing for $j > n$ follows simply by induction from the statement for $R\pi_!\mathrm{Sym}^i\mathscr{H}$, see Proposition 5.4.1 (1).

We now turn to the essential part of the statement, with $0 < j < n$. We claim that the a^j-eigenspace vanishes. By [AHP, Lemma A.6] it is enough to prove the statement after base change to geometric points $\bar{s} : k \to S$. Moreover, $\bar{s}^*$ is a tensor functor commuting with $R\pi_!$ and $\bar{s}^*\mathscr{H}_{G/S} = \mathscr{H}_{G_k/k}$ by [AHP, Proposition 2.7]. Hence we may assume without loss of generality that $S = \mathrm{Spec}\,k$ with k algebraically closed. We have been working in categories of étale motives without transfers so far. In the case of a perfect ground field k, the "adding transfer" functor is an equivalence of categories. Hence we can argue in Voevodsky's orginal category of geometric motives $\mathbf{DM}(k, \mathbb{Q})$ from now on.

We claim that the object $R\pi_!\mathcal{L}og^{(n)}$ is contained in the subcategory of abelian motives in the sense of [Wi14, Definition 1.1]. It is the thick tensor triangulated subcategory of the category of geometric motives generated by $\mathbb{Q}(r)$ for $r \in \mathbb{Z}$ and the Chow motives of abelian varieties. We can verify this by induction on n. We have computed $R\pi_!\pi^*\mathrm{Sym}^i\mathscr{H}$ in the proof of Proposition 5.4.1. Hence it suffices to establish the claim for $M_1(G)$. By [AEH, Lemma 7.4.5], the motive $M_1(G)$ agrees with the 1-motive of the semiabelian part G^{sa} of G. In the semi-abelian case, the sequence

$$1 \to T \to G^{sa} \to A$$

with T a torus and A an abelian variety induces an exact triangle $M_1(T) \to M_1(G) \to M_1(A)$. The torus T is split because we have assumed k to be algebraically closed. Hence $M_1(T) = \mathbb{Q}(1)^r$ is in the category of abelian motives. The motive $M_1(A)$ is a Chow motive as a direct summand of the motive of A, hence also in the category of abelian motives.

Let ℓ be a prime invertible in k. If S is of characteristic 0, we have verified the assumptions of [Wi14, Theorem 1.16]. By loc.cit. the ℓ-adic realization H^*R_ℓ is conservative. We have reduced the assertion to the same vanishing in the ℓ-adic setting. If G is affine, then its motive is a mixed Tate motive. Again the ℓ-adic realization is conservative; this time via the conservative slice functors c_n of [HuKa06, Section 5].

Consider the computation of $R\pi_!\mathcal{L}og_\ell^{(n)}$ in Proposition 7.1.2. The proof shows that the cohomology in degree $i < 2d$ is given by $E_2^{n,i-n}$ and a functorial quotient of $E_1^{n,i-n} = \bigwedge^{2d-i}\mathscr{H}_\ell \otimes \mathrm{Sym}^n\mathscr{H}_\ell$. The operation of $[a]$ on this term is by multiplication by a^{2d-i+n}. Recall that $0 \le i < 2d$. There is no contribution to the a^j-eigenspace for $0 < j < n$. $\qquad\square$

Proof of Theorem 4.5.1. We want to show that

$$R\pi_!\mathcal{L}og \to R\pi_!\mathbb{Q} \xrightarrow{\mathrm{res}_e} e^*\mathbb{Q}(-d)[-2d]$$

is an isomorphism in $\mathbf{DA}(S)$. We pass to generalized a-eigenspaces for the operation of $a \in \mathbb{Z}$. It suffices to show:

(1) The a^0-eigenspace of $R\pi_!\mathcal{L}og^{(n)}$ is equal to $\mathbb{Q}(-d)[-2d]$ for all n.
(2) For $i \ge 1$, the pro-object given by the generalized a^i-eigenspaces of $R\pi_!\mathcal{L}og^{(n)}$ is isomorphic to 0.

The first claim was shown in Proposition 5.4.1 (3). The second claim is a consequence of Lemma 7.2.1. $\qquad\square$

Remark 7.2.2. It is tempting to remove the characteristic 0 hypothesis from the result. It enters the argument via the proof of [Wi14, Theorem 1.13], where it is used that homological and numerical equivalence agree on abelian varieties. This is open in positive characteristic.

APPENDIX A. EIGENSPACE DECOMPOSITION

The aim of this section is verify the existence of decomposition into generalized eigenspaces in the setting of triangulated categories.

Definition A.0.3. Let $\mathcal{A}$ be a pseudo-abelian $\mathbb{Q}$-linear additive category. Let X be an object and $\varphi : X \to X$ an endomorphism. We say that X has a *finite decomposition into generalized φ-eigenspaces* if there is a φ-equivariant direct sum decomposition

$$X = \bigoplus_{i=1}^{n} X_i$$

together with a sequence $\alpha_1, \ldots, \alpha_n$ of pairwise distinct rational numbers ("eigenvalues") and a sequence $m_1, \ldots, m_n$ of positive integers such that $(\varphi - \alpha_i)^{m_i}$ vanishes on X_i. We call X_i the *generalized eigenspace for the eigenvalue α_i*.

Example A.0.4. Let $\mathcal{A}$ be the category of finitely generated $\mathbb{Q}$-vector spaces. Every object has a finite decomposition into generalized φ-eigenspaces by putting φ in Jordan normal form.

This is not the most general notion one could imagine, but it suffices for our application. The condition is equivalent to the following: We view X as a $\mathbb{Q}[T]$-module with T operating via φ. The object X has a finite decomposition into generalized φ-eigenspaces if and only if the operation of $\mathbb{Q}[T]$ factors via an Artin quotient $\mathbb{Q}[T]/I$ with I of the form $\prod_{i=1}^{n}(T - \alpha_i)^{m_i}$. By the Chinese Remainder Theorem, we have a ring isomorphism

$$\mathbb{Q}[T]/I = \prod_{i=1}^{n} \mathbb{Q}[T]/(T - \alpha_i)^{m_i} \ .$$

The decomposition of X is induced from the decomposition of $1 \in \mathbb{Q}[T]/I$ into projectors. In particular, the decomposition is unique if it exists.

Lemma A.0.5. *Let $A \to B \to C$ be an exact sequence of (possibly infinite dimensional) $\mathbb{Q}$-vector spaces with operation of an endomorphism φ. Assume that A and C admit a finite decomposition into generalized φ-eigenspaces. Then so does B.*

Proof. By assumption A is a $\mathbb{Q}[T]/I$-module and C a $\mathbb{Q}[T]/J$ module with I and J of the special shape above. It is easy to check that IJ annihilates B, hence B also admits a decomposition into generalized φ-eigenspaces. $\square$

Proposition A.0.6. *Let $\mathcal{T}$ be a $\mathbb{Q}$-linear pseudo-abelian triangulated category. Let $A \to B \to C \to A[1]$ be an exact triangle and φ an endomorphism of the triangle. Assume that A and C admit a finite decomposition into generalized φ-eigenspaces. Then so does B. Given $\alpha \in \mathbb{Q}$ the triangle of generalized eigenspaces for the eigenvalue α is distinguished.*

Proof. Consider the exact sequence of $\mathbb{Q}$-vector spaces
$$\operatorname{Hom}_{\mathcal{T}}(B,A) \to \operatorname{Hom}_{\mathcal{T}}(B,B) \to \operatorname{Hom}_{\mathcal{T}}(B,C) .$$
By functoriality, it has an operation of φ. As A and C have a decomposition, so have $\operatorname{Hom}_{\mathcal{T}}(B,A)$ and $\operatorname{Hom}_{\mathcal{T}}(B,C)$. By the lemma this implies that $\operatorname{Hom}_{\mathcal{T}}(B,B)$ has decomposition. Equivalently, $\operatorname{Hom}_{\mathcal{T}}(B,B)$ is annihilated by an ideal I of the special form above. In particular, this is the case for id_B and hence for B. This means that B is an $\mathbb{Q}[T]/I$-module, or equivalently that it admits a finite decomposition into generalized φ-eigenspaces.

The ideal I can be chosen such that it annihilates all of A, B, C. This means that $\mathbb{Q}[T]/I$ operates on the exact triangle. The decomposition of B is compatible with the exact triangle. Summing the triangles for all $\alpha \in \mathbb{Q}$ we get back the original triangle. Hence the indivual triangles for fixed α are distinguished. $\square$

Lemma A.0.7. *Let $\mathcal{A}$ be a pseudo-abelian $\mathbb{Q}$-linear additive category. Let X be an object and $\varphi : X \to X$ and $\psi : X \to X$ commuting endomorphisms. Assume that X has a finite decomposition into generalized eigenspaces for φ and ψ. Then there is a unique simultaneous decomposition.*

Proof. The operation of φ and ψ make X into a $\mathbb{Q}[T,S]$-module. By assumption X is annihilated by a polynomial $P = \prod_{i=1}^{n}(T-\alpha_i)^{n_i}$ and also by a polynomial $Q = \prod_{j=1}^{m}(S-\beta_j)^{m_i}$. Hence the operation factors via the Artinian ring $\mathbb{Q}[T,S]/(P,Q)$. By the Chinese Remainder Theorem, we have a ring isomorphism
$$\mathbb{Q}[T,S]/(P,Q) = \prod_{i,j} \mathbb{Q}[T,S]/((T-\alpha_i)^{n_i},(S-\beta_j)^{m_j}).$$

The decomposition of X is induced from the decomposition of 1 into projectors. $\square$

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(Huber) Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, 79104 Freiburg
E-mail address: annette.huber@math.uni-freiburg.de

(Kings) Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg
E-mail address: guido.kings@mathematik.uni-regensburg.de