ON THE RELATION BETWEEN NORI MOTIVES AND KONTSEVICH PERIODS

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ABSTRACT. We show that the spectrum of Kontsevich's algebra of formal periods is a torsor under the motivic Galois group for mixed motives over $\mathbb{Q}$. This assertion is stated without proof by Kontsevich ([Ko] Theorem 6) and originally due to Nori. In a series of appendices, we also provide the necessary details on Nori's category of motives (see the survey [Le]).

CONTENTS

INTRODUCTION

Let X be a smooth variety over $\mathbb{Q}$. A *period number* of X is the value of an integral

$$\int_Z \omega$$

where ω is a closed algebraic (hence rational) differential form of degree d and $Z \subset X(\mathbb{C})$ a closed real submanifold of dimension d. Periods are complex numbers. A more conceptual way is to view periods of X as the values of the period pairing between $H_d(X(\mathbb{C}), \mathbb{Q})$ (singular homology) and $H^d_{\mathrm{dR}}(X/\mathbb{Q})$ (algebraic de Rham cohomology), or – equivalently – matrix entries of the period isomorphism [D1, D2]

$$H^d(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \longrightarrow H^d_{\mathrm{dR}}(X/\mathbb{Q}) \otimes \mathbb{C}$$

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in any two $\mathbb{Q}$–bases. Stated like this, the notion generalizes to any variety over $\mathbb{Q}$ or even any mixed motive over $\mathbb{Q}$, independently of the chosen category of mixed motives.

The algebra of Kontsevich-Zagier periods [KZ], defined as integrals of algebraic functions over domains described by algebraic equations or inequalities with co-efficients in $\mathbb{Q}$ is the set of all periods of all mixed motives over $\mathbb{Q}$. It is a very interesting, countable subalgebra of $\mathbb{C}$. It contains e.g. all algebraic numbers, $2\pi i$, and all $\zeta(n)$ for $n \in \mathbb{Z}$. Indeed, if Beilinson's conjectures are true, then all special values of L-functions of mixed motives are periods.

The relations between period numbers are mysterious and intertwined with transcendence theory. A general period conjecture goes back to Grothendieck, see the footnote on page 358 in [G], and the reformulations of André in [A1] and [A2] chapitre 23. Conjecturally, the only relations are in a sense the obvious ones, i.e., coming from geometry. In order to make this statement precise, Kontsevich introduced ([Ko] Definition 20) the notion of a formal period. We recall his definition (actually, a variant, see Remark 2.9).

Definition 0.1. The space of *effective formal periods* P^+ is defined as the $\mathbb{Q}$-vector space generated by symbols (X, D, ω, γ), where X is an algebraic variety over $\mathbb{Q}$, $D \subset X$ a subvariety, $\omega \in H^d_{\mathrm{dR}}(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ with relations

    (1) linearity in ω and γ;
    (2) for every $f : X \to X'$ with $f(D) \subset D'$

$$(X, D, f^*\omega', \gamma) = (X', D', \omega', f_*\gamma)$$

    (3) for every triple $Z \subset Y \subset X$

$$(Y, Z, \omega, \partial\gamma) = (X, Y, d\omega, \gamma)$$

        with ∂ the connecting morphism for relative cohomology.

P^+ is turned into an algebra via

$$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup X' \times D, \omega \cup \omega', \gamma \cap \gamma') \ .$$

The space of *formal periods* is the localization P of P^+ with respect to the period of $(\mathbb{G}_m, \{1\}, \frac{dX}{X}, S^1)$ where S^1 is the unit circle in $\mathbb{C}^*$.

The period conjecture then predicts the injectivity of the evaluation map $P \to \mathbb{C}$. We have nothing to say about this deep conjecture, which includes for example the transcendence of π and all $\zeta(2n + 1)$ for $n \in \mathbb{N}$ [A1].

The main aim of this note is to provide the proof (see Corollary 2.11) of the following result:

Theorem 0.2 (Nori, [Ko] Theorem 6)**.** $\mathrm{Spec}(P)$ *is a torsor under the motivic Galois group of Nori's category of mixed motives over* $\mathbb{Q}$.

As already explained by Kontsevich, singular cohomology and algebraic de Rham cohomology are both fiber functors on the same Tannaka category of motives. By general Tannaka formalism, there is a pro-algebraic torsor of isomorphisms between them. The period pairing is nothing but a complex point of this torsor. Our task was to check that this torsor is nothing but the explicit $\mathrm{Spec}(P)$. While baffling at first, the statement turns out to be a corollary of the very construction of Nori's motives.

This brings us to the second part of this paper. Nori's unconditional construction of an abelian category of mixed motives has been around for some time. Notes of talks have been circulating, see [N] and [N1]. In his survey article [Le], Levine includes a sketch of the construction, bringing it into the public domain. These sources combined contain basically all the necessary ideas. We decided to work out all technical details that were not obvious to us, but were needed in order to get a full proof of our main theorem. This material is contained in the appendices. They are supposed to be self-contained and independent of each other. We tried to make clear where we are using Nori's ideas. All mistakes remain of course ours.

Section 1 is another survey on Nori motives, which brings the results of the appendices together in order to establish that Nori motives are a neutral Tannakian category. Section 2 gives an interpretation of the algebra of formal periods in terms of Nori's machine. The main result is Theorem 2.6. In section 3 this is combined with the rigidity property of Nori's category of mixed motives in order to deduce the statement on the torsor structure.

Appendix A clarifies in detail the notion of torsor used by Kontsevich in [Ko] which was unfamiliar to us. Appendix B is on the multiplicative structure on Nori's diagram categories. Appendix C establishes a criterion for an abelian tensor category with faithful fiber functor to be rigid. Both appendices are supposed to be applied to Nori motives but formulated in general.

The key geometric input in Nori's approach was the so-called basic lemma to which we refer here as Beilinson's lemma since A. Beilinson had proved a more general version earlier. It allows for "cellular" decompositions of algebraic varieties, in the sense that their cohomology looks like cohomology of a cellular decomposition of a manifold. In appendix C this is used to compare different definitions of Nori motives using "pairs" or "good pairs" or even "very good pairs". The results of this section are essential input in order to apply the rigidity criterion of section D. What is missing from our account is the proof of universality of Nori's diagram category for a diagram with a representation. The paper [vW] by J. von Wangenheim provides full details. There is also the (unfortunately unpublished) paper [Br] by A. Brughières to fill in this point.

1. Essentials of Nori Motives

We use the setup of [N] for Nori motives. The key ideas are contained in the survey [Le]. Parts of the theory and further details are also developed in appendix B. Nori works primarily with singular homology. We have switched to singular cohomology throughout. We restrict to rational coefficients for simplicity.

We fix the following notation.

- By *variety* we mean a reduced separated scheme of finite type over $\mathbb{Q}$.
- Let $\mathbb{Q}$-Mod be the category of finite dimensional $\mathbb{Q}$-vector spaces.
- A *diagram D* is a directed graph.

- A *representation* $T : D \to \mathbb{Q}$-Mod assigns to every vertex in D an object in $\mathbb{Q}$-Mod and to every edge e from v to v' a homomorphism $T(e) : T(v) \to T(v')$.
- Let $\mathcal{C}(D, T)$ its associated *diagram category*. It is the universal abelian category together with a functor $f_T : \mathcal{C}(D, T) \to \mathbb{Q}$-Mod such that T factors via f_T. We often write T instead of f_T. The category $\mathcal{C}(D, T)$ arises as the category of $A(T)$-comodules finite dimensional over $\mathbb{Q}$ for a certain coalgebra $A(T)$. For the explicit construction due to Nori, see appendix B and [vW]. When D or T are understood from the context, we often abbreviate $\mathcal{C}(T)$ or $\mathcal{C}(D)$.

The following are the cases of interest in the present paper:

Definition 1.1. (1) The diagram D^{eff} of *effective pairs* consists of triples (X, Y, i) with X a $\mathbb{Q}$-variety, $Y \subset X$ a closed subvariety and an integer i. There are two types of edges between effective good pairs:

(a) (functoriality) For every morphism $f : X \to X'$ with $f(Y) \subset Y'$ an edge
$$f^* : (X', Y', i) \to (X, Y, i) .$$

(b) (coboundary) For every chain $X \supset Y \supset Z$ of closed $\mathbb{Q}$-subschemes of X an edge
$$\partial : (Y, Z, i) \to (X, Y, i+1) .$$

The diagram is graded (see Definition B.14) by $|(X, Y, i)| = i$.

(2) The diagram $D^{\mathrm{eff}}_{\mathrm{Nori}}$ of *effective good pairs* is the full subdiagram of D^{eff} with vertices the triples (X, Y, i) such that singular cohomology satisfies
$$H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q}) = 0, \text{ unless } j = i.$$

(3) The diagram $\tilde{D}^{\mathrm{eff}}$ of *effective very good pairs* is the full subdiagram of those effective good pairs (X, Y, i) with X affine, $X \smallsetminus Y$ smooth and either X of dimension i and Y of dimension $i - 1$, or $X = Y$ of dimension less than i.

The diagrams D of *pairs*, D_{Nori} of *good pairs* and $\tilde{D}$ of *very good pairs* are obtained by localization (see Definition B.17) with respect to $(\mathbb{G}_m, \{1\}, 1)$.

We use the representation $H^* : D_{\mathrm{Nori}} \to \mathbb{Q}$-Mod which assigns to (X, Y, i) relative singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$.

Remark 1.2. For the purposes of our paper $\mathbb{Q}$-coefficients are sufficient. Nori's machine also works for integral coefficients.

Good pairs exist in abundance, see Appendix D.

Definition 1.3. The category of (effective) *mixed Nori motives* $\mathcal{MM}_{\mathrm{Nori}}$ (resp. $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$) is defined as the diagram category $\mathcal{C}(D_{\mathrm{Nori}}, H^*)$ (resp. $\mathcal{C}(D^{\mathrm{eff}}_{\mathrm{Nori}}, H^*)$). For a good pair (X, Y, i) we write $H^i_{\mathrm{Nori}}(X, Y)$ for the corresponding object in $\mathcal{MM}_{\mathrm{Nori}}$. We put
$$\mathbf{1}(-1) = H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\}) \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} .$$

Remark 1.4. In applying the theory of Appendix B.3 we need an object of even degree. We really localize with respect to the square of the Lefschetz object. This causes a conflict in twist notation. In the main text of the article, we keep writing (-1) for tensor product with $\mathbf{1}(-1)$.

Remark 1.5. It will established in Corollary 1.7 that Nori motives can equivalently be defined using D or $\tilde{D}$.

Theorem 1.6.   (1) *This definition is equivalent to Nori's original definition.*
   (2) $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} \subset \mathcal{MM}_{\mathrm{Nori}}$ *are commutative tensor categories with a faithful fiber functor* H^*.
   (3) $\mathcal{MM}_{\mathrm{Nori}}$ *is the localization of* $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ *with respect to the Lefschetz object* $\mathbf{1}(-1)$.

Proof. $D_{\mathrm{Nori}}^{\mathrm{eff}}$ is a graded diagram in the sense of Definition B.14. It carries a commutative multiplicative structure (see Definition B.14 again) by

$$(X, Y, i) \times (X', Y', i') = (X \times X', X \times Y' \cup Y \times X', i + i').$$

with unit given by $(\mathrm{Spec}\,\mathbb{Q}, \emptyset, 0)$ and

$$u : (X, Y, i) \to (\mathrm{Spec}\,\mathbb{Q}, \emptyset, 0) \times (X, Y, i) = (\mathrm{Spec}\,\mathbb{Q} \times X, \mathrm{Spec}\,\mathbb{Q} \times Y, i)$$

be given by the natural isomorphism of varieties. Let also

$$\alpha : (X, Y, i) \times (X', Y', i') \to (X', Y', i') \times (X, Y, i)$$

$$\beta : (X, Y, i) \times ((X', Y', i') \times (X'', Y'', i'')) \to ((X, Y, i) \times (X', Y', i')) \times (X'', Y'', i'')$$

be given by the natural isomorphisms of varieties.
H^* is a graded representation in the sense of Definition B.14. Properties (2) and (3) depend on a choice of a sign convention such that the boundary map ∂ is compatible with cup products in the first variable and compatible up to sign in the second variable.
Hence by Proposition B.16, the category $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ carries a tensor structure.
The Lefschetz object $(\mathbb{G}_m, \{1\}, 1)$ satisfies Assumption B.19, hence by Proposition B.21 the category $\mathcal{MM}_{\mathrm{Nori}}$ is the localization of $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ at $\mathbf{1}(-1)$ and also a tensor category.
By definition, $D_{\mathrm{Nori}}^{\mathrm{eff}}$ is the category of cohomological good pairs in the terminology of [Le]. In loc. cit. the category of Nori motives is defined as the category of comodules of finite type over $\mathbb{Q}$ for the localization of the ring A^{eff} with respect to the element $\chi \in A(\mathbf{1}(-1))$ considered in Proposition B.21. By this Proposition, the category of A_χ^{eff}-comodules agrees with $\mathcal{MM}_{\mathrm{Nori}}$. $\square$

Comparing diagrams and diagram categories. Nori establishes that the representation $T = H^*$ extends to all pairs of varieties.

Corollary 1.7. *The diagram categories of D^{eff} and $\tilde{D}^{\mathrm{eff}}$ with respect to singular cohomology are equivalent to $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ as abelian categories. The diagram categories of D and $\tilde{D}$ are equivalent to $\mathcal{MM}_{\mathrm{Nori}}$.*

Proof. In the following proof we omit the T in the notation $\mathcal{C}(D, T)$. It suffices to consider the effective case. The inclusion of diagrams induces faithful functors

$$i : \mathcal{C}(\tilde{D}^{\mathrm{eff}}) \to \mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}} \to \mathcal{C}(D^{\mathrm{eff}}).$$

We are going to represent the diagramm D^{eff} in $\mathcal{C}(\tilde{D}^{\mathrm{eff}})$ such the restriction of the representation to $\tilde{D}^{\mathrm{eff}}$ gives back H^* (up to natural isomorphism). By the universal property this induces a faithful functor

$$j : \mathcal{C}(D^{\mathrm{eff}}) \to \mathcal{C}(\tilde{D}^{\mathrm{eff}})$$

such that $j \circ i = \mathrm{id}$ (up to natural isomorphism). This implies that j is essentially surjective and full. Hence j is an equivalence of categories. This implies also that i is an equivalence of categories.

We now turn to the construction of the representation of D^{eff} in $\mathcal{C}(\tilde{D}^{\mathrm{eff}})$.

We apply Proposition D.3 to

$$H^* : \tilde{D}^{\mathrm{eff}} \to \mathcal{C}(\tilde{D}^{\mathrm{eff}})$$

and get a functor

$$R : C^b(\mathbb{Z}[\mathrm{Var}]) \to D^+(\mathcal{C}(\tilde{D}^{\mathrm{eff}})).$$

Consider an effective pair (X, Y, i) in D. It is represented by

$$H^i_{\mathrm{Nori}}(X, Y) = H^i(R(X, Y)) \in \mathcal{C}(\tilde{D}^{\mathrm{eff}})$$

where

$$R(X, Y) = R(\mathrm{Cone}(Y \to X)) .$$

The construction is functorial for morphisms of pairs. This allows to represent edges of type f^*.

Finally, we need to consider edges corresponding to coboundary maps for triples $X \supset Y \supset Z$. In this case, it follows from the construction of R that there is a natural triangle

$$R(X, Y) \to R(X, Z) \to R(Y, Z).$$

We use the connecting morphism in cohomology to represent the edge $(Y, Z, i) \to (X, Y, i+1)$. $\square$

Corollary 1.8. *Every object of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is subquotient of a direct sum of objects of the form $H^i_{\mathrm{Nori}}(X, Y)$ for a good pair (X, Y, i) where $X = W \smallsetminus W_\infty$ and $Y = W_0 \smallsetminus (W_0 \cap W_\infty)$ with W smooth projective, $W_\infty \cup W_0$ a divisor with normal crossings.*

Proof. By Proposition B.9 every object in the diagram category of $\tilde{D}^{\mathrm{eff}}$ (and hence $\mathcal{MM}_{\mathrm{Nori}}$) is subquotient of a direct sum of some $H^i_{\mathrm{Nori}}(X, Y)$ with (X, Y, i) very good.

We follow Nori: By resolution of singularities there is a smooth projective variety W and a normal crossing divisors $W_0 \cup W_\infty \subset W$ together with a proper, surjective morphism $\pi : W \smallsetminus W_\infty \to X$ such that one has $\pi^{-1}(Y) = W_0 \smallsetminus W_\infty$ and $\pi : W \smallsetminus \pi^{-1}(Y) \to X \smallsetminus Y$ is an isomorphism. This implies that

$$H^i_{\mathrm{Nori}}(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)) \to H^i_{\mathrm{Nori}}(X, Y)$$

is also an isomorphism by proper base change, i.e., excision. $\square$

Remark 1.9. Note that the pair $(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)$ is good, but not very good in general. Replacing Y by a larger closed subset Z, one may, however, assume that $W_0 \smallsetminus (W_0 \cap W_\infty)$ is affine. Therefore, by Lemma 1.13, the dual of each generator can be assumed to be very good.

It is not clear to us if it suffices to construct Nori's category using the diagram of (X, Y, i) with X smooth, Y a divisor with normal crossings. The corollary says that the diagram category has the right "generators", but there might be too few "relations".

Corollary 1.10. *Let $Z \subset X$ be a closed immersion. Then there is a natural object $H_Z^i(X)$ in $\mathcal{MM}_{\mathrm{Nori}}$ representing cohomology with supports. There is a natural long exact sequence*

$$\cdots \to H_Z^i(X) \to H_{\mathrm{Nori}}^i(X) \to H_{\mathrm{Nori}}^i(X \smallsetminus Z) \to H_Z^{i+1}(X) \to \cdots$$

Proof. Let $U = X \smallsetminus Z$. Put

$$R_Z(X) = R(\mathrm{Cone}(U \to X)), \quad H_Z^i(X) = H^i(R_Z(X)) \ .$$

$\square$

Rigidity. In order to establish duality, we need to check that Poincaré duality is motivic, at least in a weak sense.

Definition 1.11. Let $\mathbf{1}(-1) = H_{\mathrm{Nori}}^1(\mathbb{G}_m)$ and $\mathbf{1}(-n) = \mathbf{1}(-1)^{\otimes n}$.

Lemma 1.12.   (1) $H_{\mathrm{Nori}}^{2n}(\mathbb{P}^N) = \mathbf{1}(-n)$ *for $N \geq n \geq 0$.*
   (2) *Let Z be a projective variety of dimension n. Then $H_{\mathrm{Nori}}^{2n}(Z) \cong \mathbf{1}(-n)$.*
   (3) *Let X be a smooth variety, $Z \subset X$ a smooth, irreducible, closed subvariety of pure codimension n. Then*

$$H_Z^{2n}(X) \cong \mathbf{1}(-n).$$

Proof. (1) Embedding projective spaces linearly into higher dimensional projective spaces induces isomorphisms on cohomology. Hence it suffices to check the top cohomology of $\mathbb{P}^N$.
We start with $\mathbb{P}^1$. Consider the standard cover of $\mathbb{P}^1$ by $U_1 = \mathbb{A}^1$ and $U_2 = \mathbb{P}^1 \smallsetminus \{0\}$. We have $U_1 \cap U_2 = \mathbb{G}_m$. By Corollary D.19

$$R(\mathbb{P}^1) \to \mathrm{Cone}\bigg(R(U_1) \oplus R(U_2) \to R(\mathbb{G}_m) \bigg)[-1]$$

is an isomorphism in the derived category. This induces the isomorphism $H_{\mathrm{Nori}}^2(\mathbb{P}^1) \to H_{\mathrm{Nori}}^1(\mathbb{G}_m)$. Similarly, the Čech complex (see Definition D.10) for the standard affine cover of $\mathbb{P}^N$ relates $H_{\mathrm{Nori}}^{2N}(\mathbb{P}^N)$ with $H_{\mathrm{Nori}}^N(\mathbb{G}_m^N)$.
(2) Let $Z \subset \mathbb{P}^N$ be a closed immersion with N large enough. Then $H_{\mathrm{Nori}}^{2n}(Z) \to H_{\mathrm{Nori}}^{2n}(\mathbb{P}^N)$ is an isomorphism in $\mathcal{MM}_{\mathrm{Nori}}$ because it is in singular cohomology.
(3) We note first that (3) holds in singular cohomology by the Gysin isomorphism

$$H^0(Z) \xrightarrow{\cong} H_Z^{2n}(X)$$

under our assumptions. For the embedding $Z \subset X$ one has the deformation to the normal cone [Fu, Sec. 5.1], i.e., a smooth scheme $D(X, Z)$ together with a morphism to $\mathbb{A}^1$ such that the fiber over 0 is given by the normal bundle $N_Z X$ of Z in X, and the other fibers by X. The product $Z \times \mathbb{A}^1$ can be embedded into $D(X, Z)$ as a closed subvariety of codimension n, inducing the embeddings of $Z \subset X$ as well as the embedding of the zero section $Z \subset N_Z X$ over 0. Hence, using the three Gysin isomorphisms and homotopy invariance, it follows that there are isomorphisms

$$H_Z^{2n}(X) \leftarrow H_{Z \times \mathbb{A}^1}^{2n}(D(X, Z)) \to H_Z^{2n}(N_Z X)$$

in singular cohomology and hence in our category. Thus, we have reduced the problem to the embedding of the zero section $Z \hookrightarrow N_Z X$. However, the normal bundle $\pi : N_Z X \to Z$ trivializes on some dense open subset $U \subset Z$. This induces an isomorphism

$$H_Z^{2n}(N_Z X) \to H_U^{2n}(\pi^{-1}(U)),$$

and we may assume that the normal bundle $N_Z X$ is trivial. In this case, we have

$$N_Z(X) = N_{Z \times \{0\}}(Z \times \mathbb{A}^n) = N_{\{0\}}(\mathbb{A}^n),$$

so that we have reached the case of $Z = \{0\} \subset \mathbb{A}^n$. Using the Künneth formula with supports and induction on n, it suffices to consider $H^2_{\{0\}}(\mathbb{A}^1)$ which is isomorphic to $H^1(\mathbb{G}_m) = \mathbf{1}(-1)$ by Cor. 1.10. □

The following lemma (more precisely, its dual) is formulated implicitly in [N] in order to establish rigidity of $\mathcal{MM}_{\mathrm{Nori}}$.

Lemma 1.13. *Let W be a smooth projective variety of dimension i, $W_0, W_\infty \subset W$ divisors such that $W_0 \cup W_\infty$ is a normal crossings divisor. Let*

$$
\begin{aligned}
X &= W \smallsetminus W_\infty \\
Y &= W_0 \smallsetminus W_0 \cap W_\infty \\
X' &= W \smallsetminus W_0 \\
Y' &= W_\infty \smallsetminus W_0 \cap W_\infty
\end{aligned}
$$

We assume that (X, Y) is a very good pair.
Then there is a morphism in $\mathcal{MM}_{\mathrm{Nori}}$

$$q : \mathbf{1} \to H^i_{\mathrm{Nori}}(X, Y) \otimes H^i_{\mathrm{Nori}}(X', Y')(i)$$

such that the dual of $H^*(q)$ is a perfect pairing.

Proof. We follow Nori's construction. The two pairs are Poincaré dual to each other in singular cohomology. (This is easily seen by computing with sheaves on W and the duality between j_* and $j_!$). This implies that they are both good pairs. Hence

$$H^i_{\mathrm{Nori}}(X, Y) \otimes H^i_{\mathrm{Nori}}(X', Y') \to H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X')$$

is an isomorphism. Let $\Delta = \Delta(W \smallsetminus (W_0 \cup W_\infty))$ via the diagonal map. Note that

$$X \times Y' \cup X' \times Y \subset X \times X' \smallsetminus \Delta$$

Hence by functoriality and the definition of cohomology with support, there is a map

$$H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X') \leftarrow H^{2i}_\Delta(X \times X').$$

Again, by functoriality, there is a map

$$H^{2i}_\Delta(X \times X') \leftarrow H^{2i}_{\bar{\Delta}}(W \times W)$$

with $\bar{\Delta} = \Delta(W)$. By Lemma 1.12 it is isomorphic to $\mathbf{1}(-i)$. The map q is defined by twisting the composition by (i). The dual of this map realizes Poincaré duality, hence it is a perfect pairing. □

Theorem 1.14 (Nori)**.** $\mathcal{MM}_{\mathrm{Nori}}$ *is rigid, hence a neutral Tannakian category. Its Tannaka dual is given by $G_{\mathrm{mot}} = \mathrm{Spec}(A(D_{\mathrm{Nori}}, H^*))$.*

Proof. By Corollary 1.8 every object of $\mathcal{MM}_{\mathrm{Nori}}$ is subquotient of $M = H^i_{\mathrm{Nori}}(X, Y)(j)$ for a good pair (X, Y, i) of the particular form occurring in Lemma 1.13. By this Lemma they all admit a perfect pairing.
By Proposition C.4, the category $\mathcal{MM}_{\mathrm{Nori}}$ is neutral Tannakian. The Hopf algebra of its Tannaka dual agrees with Nori's algebra by Theorem B.10. □

2. Main Theorem

We describe the strategy of proof for Theorem 6 in [Ko]: The period algebra is given by the comparison of Nori motives with respect to singular and de Rham cohomology. The argument seems to be formal, so we do it abstractly. Let D be a graded diagram with commutative product structure (see Definition B.14), $T_1, T_2 : D \to \mathbb{Q}$-Mod two representations.

Definition 2.1. Let $A_1 = A(D, T_1)$, $A_2 = A(D, T_2)$. Put

$$A_{1,2} = \operatorname{colim}_F \operatorname{Hom}(T_1|_F, T_2|_F)^\vee$$

where $^\vee$ denotes the $\mathbb{Q}$-dual and F runs through all finite subdiagrams of D.

Lemma 2.2. $A_{1,2}$ *is a commutative ring with multiplication induced by the tensor structure of the diagram category. The operation*

$$\operatorname{End}(T_1|_F) \times \operatorname{Hom}(T_1|_F, T_2|_F) \to \operatorname{Hom}(T_1|_F, T_2|_F)$$

induces a compatible comultiplication

$$A_1 \otimes A_{1,2} \leftarrow A_{1,2}.$$

Proof. The hard part is the existence of the multiplication. This follows by going through the proof of Proposition B.16, replacing $\operatorname{End}(T|_F)$ by $\operatorname{Hom}(T_1|_F, T_2|_F)$ in the appropriate places. $\square$

Example 2.3. For D_{Nori}, $T_1 = H^*$ (singular cohomology) as before and $T_2 = H^*_{\mathrm{dR}}$ (de Rham cohomology) this is going to induce the operation of the motivic Galois group G_{mot} on the torsor $X = \operatorname{Spec} A_{1,2}$.

Definition 2.4. We define the space of *periods* $P_{1,2}$ as the $\mathbb{Q}$-vector space generated by symbols

$$(p, \omega, \gamma)$$

where p is a vertex of D, $\omega \in T_1(p)$, $\gamma \in T_2(p)^\vee$ with the following relations:
   (1) linearity in ω, γ;
   (2) (change of variables) If $f : p \to p'$ is an edge in D, $\gamma \in T_2(p')^\vee$, $\omega \in T_1(p)$, then

$$(p', T_1(f)(\omega), \gamma) = (p, \omega, T_2(f)^\vee(\gamma)).$$

Proposition 2.5. $P_{1,2}$ *is a commutative $\mathbb{Q}$-algebra with multiplication given on generators by*

$$(p, \omega, \gamma)(p', \omega', \gamma') = (p \times p', \omega \otimes \omega', \gamma \otimes \gamma')$$

Proof. It is obvious that the relations of $P_{1,2}$ are respected by the formula. $\square$

There is a natural transformation

$$\Psi : P_{1,2} \to A_{1,2}$$

defined as follows: let $(p, \omega, \gamma) \in P_{1,2}$. Let F be a finite diagram containing p. Then

$$\Psi(p, \omega, \gamma) \in A_{1,2}(F) = \operatorname{Hom}(T_1|_F, T_2|_F)^\vee,$$

is the map

$$\operatorname{Hom}(T_1|_F, T_2|_F) \to \mathbb{Q}$$

which maps $\phi \in \operatorname{Hom}(T_1|_F, T_2|_F)$ to $\gamma(\phi(p)(\omega))$. Clearly this is independent of F and respects relations of $P_{1,2}$.

Theorem 2.6. *The above map*

$$\Psi : P_{1,2} \to A_{1,2}$$

is an isomorphism of $\mathbb{Q}$-algebras.

Proof. For a finite subdiagram $F \subset D$ let $P_{1,2}(F)$ be the space of periods. By definition $P = \operatorname{colim}_F P(F)$. The statement is compatible with these direct limits. Hence without loss of generality $D = F$ is finite.

By definition $P_{1,2}$ is the subspace of

$$\prod_{p \in D} T_1(p) \otimes T_2(p)^\vee$$

of elements satisfying the relations induced by D. By definition $A_{1,2}(T)$ is the subspace of

$$\prod_{p \in D} \operatorname{Hom}(T_1(p), T_2(p))^\vee$$

of elements satisfying the relations induced by D. As all $T_i(p)$ are finite dimensional, this is the same thing.

The compatibility with coproducts is easy to see. $\square$

Remark 2.7. This works for coefficients in Dedekind rings as long as the representations take values in projective modules of finite type. The theorem is also of interest in the case $T_1 = T_2$. It then gives an explicit description of Nori's coalgebra by generators and relations.

Recall that de Rham cohomology of a smooth algebraic variety is defined as hypercohomology of the complex of differential forms. It is possible to extend the definition naturally not only to singular varieties, but to pairs of varieties. A possible reference is [Hu1] Section 7.

Definition 2.8. The space of *effective formal periods* P^+ is defined as the $\mathbb{Q}$-vector space generated by symbols (X, D, ω, γ), where X is an algebraic variety over $\mathbb{Q}$, $D \subset X$ a subvariety, $\omega \in H_{\mathrm{dR}}^d(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ with relations

(1) linearity in ω and γ;
(2) for every $f : X \to X'$ with $f(D) \subset D'$

$$(X, D, f^*\omega', \gamma) = (X', D', \omega', f_*\gamma)$$

(3) for every triple $Z \subset Y \subset X$

$$(Y, Z, \omega, \partial\gamma) = (X, Y, d\omega, \gamma)$$

with ∂ the connecting morphism for relative cohomology.

P^+ is turned into an algebra via

$$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup D' \times X, \omega \cup \omega', \gamma \cap \gamma')$$

The space of *formal periods* is the localization P of P^+ with respect to the period of $(\mathbb{G}_m, \{1\}, \frac{dX}{X}, S^1)$ where S^1 is the unit circle in $\mathbb{C}^*$.

Remark 2.9. This is modeled after Kontsevich [Ko] Definition 20 but does not agree with it. He restricts to smooth X and D a divisor with normal crossings.

The above definition uses effective pairs (X, D, d) in the sense of Definition 1.1. By Corollary 1.7, it is clear that it suffices to take good or even very good pairs. By Corollary 1.8, it then suffices even to take generators of Kontsevich's form. However,

we do not know, if these pairs generate all relations. Also, he only imposes relation (3) in a special case.

Moreover, Kontsevich considers differential forms of top degree rather than cohomology classes. This change is harmless. They are automatically closed. He imposes Stokes's formula as an additional relation, hence this amounts to considering cohomology classes.

Theorem 2.10. *Let D be the diagram of pairs (see Definition 1.1). Let*

$$T_1 = H^* : D \to \mathbb{Q}\text{-}Mod \qquad\qquad \text{(singular cohomology)}$$
$$T_2 = H_{\mathrm{dR}}^* : D \to \mathbb{Q}\text{-}Mod \qquad\qquad \text{(de Rham cohomology)}$$

then the space of formal periods P (Definition 2.4) agrees with the comparison algebra $A_{1,2}$ (Definition 2.8).

$$P = P_{1,2} = A_{1,2} \ .$$

Proof. We first restrict to the effective situation. Let $P_{1,2}^{\mathrm{eff}}$ and $A_{1,2}^{\mathrm{eff}}$ be the algebras for this diagram. Note that $H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ is dual to $H^d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$. Hence by definition $P^+ = P_{1,2}^{\mathrm{eff}}$.

Recall that there is a natural comparison isomorphism between singular cohomology and de Rham cohomology (see e.g. [Hu2] §8) over $\mathbb{C}$. Hence by Theorem 2.6 $P_{1,2}^{\mathrm{eff}} = A_{1,2}^{\mathrm{eff}}$ and $P_{1,2} = A_{1,2}$. By localization and the analogue of Proposition B.21, this implies $P = A_{1,2}$. $\qquad\square$

Corollary 2.11. *The algebra of formal periods P remains unchanged when we restrict in Definition 2.8 to (X, D, ω, γ) with X affine of dimension d, D of dimension $d-1$ and $X \smallsetminus D$ smooth, $\omega \in H_{\mathrm{dR}}^d(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$.*

Proof. By the same proof, Theorem 2.10 holds also for the diagram $\tilde{D}$ of very good pairs. By the analogue of Corollary 1.7, the comparison algebra $A_{1,2}$ is the same for both diagrams. $\qquad\square$

All formal effective periods (X, D, ω, γ) can be evaluated by "integrating" ω along γ. More precisely, there is a natural pairing

$$H_{\mathrm{dR}}^d(X, D) \times H_d(X(\mathbb{C}), D(\mathbb{C})) \to \mathbb{C}$$

This induces a ring homomorphism

$$\mathrm{ev} : P \to \mathbb{C}$$

which maps $(\mathbb{G}_m, \{1\}, dX/X, S^1)$ to $2\pi i$. Numbers in the image of ev are called *Kontsevich-Zagier periods*.

Corollary 2.12. *The algebra of Kontsevich-Zagier periods is generated by $(2\pi i)^{-1}$ together with periods of (X, D, ω, γ) with X smooth affine, D a divisor with normal crossings, $\omega \in \Omega^d(X)$.*

Proof. Note that the period $2\pi i$ is of this shape.

By Corollary 1.8 the category $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ is generated by motives of good pairs (X, Y, d) of the form $X = W \setminus W_\infty$, $Y = W_0 \setminus (W_\infty \cap W_0)$ with W smooth projective of dimension d and $W_0 \cup W_\infty$ a divisor with normal crossings. Hence their periods together with $(2\pi i)^{-1}$ generate the algebra of Kontsevich-Zagier periods.

By Remark 1.9 we can assume that $X' = W \setminus W_0$ is affine. Let $Y' = W_\infty \setminus (W_0 \cap W_\infty)$. By Lemma 1.13 the motive $H_{\mathrm{Nori}}^d(X, Y)$ is dual to $H_{\mathrm{Nori}}^d(X', Y')(d)$.

Hence the periods of $H_{\text{Nori}}^d(X, Y)$ are in the algebra generated by $(2\pi i)^{-1}$ and the periods of $H_{\text{Nori}}^d(X', Y')$. As X' is affine and Y' a divisor with normal crossings, $H_{\text{dR}}^d(X', Y')$ is generated by $\Omega(X')$. $\qquad\square$

3. Torsor structure on $P_{1,2}$

We return to the abstract setting. Let D be a graded diagram with commutative product structure, $T_1, T_2 \to \mathbb{Q}$-Mod two representations which become isomorphic after some field extension $K/\mathbb{Q}$. The isomorphism is denoted by

$$\varphi : T_1 \otimes K \to T_2 \otimes K.$$

Using the universal property of $\mathcal{C}(T_1)$ we immediately obtain:

Lemma 3.1. T_2 extends to a fiber functor

$$T_2 : \mathcal{C}(T_1) \to \mathbb{Q}\text{-Mod}.$$

Proof. Consider the abelian category $\mathcal{A}$ whose objects are pairs (V_1, V_2) of $\mathbb{Q}$-vector spaces together with an isomorphism $v : V_1 \otimes K \to V_2 \otimes K$. The data T_1, T_2 and φ together define a representation

$$T : D \to \mathcal{A}.$$

Via the projection to the first component T is compatible with T_1. The universal property (appendix B.10) implies that we have a commutative diagram of functors:

The composition of $\mathcal{C}(T_1) \to \mathcal{A}$ with the projection p_2 to the second component is the extension T_2. $\qquad\square$

In particular we have two fiber functors $T_1, T_2 : \mathcal{C}(T_1) \to \mathbb{Q}$-Mod.

Lemma 3.2. Assume $\mathcal{C}(T_1)$ is a neutral Tannakian category. In this way we obtain two affine group schemes $G_1 = \text{Aut}^\otimes(T_1)$, $G_2 = \text{Aut}^\otimes(T_2)$ and an affine scheme $X = X_{1,2} = \text{Iso}^\otimes(T_1, T_2)$. If A_1, A_2 and $A_{1,2}$ denote the Hopf algebras defined above then we have

$$G_1 = \text{Spec}(A_1), \; G_2 = \text{Spec}(A_2), \quad and \; X = \text{Spec}(A_{1,2}).$$

Proof. This follows (almost verbatim) the Tannakian pattern, see [DM]. $\qquad\square$

In a similar way we can define an affine scheme $X_{2,1} = \text{Iso}^\otimes(T_2, T_1)$. These schemes are related via natural morphisms

$$X_{1,2} \times X_{2,1} \longrightarrow G_1, \quad X_{2,1} \times X_{1,2} \longrightarrow G_2,$$

and

$$G_1 \times X_{1,2} \longrightarrow X_{1,2}, \quad X_{1,2} \times G_2 \longrightarrow X_{1,2}.$$

Theorem 3.3. *There is a natural isomorphism of affine schemes*

$$\iota : X_{1,2} \longrightarrow X_{2,1}$$

given by $f \mapsto f^{-1}$*. Furthermore,* $X_{1,2}$ *and* $X_{2,1}$ *carry the structure of affine torsors in the sense of appendix A.*

Proof. The assertion about ι is clear. Using the natural maps above, one obtains a commutative diagram

$$
\begin{array}{ccccc}
 & & X_{1,2} \times X_{1,2} \times X_{1,2} & & \\
 & & \downarrow{\scriptstyle id \times \iota \times id} & & \\
 & & X_{1,2} \times X_{2,1} \times X_{1,2} & & \\
 & \swarrow & \vdots & \searrow & \\
G_1 \times X_{1,2} & & \vdots & & X_{1,2} \times G_2 \\
 & \searrow & \downarrow & \swarrow & \\
 & & X_{1,2} & &
\end{array}
$$

with, as the composition of the two vertical maps, an induced morphism in the category of affine schemes

$$(\cdot, \cdot, \cdot) : X_{1,2} \times X_{1,2} \times X_{1,2} \longrightarrow X_{1,2}$$

fitting into the diagram. It satisfies the axioms of a affine torsor, as defined in appendix A. The assertion about $X_{2,1}$ is proved in a similar way. $\qquad\square$

Corollary 3.4. *The algebra of formal periods P (see Definition 2.8) has a natural torsor structure under G_{mot}.*

Proof. By Theorems 2.6 and 2.9 we have $A_{1,2} = P = P_{1,2}$ The previous theorem defines a map

$$A_{1,2} \longrightarrow A_{1,2} \otimes A_{1,2} \otimes A_{1,2},$$

i.e., a natural map

$$P \longrightarrow P \otimes P \otimes P.$$

Finally note that $G_1 = G_{\mathrm{mot}}$ is the motivic fundamental group by definition. $\quad\square$

Remark 3.5. In terms of period matrices this is given by the formula in [Ko]:

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P_{k\ell}^{-1} \otimes P_{\ell j}.$$

The torsors in this section are naturally topological torsors in the pro-fppf topology.

Appendix A. Torsors

Kontsevich uses the following definition of torsors in [Ko]. This notion at least goes back to a paper of R. Baer [Ba] from 1929, see the footnote on page 202 of loc. cit. where Baer explains how the notion of a torsor comes up in the context of earlier work of H. Prüfer [P]. In yet another context, ternary operations satisfying these axioms are called associative Malcev operations, see [J] for a short account.

Definition A.1 ([Ba] p. 202, [Ko] p. 61, [Fr] Definition 7.2.1)**.** A *torsor* is a set X together with a map

$$(\cdot,\cdot,\cdot) : X \times X \times X \to X$$

satisfying:

(1) $(x,y,y) = (y,y,x) = x$ for all $x, y \in X$
(2) $((x,y,z),u,v) = (x,(u,z,y),v) = (x,y,(z,u,v))$ for all $x,y,z,u,v \in X$.

Morphisms are defined in the obvious way, i.e., maps $X \to X'$ of sets commuting with the torsor structure.

Lemma A.2. *Let G be a group. Then $(g,h,k) = gh^{-1}k$ defines a torsor structure on G.*

Proof. This is a direct computation:

$$(x,y,y) = xy^{-1}y = x = yy^{-1}x = (y,y,x),$$

$$((x,y,z),u,v) = (xy^{-1}z,u,v) = xy^{-1}zu^{-1}v = (x,y,zu^{-1}v) = (x,y,(z,u,v)),$$

$$(x,(u,z,y),v) = (x,uz^{-1}y,v) = x(uz^{-1}y)^{-1}v = xy^{-1}zu^{-1}v.$$

$\square$

Lemma A.3 ([Ba] page 202)**.** *Let X be a torsor, $e \in X$ an element. Then $G_e := X$ carries a group structure via*

$$gh := (g,e,h), \quad g^{-1} := (e,g,e).$$

Moreover, the torsor structure on X is given by the formula $(g,h,k) = gh^{-1}k$ in G_e.

Proof. First we show associativity:

$$(gh)k = (g,e,h)k = ((g,e,h),e,k) = (g,e,(h,e,k)) = g(h,e,k) = g(hk).$$

e becomes the neutral element:

$$eg = (e,e,g) = g; ge = (g,e,e) = g.$$

We also have to show that g^{-1} is indeed the inverse element:

$$gg^{-1} = g(e,g,e) = (g,e,(e,g,e)) = ((g,e,e),g,e) = (g,g,e) = e.$$

Similarly one shows that $g^{-1}g = e$. One gets the torsor structure back, since

$$gh^{-1}k = g(e,h,e)k = (g,e,(e,h,e))k = ((g,e,(e,h,e)),e,k)$$
$$= (g,(e,(e,h,e),e),k) = (g,((e,e,h),e,e),k)$$
$$= (g,(h,e,e),k) = (g,h,k).$$

$\square$

Proposition A.4. *Let $\mu_l : X^2 \times X^2 \to X^2$ be given by*

$$\mu_l((a,b),(c,d)) = ((a,b,c),d).$$

Then μ_l is associative and has (x,x) for $x \in X$ as left-neutral elements. Let $G^l = X^2/\sim_l$ where $(a,b) \sim_l (a,b)(x,x)$ for all $x \in X$ is an equivalence relation. Then μ_l is well-defined on G^l and turns G^l into a group. Moreover, the torsor structure map factors via a simply transitive left G^l-operation on X which is defined by

$$(a,b)x := (a,b,x).$$

Let $e \in X$. Then

$$i_e : G_e \to G^l, \qquad x \mapsto (x, e)$$

is group isomorphism inverse to $(a, b) \mapsto (a, b, e)$.
In a similar way, using $\mu_r\left((a, b), (c, d)\right) := (a, (b, c, d))$ we obtain a group G^r with analogous properties acting transitively on the right on X and such that μ_r factors through the action $X \times G^r \to X$.

Proof. First we check associativity of μ_l:

$$(a, b)[(c, d)(e, f)] = (a, b)((c, d, e), f) = ((a, b, (c, d, e)), f) = (((a, b, c), d, e), f)$$
$$[(a, b)(c, d)](e, f) = ((a, b, c), d)(e, f) = (((a, b, c), d, e), f)$$

(x, x) is a left neutral element for every $x \in X$:

$$(x, x)(a, b) = ((x, x, a), b) = (a, b)$$

We also need to check that $\sim_l$ is an equivalence relation: $\sim_l$ is reflexive, since one has $(a, b) = ((a, b, b), b) = (a, b)(b, b)$ by the first torsor axiom and the definition of μ. For symmetry, assume $(c, d) = (a, b)(x, x)$. Then

$$(a, b) = ((a, b, b), b) = ((a, b, (x, x, b)), b) = (((a, b, x), x, b), b)$$
$$= ((a, b, x), x)(b, b) = (a, b)(x, x)(b, b) = (c, d)(b, b)$$

again by the torsor axioms and the definition of μ_l. For transitivity observe that

$$(a, b)(x, x)(y, y) = (a, b)((x, x, y), y) = (a, b)(y, y).$$

Now we show that μ_l is well-defined on G^l:

$$[(a, b)(x, x)][(c, d)(y, y)] = (a, b)[(x, x)(c, d)](y, y) = (a, b)(c, d)(y, y).$$

The inverse element to (a, b) in G^l is given by (b, a), since

$$(a, b)(b, a) = ((a, b, b), a) = (a, a).$$

Define the left G^l-operation on X by $(a, b)x := (a, b, x)$. This is compatible with μ_l, since

$$[(a, b)(c, d)]x = ((a, b, c), d)x = ((a, b, c), d, x),$$
$$(a, b)[(c, d)x] = (a, b)(c, d, x) = ((a, b, (c, d, x))$$

are equal by the second torsor axiom. The left G^l-operation is well-defined with respect to $\sim_l$:

$$[(a, b)(x, x)]y = ((a, b, x), x)y = ((a, b, x), x, y) = (a, (x, x, b), y) = (a, b, y) = (a, b)y.$$

Now we show that i_e is a group homomorphism:

$$ab = (a, e, b) \mapsto ((a, e, b), e) = (a, e)(b, e)$$

The inverse group homomorphism is given by

$$(a, b)(c, d) = ((a, b, c), d) \mapsto ((a, b, c), d, e).$$

On the other hand in G_e one has:

$$(a, b, e)(c, d, e) = ((a, b, e), e, (c, d, e)) = (a, b, (e, e, (c, d, e))) = (a, b, (c, d, e)).$$

This shows that i_e is an isomorphism. The fact that G_e is a group implies that the operation of G^l on X is simply transitive. Indeed the group structure on $G_e = X$ is the one induced by the operation of G^l. The analogous group G^r

is constructed using μ_r and an equivalence relation $\sim_r$ with opposite order, i.e., $(a, b) \sim_r (x, x)(a, b)$ for all $x \in X$. The properties of G^r can be verified in the same way as for G^l and are left to the reader. $\square$

Definition A.5. A *torsor* in the category of schemes is a scheme X and a morphism

$$X \times X \times X \to X$$

which on S-valued points is a torsor for all S.

This simply means that the diagrams of the previous definition commute as morphisms of schemes. The following is the scheme theoretic version of Lemma A.4.

Proposition A.6. *Let X be a torsor in the category of affine schemes. Then there are affine group schemes G^l and G^r operating from the left and right on X respectively such that X is a G^l- and G^r-torsor.*

Proof. We use Thm. 1.4 from [SGA3, exposé VII] to obtain affine scheme quotients by equivalence relations. In the case of G^l we have to construct the quotient $G^l = X^2 / \sim_l$. The case of G^r is similar. Then it follows also that the action homomorphism $G^l \times X \to X$ will be algebraic as the morphism μ_l also descends. In order to apply [SGA3] we need to construct two morphisms

$$p_0, p_1 : R \to X^2$$

from an affine scheme R to X^2, such that

$$R \to X^2 \times X^2$$

is a closed embedding, for all T the image $R(T) \to (X^2 \times X^2)(T)$ is an equivalence relation, i.e., reflexive, symmetric and transitive and such that the first projection $R \to X^2$ is flat. We choose

$$R := X^2 \times X$$

and set

$$p_0(a, b, x) := (a, b), \quad p_1(a, b, x) := (a, b)(x, x) = ((a, b, x), x).$$

Then the first projection $R \to X^2$ is flat. The image of R is an equivalence relation by Proposition A.4. It remains to show that $(p_0, p_1) : R \to X^2 \times X^2$ is a closed embedding. But this follows as $(p_0, p_1)(a, b, x) = ((a, b), ((a, b, x), x))$ is a graph type morphism. $\square$

APPENDIX B. LOCALIZATION AND MULTIPLICATION IN DIAGRAMS

A sketch of Nori's construction of motives is contained in Levine's survey [Le] §5.3. We use this as a starting point and develop the theory further to the extent needed for the proof of the main theorem 2.10 on periods. Full proofs for the basics of diagram categories can be found in von Wangenheim's diploma thesis [vW].

B.1. **Diagrams and diagram categories.** Let R be a noetherian ring.

Definition B.1. A *small diagram D* is a directed graph on a set of vertices such that for every vertex there is a distinguished edge id : $v \to v$. A diagram is called *finite* if it has only finitely many vertices. A *finite subdiagram* of a small diagram D is a diagram containing a finite subset of vertices of D and all edges (in D) between them.

Remark B.2. We added the notion of identity edges to Nori's definition given in [Le]. It is useful when considering multiplicative structures.

Example B.3. Let $\mathcal{C}$ be a small category. Then we can associate a diagram $D(\mathcal{C})$ with vertices the set of objects in $\mathcal{C}$ and edges given by morphisms.

Definition B.4 (Nori)**.** A *representation* T of a diagram D in a category $\mathcal{C}$ is a map T of directed graphs from D to $D(\mathcal{C})$ such that id is mapped to id.

We are particularly interested in categories of modules.

Definition B.5. By R-Mod we denote the category of finitely generated R-modules. By R-Proj we denote the subcategory of projective R-modules of finite type.

Nori constructs a certain universal abelian category $\mathcal{C}(T)$ attached to a diagram and a representation T. For later use we recall Nori's construction.

Definition B.6 (Nori)**.** Let T be a representation of D in R-Mod. For each finite subdiagram $F \subset D$ let $\mathrm{End}(T|_F)$ be the ring of endomorphisms of the functor $T|_F$, more precisely, as

$$\mathrm{End}(T|_F) := \left\{ (e_p)_{p \in F} \in \prod_{p \in F} \mathrm{End}_R(T(p)) \mid e_q \circ T(m) = T(m) \circ e_p \ \forall p, q \in F \ \forall m \in \mathrm{Mor}(p, q) \right\}.$$

Let

$$\mathcal{C}(T|_F) = \mathrm{End}(T|_F) - \mathrm{Mod}$$

be the category of finitely generated R-modules equipped with an R-linear operation of algebra $\mathrm{End}(T|_F)$. Finally let

$$\mathcal{C}(T) = \mathrm{colim}_F \mathcal{C}(T|_F) .$$

In the cases of most interest, there is more direct description.

Proposition B.7. *Suppose R is a Dedekind domain or a field and T takes values in R-Proj. Let $A(F, T) := \mathrm{Hom}_R(\mathrm{End}(T|_F), R)$. We set*

$$A(T) := \mathrm{colim}_F A(F, T).$$

Then $\mathcal{C}(T)$ is the category of finitely generated R-modules with an R-linear $A(T)$-comodule structure.

Proof. The assumptions on R and T ensure that $\mathrm{End}(T|_F, R)$ is a locally free R-module. This allows to pass to the comodule description for finite diagrams, then pass to the limit.
For full details see [vW] Satz 5.23. (He uses principal ideal domains. The arguments work without changes for Dedekind domains.) $\square$

The main step in Nori's construction of an abelian category of motives is the following result:

Proposition B.8 (Nori's diagram category)**.** *Let R be a noetherian ring. Let D be a diagram and $T : D \to R$-Mod be a representation. Then there is a category $\mathcal{C}(T)$ together with a faithful exact R-linear functor $f_T : \mathcal{C}(T) \to R$-Mod and a representation $\tilde{T} : D \to \mathcal{C}(T)$ such that $f_T \circ \tilde{T} = T$ and $\tilde{T}$ is universal with respect to this property, i.e., for any other representation $F : D \to \mathcal{A}$ into some R-linear*

abelian category $\mathcal{A}$ with a faithful exact functor $\mathcal{A} \to R\text{-}Mod$ there is a unique functor (up to unique isomorphism) $\mathcal{C}(T) \to \mathcal{A}$ such that

commutes up to unique isomorphism.
$\mathcal{C}(T)$ *is functorial in D and T in the obvious way.*

We are going to view f_T as an extension of T from D to $\mathcal{C}(T)$ and write simply T instead of f_T.

Proof. [Le, N]. A detailed proof is given in [Br] or in [vW] Theorem 2.4. The condition on R being noetherian is needed to ensure that $\text{End}(T|_F)$ is a finitely generated R-module. $\qquad\square$

The following properties hopefully allow a better understanding of the nature of $\mathcal{C}(T)$.

Proposition B.9.   (1) *As an abelian category $\mathcal{C}(T)$ is generated by the $\tilde{T}(v)$ where v runs through the set of vertices of D, i.e., it agrees with its smallest full subcategory containing all such $\tilde{T}(v)$.*
   (2) *Each object of $\mathcal{C}(T)$ is a subquotient of a finite direct sum of objects of the form $\tilde{T}(v)$.*
   (3) *If $\alpha : v \to v'$ is an edge in D such that $T(\alpha)$ is an isomorphism, then $\tilde{T}(\alpha)$ is also an isomorphism.*

Proof. Let $\mathcal{C}' \subset \mathcal{C}(T)$ be the subcategory generated by all $\tilde{T}(v)$. By definition the representation $\tilde{T}$ factors through $\mathcal{C}'$. By the universal property of $\mathcal{C}(T)$ we obtain a functor $\mathcal{C}(T) \to \mathcal{C}'$, hence an equivalence of subcategories of R-Mod.
The second statement follows from the first criterion since the full subcategory in $\mathcal{C}(T)$ of subquotients of finite direct sums is abelian hence agrees with $\mathcal{C}(T)$.
The assertion on morphisms follows since the functor $f_T : \mathcal{C}(T) \to R$-Mod is faithful and exact between abelian categories. $\qquad\square$

Theorem B.10. *Let R be a field and $\mathcal{A}$ be a neutral R-linear Tannakian category with fiber functor $T : D(\mathcal{A}) \to R\text{-}Mod$. Then $A(T)$ is equal to the Hopf algebra of the Tannakian dual.*

Proof. By construction, see [DM] Theorem 2.11 and its proof. $\qquad\square$

We need to understand the behavior under base-change. Let S be a noetherian R-algebra. Then $T \otimes S$ is an S-representation of D.

Lemma B.11 (Base change)**.** *Let S be a flat noetherian R-algebra and $T : D \to R\text{-}Proj$ a representation. Let $F \subset D$ be a finite subdiagram. Then:*
   (1)
$$\text{End}_S(T \otimes S|_F) = \text{End}_R(T|_F) \otimes S$$

(2) *If R is a field or a Dedekind domain and T takes values in R-Proj, then*

$$A(F, T \otimes S) = A(F,T) \otimes S \ , \qquad A(T \otimes S) = A(T) \otimes S \ .$$

Proof. We write $T_S = T \otimes S$.
$\mathrm{End}(T|_F)$ is defined as a limit, i.e., a kernel

$$0 \to \mathrm{End}(T|_F) \to \prod_{p \in O(D)} \mathrm{End}_R(T(p)) \xrightarrow{\phi} \prod_{m \in \mathrm{Mor}(p,q)} \mathrm{Hom}_R(T(p), T(q))$$

with $\phi(p)(m) = e_q \circ T(m) - T(m) \circ e_p$. As S is flat over R, this remains exact after $\otimes S$. As $T(p)$ is projective, we have

$$\mathrm{End}_R(T(p)) \otimes S = \mathrm{End}_S(T \otimes S(p))$$

Hence we get

$$0 \to \mathrm{End}(T|_F) \otimes S \to \prod_{p \in O(D)} \mathrm{End}_S(T_S(p)) \xrightarrow{\phi} \prod_{m \in \mathrm{Mor}(p,q)} \mathrm{Hom}_S(T_S(p), T_S(q))$$

This is the defining sequence for $\mathrm{End}(T_S|_F)$. This finishes the proof of the first statement.
Recall that $A(F,T) = \mathrm{Hom}_R(\mathrm{End}(T|_F), R)$. Both R and $\mathrm{End}_R(T|_F)$ are projective because R is now a field or a Dedekind domain. Hence

$$\mathrm{Hom}_R(\mathrm{End}_R(T|_F), R) \otimes S \cong \mathrm{Hom}_S(\mathrm{End}_R(T|_F) \otimes S, S) \cong \mathrm{Hom}_S(\mathrm{End}_S((T_S)|_F), S).$$

This is nothing but $A(F, T_S)$.
Tensor products commute with direct limit, hence the statement for $A(T)$ follows immediately. $\qquad\square$

B.2. **Multiplicative structure.** Construction and properties of the tensor structure are not worked out in detail in [N], [N1] or [Le]. In particular, we were puzzled by the question how the graded commutativity of the Künneth formula is dealt with in the construction. The following is an attempt to clarify this on the formal level. The first version of this section contained a serious mistake. We are particularly thankful to Gallauer for pointing out both the mistake and the correction.
Recall that R-Proj is the category of projective R-modules of finite type for a fixed noetherian ring R.

Definition B.12. Let D_1, D_2 be small diagrams. Then $D_1 \times D_2$ is the small diagram with vertices of the form (f,g) for f a vertex of D_1, g a vertex of D_2, and with edges of the form $\alpha \times \mathrm{id}$ and $\mathrm{id} \times \beta$ for α an edge of D_1 and β an edge of D_2 and with $\mathrm{id} = \mathrm{id} \times \mathrm{id}$.

Remark B.13. Levine in [Le] p.466 seems to define $D_1 \times D_2$ by taking the product of the graphs in the ordinary sense. He claims (in the notation of loc. cit.) a map of diagrams

$$H_* \mathrm{Sch}' \times H_* \mathrm{Sch}' \to H_* \mathrm{Sch}'.$$

We do not understand it on general pairs of edges. If α, β are edges corresponding to boundary maps and hence lower the degree by 1, then we would expect $\alpha \times \beta$ to lower the degree by 2. However, there are no such edges in $H_* \mathrm{Sch}'$.
Our restricted version of products of diagrams is enough to get the implication.

Definition B.14. A *graded diagram* is a small diagram D together with a map

$$| \cdot | : \{\text{vertices of } D \} \to \mathbb{Z}/2\mathbb{Z} \ .$$

For an edge $\gamma : e \to e'$ we put $|\gamma| = |e| - |e'|$. If D is a graded diagram, $D \times D$ is equipped with the grading $|(f, g)| = |f| + |g|$.

A *commutative product structure* on a graded D is a map of graded diagrams

$$\times : D \times D \to D$$

together with choices of edges

$$\alpha_{f,g} : f \times g \to g \times f$$
$$\beta_{f,g,h} : f \times (g \times h) \to (f \times g) \times h$$

for all vertices f, g, h of D.

A *graded representation* T of a graded diagram with commutative product structure is a representation of T in R-Proj together with a choice of isomorphism

$$\tau_{(f,g)} : T(f \times g) \to T(f) \otimes T(g)$$

such that:

(1) The composition

$$T(f) \otimes T(g) \xrightarrow{\tau^{-1}_{(f,g)}} T(f \times g) \xrightarrow{\alpha_{f,g}} T(g \times f) \xrightarrow{\tau_{(g,f)}} T(g) \otimes T(f)$$

  is $(-1)^{|f||g|}$ times the natural map of R-modules.

(2) If $\gamma : f \to f'$ is an edge, then the diagram

$$\begin{array}{ccc} T(f \times g) & \xrightarrow{T(\gamma \times \mathrm{id})} & T(f' \times g) \\ \tau \downarrow & & \downarrow \tau \\ T(f) \otimes T(g) & \xrightarrow{T(\gamma) \otimes \mathrm{id}} & T(f') \otimes T(g) \end{array}$$

  commutes.

(3) If $\gamma : f \to f'$ is an edge, then the diagram

$$\begin{array}{ccc} T(g \times f) & \xrightarrow{T(\gamma \times \mathrm{id})} & T(g \times f') \\ \tau \downarrow & & \downarrow \tau \\ T(g) \otimes T(f) & \xrightarrow{(-1)^{|\gamma|} \mathrm{id} \otimes T(\gamma)} & T(g) \otimes T(f') \end{array}$$

  commutes.

(4) The diagram

$$\begin{array}{ccc} T((f \times g) \times h) & \xrightarrow{T(\beta_{f,g,h})} & T(f \times (g \times h)) \\ \downarrow & & \downarrow \\ T(f) \otimes T(g \times h) & & T(f \times g) \otimes T(h) \\ \downarrow & & \downarrow \\ T(f) \otimes (T(g) \otimes T(h)) & \longrightarrow & (T(f) \otimes T(g)) \otimes T(h) \end{array}$$

  commutes under the standard identification

$$T(f) \otimes (T(g) \otimes T(h)) \cong (T(f) \otimes T(g)) \otimes T(h).$$

A *unit* for a graded diagram with commutative product structure D is a vertex $\mathbf{1}$ of degree 0 together with a choice of edges

$$u_f : f \to \mathbf{1} \times f$$

for all vertices of f. A graded representation is *unital* if $T(u_f)$ is an isomorphism for all vertices f.

Remark B.15. In particular, $T(\alpha_{f,g})$ and $T(\beta_{f,g,h})$ are isomorphisms. If $f = g$ then $T(\alpha_{f,f}) = (-1)^{|f|}$. If $\mathbf{1}$ is a unit, then $T(\mathbf{1})$ satisfies $T(\mathbf{1}) \cong T(\mathbf{1}) \otimes T(\mathbf{1})$. Hence it is a free R-module of rank 1.

Proposition B.16. *Let D be a graded diagram with commutative product structure with unit and T a unital graded representation of D in R-Proj.*

(1) *Then $\mathcal{C}(T)$ is a commutative and associative tensor category with unit and $T : \mathcal{C}(T) \to R$-Mod is a tensor functor.*

(2) *If in addition R is a field or a Dedekind domain, the coalgebra $A(T)$ carries a natural structure of commutative bialgebra (with unit and counit).*

The unit object is going to be denoted $\mathbf{1}$.

Proof. We consider finite diagrams F and F' such that

$$\{f \times g | f, g \in F\} \subset F' \ .$$

We are going to define natural maps

$$\mu_F^* : \text{End}(T|_{F'}) \to \text{End}(T|_F) \otimes \text{End}(T|_F).$$

Assume this for a moment. Let $X, Y \in \mathcal{C}(T)$. We want to define $X \otimes Y$ in $\mathcal{C}(T) = \text{colim}_F \mathcal{C}(T|_F)$. Let F such that $X, Y \in \mathcal{C}(T|_F)$. This means that X and Y are finitely generated R-modules with an action of $\text{End}(T|_F)$. We equip the R-module $X \otimes Y$ with a structure of $\text{End}(T|_F')$-module. It is given by

$$\text{End}(T|_F') \otimes X \otimes Y \to \text{End}(T|_F) \otimes \text{End}(T|_F) \otimes X \otimes Y \to X \otimes Y$$

where we have used the comultiplication map μ_F^* and the module structures of X and Y. This will be independent of the choice of F and F'. Properties of $\otimes$ on $\mathcal{C}(T)$ follow from properties of μ_F^*.

If R is a field or a Dedekind domain, let

$$\mu_F : A(F, T) \otimes A(F, T) \to A(F', T)$$

be dual to μ_F^*. Passing to the direct limit defines a multiplication μ on $A(T)$.

We now turn to the construction of μ_F^*. Let $a \in \text{End}(T|_{F'})$, i.e., a compatible system of endomorphisms $a_f \in \text{End}(T(f))$ for $f \in F'$. We describe its image $\mu_F^*(a)$. Let $(f, g) \in F \times F$. The isomorphism

$$\tau : T(f \times g) \to T(f) \otimes T(g)$$

induces an isomorphism

$$\text{End}(T(f \times g)) \cong \text{End}(T(f)) \otimes \text{End}(T(g)).$$

We define the (f, g)-component of $\mu^*(a)$ by the image of $a_{f \times g}$ under this isomorphism.

In order to show that this is a well-defined element of $\mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F)$, we need to check that diagrams of the form

$$
\begin{array}{ccc}
T(f) \otimes T(g) & \xrightarrow{\mu^*(a)_{(f,g)}} & T(f) \otimes T(g) \\
{\scriptstyle T(\alpha) \otimes T(\beta)}\big\downarrow & & \big\downarrow{\scriptstyle T(\alpha) \otimes T(\beta)} \\
T(f') \otimes T(g') & \xrightarrow[\mu^*(a)_{(f',g')}]{} & T(f') \otimes T(g')
\end{array}
$$

commute for all edges $\alpha : f \to f'$, $\beta : g \to g'$ in F. We factor

$$T(\alpha) \otimes T(\beta) = (T(\mathrm{id}) \otimes T(\beta)) \circ (T(\alpha) \circ T(\mathrm{id}))$$

and check the factors separately.
Consider the diagram

$$
\begin{array}{ccccc}
T(f \times g) & \xrightarrow{\qquad\qquad a_{f \times g} \qquad\qquad} & & & T(f \times g) \\
\big\downarrow & \searrow{\scriptstyle \tau} & & {\scriptstyle \tau}\swarrow & \big\downarrow \\
 & & T(f) \otimes T(g) \xrightarrow{\mu^*(a)_{(f,g)}} T(f) \otimes T(g) & & \\
{\scriptstyle T(\mathrm{id}\times\beta)} & {\scriptstyle T(\mathrm{id})\otimes T(\beta)}\big\downarrow & & \big\downarrow{\scriptstyle T(\mathrm{id})\otimes T(\beta)} & {\scriptstyle T(\mathrm{id}\times\beta)} \\
 & & T(f) \otimes T(g') \xrightarrow[\mu^*(a)_{(f,g')}]{} T(f) \otimes T(g') & & \\
\big\downarrow & \nearrow{\scriptstyle \tau} & & {\scriptstyle \tau}\nwarrow & \big\downarrow \\
T(f \times g') & \xrightarrow[\qquad\qquad a_{f \times g'} \qquad\qquad]{} & & & T(f \times g')
\end{array}
$$

The outer square commutes because a is a diagram endomorphism. Top and bottom commute by definition of $\mu^*(a)$. Left and right commute by property (3) up to the same sign $(-1)^{|f||\beta|}$. Hence the middle square commutes without signs. The analogous diagram for $\alpha \times \mathrm{id}$ commutes on the nose. Hence $\mu^*(a)$ is well-defined. We now want to compare the (f,g)-component to the (g,f)-component. Recall that there is a distinguished edge $\alpha_{f,g} : f \times g \to g \times f$. Consider the diagram

$$
\begin{array}{ccc}
 & T(f) \otimes T(g) \xrightarrow{\mu^*(a)_{(f,g)}} T(f) \otimes T(g) & \\
{\scriptstyle \tau}\nearrow \qquad\quad \big\downarrow \qquad\qquad\qquad \big\downarrow \qquad\quad \nwarrow{\scriptstyle \tau} & & \\
T(f \times g) \xrightarrow{\quad a_{f\times g} \quad} T(f \times g) & & \\
{\scriptstyle T(\alpha_{f,g})}\big\downarrow \qquad\qquad\qquad\qquad \big\downarrow{\scriptstyle T(\alpha_{f,g})} & & \\
T(g \times f) \xrightarrow{\quad a_{f\times g} \quad} T(g \times f) & & \\
{\scriptstyle \tau}\searrow \qquad\quad \big\downarrow \qquad\qquad\qquad \big\downarrow \qquad\quad \swarrow{\scriptstyle \tau} & & \\
 & T(g) \otimes T(f) \xrightarrow[\mu^*(a)_{(g,f)}]{} T(g) \otimes T(f) &
\end{array}
$$

By the construction of $\mu^*(a)_{(f,g)}$ (resp. $\mu^*(a)_{(g,f)}$) the upper (resp. lower) tilted square commutes. By naturality the middle rectangle with $\alpha_{f,g}$ commutes. By property (1) of a representation of a graded diagram with commutative product, the left and right faces commute where the vertical maps are $(-1)^{|f||g|}$ times the

natural commutativity of tensor products of T-modules. Hence the inner square also commutes without the sign factors. This is cocommutativity of μ^*.

The associativity assumption (3) for representations of diagrams with product structure implies the coassociativity of μ^*.

The compatibility of multiplication and comultiplication is built into the definition. In order to define a unit object in $\mathcal{C}(T)$ it suffices to define a counit for $\mathrm{End}(T|_F)$. Assume $\mathbf{1} \in F$. The counit

$$u^* : \mathrm{End}(T|_F) \subset \prod_{f \in F} \mathrm{End}(T(f)) \to \mathrm{End}(T(\mathbf{1})) = R$$

is the natural projection. The assumption on unitality of T allows to check that the required diagrams commute. $\qquad\square$

B.3. **Localization.** The purpose of this section is to give a diagram version of the localization of a tensor category with respect to one object, i.e., a distinguished object X becomes invertible with respect to tensor product. This is the standard construction used to pass e.g. from effective Chow motives to all motives. Again we thank Gallauer for pointing out a mistake in the original version as well as the correction.

We restrict to the case when R is a field or a Dedekind domain and all representations of diagrams take values in R-Proj.

Definition B.17 (Localization of diagrams). Let D^{eff} be a graded diagram with a commutative product structure with unit $\mathbf{1}$. Let $f_0 \in D^{\mathrm{eff}}$ be a vertex of even degree. The *localized diagram* D has vertices and edges as follows:

(1) for every f a vertex of D^{eff} and $n \in \mathbb{Z}$ a vertex denoted $f(n)$;
(2) for every edge $\alpha : f \to g$ in D^{eff} and every $n \in \mathbb{Z}$, an edge denoted $\alpha(n) : f(n) \to g(n)$ in D;
(3) for every vertex f in D^{eff} and every $n \in \mathbb{Z}$ an edge denoted $(f \times f_0)(n) \to f(n+1)$.

Put $|f(n)| = |f|$.

We equip D with a commutative product structure

$$\times : D \times D \to D \qquad f(n) \times g(m) \mapsto f \times g(n+m)$$

together with

$$\alpha_{f(n),g(m)} = \alpha_{f,g}(n+m)$$
$$\beta_{f(n),g(m),h(r)} = \beta_{f,g,h}(n+m+r)$$

Let $\mathbf{1}(0)$ together with

$$u_{f(n)} = u_f(n)$$

be the unit.

Note that there is a natural inclusion of multiplicative diagrams $D^{\mathrm{eff}} \to D$ which maps a vertex f to $f(0)$.

Remark B.18. The restriction to f_0 of even degree is not serious. If f_0 is odd, we consider localization with respect to $f_0 \times f_0$ instead.

Assumption B.19. Let T be a multiplicative unital representation of D^{eff} with values in R-Proj such that $T(f_0)$ is locally free of rank 1 as R-module.

Lemma B.20. *T extends uniquely to a graded representation of D such that $T(f(n)) = T(f) \otimes T(f_0)^{\otimes n}$ for all vertices and $T(\alpha(n) = T(\alpha) \otimes T(\mathrm{id})^{\otimes n}$ for all edges. It is multiplicative and unital with the choice*

$$
\begin{array}{ccc}
T(f(n) \times g(m)) & \xrightarrow{\ \tau_{f(n),g(m)}\ } & T(f(n)) \otimes T(g(m)) \\
{\scriptstyle \tau_{f,g}(n+m)}\Big\downarrow & & \Big\downarrow{\scriptstyle =} \\
T(f) \otimes T(g) \otimes T(f_0)^{\otimes n+m} & \xrightarrow{\ \cong\ } & T(f) \otimes T(f_0)^{\otimes n} \otimes T(g) \otimes T(f_0)^{\otimes m}
\end{array}
$$

where the last line is the natural isomorphism.

Proof. Define T on the vertices and edges of D via the formula. The conditions of $\tau_{f(n),g(m)}$ are satisfied because f_0 is even. $\qquad\square$

Proposition B.21. *Let D^{eff}, D and T be as above. Let A and A^{eff} be the corresponding bialgebras. Then:*

(1) *$\mathcal{C}(D,T)$ is the localization of $\mathcal{C}(D^{\mathrm{eff}},T)$ with respect to the object $\tilde{T}(f_0)$.*
(2) *Let $\chi \in \mathrm{End}(T(f_0))^\vee = A(\{f_0\},T)$ be the dual of $\mathrm{id} \in \mathrm{End}(T(f_0))$. We view it in A^{eff}. Then $A = A^{\mathrm{eff}}_\chi$ (localization of algebras).*

Proof. Let $D^{\geq n} \subset D$ be the subdiagram with vertices of the form $f(n')$ with $n' \geq n$. Clearly, $D = \mathrm{colim}_n D^{\geq n}$ and hence

$$\mathcal{C}(D,T) \cong \mathrm{colim}_n \mathcal{C}(D^{\geq n},T) \ .$$

Consider the morphism of diagrams

$$D^{\geq n} \to D^{\geq n+1}, \ \ f(m) \mapsto f(m+1).$$

It is clearly an isomorphism. We equip $\mathcal{C}(D^{\geq n+1})$ with a new fibre functor $f_T \otimes T(f_0)^\vee$. It is faithful exact. The map $f(m) \mapsto \tilde{T}(f(m+1))$ is a representation of $D^{\geq n+1}$ in the abelian category $\mathcal{C}(D^{\geq n+1},T)$ with fibre functor $f_T \otimes T(f_0)^\vee$. By the universal property this induces a functor

$$\mathcal{C}(D^{\geq n},T) \to \mathcal{C}(D^{\geq n+1},T) \ .$$

The converse functor is constructed in the same way. Hence

$$\mathcal{C}(D^{\geq n},T) \cong \mathcal{C}(D^{\geq n},T), \quad A^{\geq n} \cong A^{\geq n+1}.$$

The map of graded diagrams with commutative product and unit

$$D^{\mathrm{eff}} \to D^{\geq 0}$$

induces an equivalence on tensor categories. Indeed, we represent $D^{\geq 0}$ in $\mathcal{C}(D^{\mathrm{eff}},T)$ by mapping $f(m)$ to $\tilde{T}(f) \otimes \tilde{T}(f_0)^m$. By the universal property, this implies that there is a faithful exact functor

$$\mathcal{C}(D^{\geq 0},T) \to \mathcal{C}(D^{\mathrm{eff}},T)$$

inverse to the obvious inclusion. Hence we also have $A^{\mathrm{eff}} \cong A^{\geq 0}$ as unital bialgebras. On the level of coalgebras, this implies

$$A = \mathrm{colim} A^{\geq n} = \mathrm{colim} A^{\mathrm{eff}}$$

with $A^{\geq n} = A(D^{\geq n},T)$ isomorphic to A^{eff} as coalgebras. A^{eff} also has a multiplication, but the $A^{\geq n}$ do not. However, they carry an A^{eff}-module structure corresponding to the map of graded diagrams

$$D^{\mathrm{eff}} \times D^{\geq n} \to D^{\geq n}.$$

We want to describe the transition maps of the direct limit. From the point of view of $D^{\text{eff}} \to D^{\text{eff}}$ it is given by $f \mapsto f \times f_0$.

In order to describe $A^{\text{eff}} \to A^{\text{eff}}$ it suffices to describe $\text{End}(T|_F) \to \text{End}(T|_{F'})$ where F, F' are finite subdiagrams of D^{eff} such that $f \times f_0 \in O(F')$ for all vertices $f \in O(F)$. It is induced by

$$\text{End}(T(f)) \to \text{End}(T(f \times f_0)) \xrightarrow{\tau} \text{End}(T(f)) \otimes \text{End}(T(f_0)) \quad a \mapsto a \otimes \text{id}.$$

On the level of coalgebras this corresponds to the map

$$A^{\text{eff}} \to A^{\text{eff}}, \quad x \mapsto \chi x.$$

Note finally, that the direct limit $\text{colim} A^{\text{eff}}$ with transition maps given by multiplication by χ agrees with the localization A^{eff}_χ. $\qquad\square$

Appendix C. Nori's Rigidity Criterion

Implicit in Nori's construction of motives is a rigidity criterion, which we are now going to formulate and prove explicitly.

Let R be a Dedekind domain or a field and $\mathcal{C}$ an R-linear tensor category. Recall that R-Mod is the category of finitely generated R-modules and R-Proj the category of finitely generated projective R-modules.

We assume that the tensor product is associative, commutative and unital. Let $\mathbf{1}$ be the unit object. Let $T : \mathcal{C} \to R$-Mod be a faithful tensor functor with values in R-Mod. In particular, $T(\mathbf{1}) = R$.

We introduce an ad-hoc notion.

Definition C.1. Let V be an object of $\mathcal{C}$. We say that V *admits a perfect duality* if there is morphism

$$q : V \otimes V \to \mathbf{1}$$

or

$$\mathbf{1} \to V \otimes V$$

such that $T(V)$ is projective and $T(q)$ (respectively its dual) is a non-degenerate bilinear form.

Definition C.2. Let V be an object of $\mathcal{C}$. By $\langle V \rangle_\otimes$ we denote the smallest full abelian unital tensor subcategory of $\mathcal{C}$ containing V.

We start with the simplest case of the criterion.

Lemma C.3. *Let V be an object such that $\mathcal{C} = \langle V \rangle_\otimes$ and such that V admits a perfect duality. Then $\mathcal{C}$ is rigid.*

Proof. By standard Tannakian formalism, $\mathcal{C}$ is the category of comodules for a bialgebra A, which is commutative and of finite type as an R-algebra. Indeed, the construction of A as a coalgebra was explained in Proposition B.7. We want to show that A is a Hopfalgebra, or equivalently, that the algebraic monoid $M = \text{Spec} A$ is an algebraic group.

By Lemma C.6 it suffices to show that there is a closed immersion $M \to G$ of monoids into an algebraic group G. We are going to construct this group or rather its ring of regular functions. We have

$$A = \lim A_n$$

with A_n the Tannakian dual of $\mathcal{C}_n = \langle \mathbf{1}, V, V^{\otimes 2}, \ldots, V^{\otimes n} \rangle$, the smallest full abelian subcategory containing $\mathbf{1}, V, \ldots, V^{\otimes n}$. By construction there is a surjective map

$$\bigoplus_{i=0}^{n} \operatorname{End}_R((T(V)^{\otimes i})^{\vee}) \to A_n$$

or dually an injective map

$$A_n^{\vee} \to \bigoplus_{i=0}^{n} \operatorname{End}_R(T(V)^{\otimes i})$$

where $A_n^{\vee}$ consists of those endomorphisms compatible with all morphisms in $\mathcal{C}_n$. In the limit there is a surjection of bialgebras

$$\bigoplus_{i=0}^{\infty} \operatorname{End}_R((T(V)^{\otimes i})^{\vee}) \to A$$

and the kernel is generated by the relation defined by compatibility with morphisms in $\mathcal{C}$. One such relation is the commutativity constraint, hence the map factors via the symmetric algebra

$$S^*(\operatorname{End}(T(V)^{\vee}) \to A .$$

Note that $S^*(\operatorname{End}(T(V)^{\vee})$ is canonically the ring of regular functions on the algebraic monoid $\operatorname{End}(T(V))$. Another morphism in $\mathcal{C}$ is the pairing $q : V \otimes V \to \mathbf{1}$. We want to work out the explicit equation induced by q.

We choose a basis $e_1, \ldots, e_r$ of $T(V)$. Let

$$a_{i,j} = T(q)(e_i, e_j) \in R$$

By assumption the matrix is invertible. Let X_{st} be the matrix coefficients on $\operatorname{End}(T(V))$ corresponding to the basis e_i. Compatibility with q gives for every pair (i, j) the equation

$$\begin{aligned}
a_{ij} &= q(e_i, e_j) \\
&= q((X_{rs})e_i, (X_{r's'})e_j) \\
&= q\left(\sum_r X_{ri}e_r, \sum_{r'} X_{r'j}e_{r'} \right) \\
&= \sum_{r,r'} X_{ri}X_{r'j}q(e_r, e_{r'}) \\
&= \sum_{r,r'} X_{ri}X_{r'j}a_{rr'}
\end{aligned}$$

Note that the latter is the (i, j)-term in the product of matrices

$$(X_{ir})^t (a_{rr'})(X_{r'j}) .$$

Let $(b_{ij}) = (a_{ij})^{-1}$. With

$$(Y_{ij}) = (b_{ij})(X_{i'r})^t(a_{rr'})$$

we have the coordinates of the inverse matrix. In other words, our set of equations defines the isometry group $G(q) \subset \operatorname{End}(T(V))$. We now have expressed A as quotient of the ring of regular functions of $G(q)$.

The argument works in the same way, if we are given

$$q : \mathbf{1} \to V \otimes V$$

instead. $\qquad\square$

Proposition C.4 (Nori)**.** *Let $\mathcal{C}$ and $T : \mathcal{C} \to R\text{-}Mod$ be as defined at the beginning of the section. Let $\{V_i | i \in I\}$ be a set of objects of $\mathcal{C}$ with the properties:*

(1) *It generates $\mathcal{C}$ as an abelian tensor category, i.e., the smallest full abelian tensor subcategory of $\mathcal{C}$ containing all V_i is equal to $\mathcal{C}$.*

(2) *For every V_i there is an object W_i and a morphism*

$$q_i : V_i \otimes W_i \to \mathbf{1}$$

such that $T(q_i) : T(V_i) \otimes T(W_i) \to T(\mathbf{1}) = R$ is a perfect pairing of free R-modules.

Then $\mathcal{C}$ is rigid, i.e., for every object V there is a dual object $V^\vee$ such that

$$\mathrm{Hom}(V \otimes A, B) = \mathrm{Hom}(A, V^\vee \otimes B) \ , \qquad \mathrm{Hom}(V^\vee \otimes A, B) = \mathrm{Hom}(A, V \otimes B) \ .$$

This means that the Tannakian dual of $\mathcal{C}$ is not only a monoid but a group.

Remark C.5. The Proposition also holds with the dual assumption, existence of morphisms

$$q_i : \mathbf{1} \to V_i \otimes W_i$$

such that $T(q_i)^\vee : T(V)^\vee \otimes T(W_i)^\vee \to R$ is a perfect pairing.

Proof. Consider $V_i' = V_i \oplus W_i$. The pairing q_i extends to a symmetric map q_i' on $V_i' \otimes V_i'$ such that $T(q_i')$ is non-degenerate. We now replace V_i by V_i'. Without loss of generality, we can assume $V_i = W_i$.

For any finite subset $J \subset I$ let $V_J = \bigoplus_{j \in J} V_j$. Let q_J be the orthogonal sum of the q_j for $j \in J$. It is again a symmetric perfect pairing.

For every object V of $\mathcal{C}$ we write $\langle V \rangle_\otimes$ for the smallest full abelian tensor subcategory of $\mathcal{C}$ containing V. By assumption we have

$$\mathcal{C} = \bigcup_J \langle V_J \rangle_\otimes$$

We apply the standard Tannakian machinery. It attaches to every $\langle V_J \rangle_\otimes$ an R-bialgebra A_J such that $\langle V_J \rangle_\otimes$ is equivalent to the category of A_J-comodules. If we put

$$A = \lim A_J$$

then $\mathcal{C}$ will be equivalent to the category of A-comodules. It suffices to show that A_J is a Hopf-algebra. This is the case by Lemma C.3. $\qquad\square$

Finally, the missing lemma on monoids.

Lemma C.6. *Let R be noetherian ring, G be an algebraic group scheme of finite type over R and $M \subset G$ a closed immersion of a submonoid with $1 \in M(R)$. Then M is an algebraic group scheme over R.*

Proof. This seems to be well-known. It is appears as an exercise in [Re] 3.5.1 2. We give the argument:

Let S be any finitely generated R-algebra. We have to show that the functor $S \mapsto M(S)$ is a group. We take base change of the situation to S. Hence without loss of generality, it suffices to consider $R = S$. If $g \in G(R)$, we denote the isomorphism $G \to G$ induced by left multiplication with g also by $g : G \to G$. Take any $g \in G(R)$ such that $gM \subset M$ (for example $g \in M(R)$). Then one has

$$M \supseteq gM \supseteq g^2 M \supseteq \cdots$$

As G is Noetherian, this sequence stabilizes, say at $s \in \mathbb{N}$:

$$g^s M = g^{s+1} M$$

as closed subschemes of G. Since every g^s is an isomorphism, we obtain that

$$M = g^{-s} g^s M = g^{-s} g^{s+1} M = gM$$

as closed subschemes of G. So for every $g \in M(R)$ we showed that $gM = M$. Since $1 \in M(R)$, this implies that $M(R)$ is a subgroup. $\qquad\square$

Appendix D. Yoga of good pairs

We recall the definition of good pairs from the main text.

Let R be noetherian ring, k a subfield of $\mathbb{C}$. A variety is a separated reduced scheme of finite type over k. We denote by $X(\mathbb{C})$ the set of complex points equipped with the analytic topology.

We denote by $\mathbb{Z}[\mathrm{Var}]$ the additive category whose objects are varieties over k and whose morphisms on connected varieties are formal $\mathbb{Z}$-linear combinations of morphisms between varieties. We denote $\mathbb{Z}[\mathrm{Aff}]$ the full subcategory whose objects are affine varieties.

By base change to $\mathbb{C}$ we can consider the corresponding analytic space and its singular cohomology.

Definition D.1. (1) A triple (X, Y, i) of a variety X and a closed subvariety Y and an integer i is called *good pair* if singular cohomology satisfies

$$H^j(X(\mathbb{C}), Y(\mathbb{C}); R) = 0, \text{ unless } j = i.$$

and $H^i(X(\mathbb{C}), Y(\mathbb{C}); R)$ is free.
  (2) The diagram D^{eff} of good pairs has as vertices good pairs. There are two types of edges between effective good pairs: first the edges induced by morphisms $f^* : (X', Y', i) \to (X, Y, i)$ of triples for $f : X \to X'$ and $f(Y) \subset Y'$. The second type of edges $\partial : (Y, Z, i) \to (X, Y, i+1)$ arises for every chain $X \supset Y \supset Z$ of closed k-subvarieties of X (coboundary).
  (3) A good pair is called *very good* if X is affine and $X \smallsetminus Y$ smooth and either X of dimension i and Y of dimension $i-1$ or $X = Y$ of dimension less than i.
  (4) The diagram $\tilde{D}^{\mathrm{eff}}$ of very good pairs has as vertices the very good pairs and edges as in D^{eff}.

Lemma D.2 (Basic Lemma of Nori). *Let X be an affine scheme of finite type over $k \subset \mathbb{C}$ of dimension n and $Z \subset X$ be a closed subscheme of dimension $\leq n-1$. Then there is a closed subscheme $Y \supset Z$ such that (X, Y, n) is a good pair.*

  - $\dim(Y) \leq n-1$.
  - $H^i(X(\mathbb{C}), Y(\mathbb{C}); R) = 0$ *for* $i \neq n$.
  - $H^n(X(\mathbb{C}), Y(\mathbb{C}); R)$ *is a free R-module.*

Moreover, $X \smallsetminus Y$ can be chosen smooth.

A similar result holds in arbitrary characteristic by work of Beilinson [B, Lemma 3.3] and Kari Vilonen apparently used similar methods in his Master thesis [V]. The aim of this appendix is to establish the following result.

Proposition D.3. *Let $\mathcal{A}$ be an R-linear abelian category with a faithful forgetful functor to R-Mod. Let $T : \tilde{D}^{\mathrm{eff}} \to \mathcal{A}$ be a representation. Then there is a natural contravariant triangulated functor*

$$R : C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$$

on the category of bounded homological complexes in $\mathbb{Z}[\mathrm{Var}]$ such that for every good pair (X, Y, i) we have

$$H^j(R(\mathrm{Cone}(Y \to X))) = \begin{cases} 0 & j \neq i \\ T(X, Y, i) & j = i \end{cases}$$

Moreover, the image of $R(X)$ in $D^b(R\text{-}Mod)$ computes singular cohomology of $X(\mathbb{C})$.

We are mostly interested in two explicit examples of complexes.

Definition D.4. Consider the situation of Proposition D.3. Let $Y \subset X$ be a closed subvariety with open complement U, $i \in \mathbb{Z}$. Then we put

$$R(X, Y) = R(\mathrm{Cone}(Y \to X)), \quad R_Y(X) = R(\mathrm{Cone}(U \to X)) \in D^b(\mathcal{A})$$

$$T(X, Y, i) = H^i(R(X, Y)), \quad T_Y(X, i) = H^i(R_Y(X)) \in \mathcal{A}$$

$T(X, Y, i)$ is called *relative cohomology*. $T_Y(X, i)$ is called *cohomology with support.*

The strategy of the proof combines a variation of constructions in [Hu2] and a key idea of Nori.

The first step is to replace arbitrary complexes by affine ones. The idea for the following construction is from the étale case, see [F] Definition 4.2.

Definition D.5. Let X a variety. A *rigidified* affine cover is a finite open affine covering $\{U_i\}_{i \in I}$ together with a choice of an index i_x for every closed point $x \in X$ such that $x \in U_{i_x}$. We also assume that in the covering every index $i \in I$ occurs as i_x for some $x \in X$.

Let $f : X \to Y$ be a morphism of varieties, $\{U_i\}_{i \in I}$ a rigidified open cover of X and $\{V_j\}_{j \in J}$ a rigidified open cover of Y. A *morphism* of rigidified covers (over f)

$$\phi : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$$

is a map of sets $\phi : I \to J$ such that $f(U_i) \subset V_{\phi(i)}$ and for all $x \in X$ we have $\phi(i_x) = j_{f(x)}$.

Remark D.6. Under these conditions the rigidification makes ϕ unique if it exists.

Lemma D.7. *The projective system of rigidified affine covers is filtered and strictly functorial, i.e., if $f : X \to Y$ is a morphism of varieties, pull-back defines a map of projective systems.*

Proof. Any two covers have their intersection as common refinement. The rigidification extends in the obvious way. Preimages of rigidified covers are rigidified open covers. $\square$

Definition D.8. Let $F = \sum a_i f_i : X \to Y$ be a morphism in $\mathbb{Z}[\mathrm{Var}]$. The *support* of F is the set of f_i with $a_i \neq 0$.

Let X_* be a homological complex of varieties, i.e., an object in $C_b(\mathbb{Z}[\mathrm{Var}])$. An *affine cover* of X_* is a complex of rigidified affine covers, i.e., for every X_n the choice of a rigidified open cover $\tilde{U}_{X_n}$ and for every $g : X_n \to X_{n-1}$ in the support

of the differential $X_n \to X_{n-1}$ in the complex X_* a morphism of rigidified covers $\tilde{g} : \tilde{U}_{X_n} \to \tilde{U}_{X_{n-1}}$ over g.

Let $F_* : X_* \to Y_*$ be a morphism in $C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$, $\tilde{U}_{Y_*}$ affine covers of X_* and Y_*. A morphism of affine covers over F_* is a morphism of ridigied affine covers $f_n : \tilde{U}_{X_n} \to \tilde{U}_{Y_n}$ over every morphism in the support of F_n.

Lemma D.9. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Then the projective system of rigidified affine covers of X_* is non-empty, filtered and functorial, i.e. if $f_* : X_* \to Y_*$ is a morphism of complexes and $\tilde{U}_{X_*}$ an affine cover of X_*, then there is affine cover $\tilde{U}_{Y_*}$ and a morphism or complexes of rigidified affine covers. Any two choices are compatible in the projective system of covers.*

Proof. Let n be minimal with $X_n \neq \emptyset$. Choose a rigidified cover of X_n. The support of $X_{n+1} \to X_n$ has only finitely many elements. Choose a rigidified cover of X_{n+1} compatible with all of them. Continue inductively.

Similar constructions show the rest of the assertion. $\qquad\square$

Definition D.10. Let X be a variety and $\tilde{U}_X = \{U_i\}_{i \in I}$ a rigidified affine cover of X. We put
$$C_\star(\tilde{U}_X) \in C_-(\mathbb{Z}[\mathrm{Aff}]),$$
the Čech complex associated to the cover, i.e.,
$$C_n(\tilde{U}_X) = \coprod_{\underline{i} \in I_n} \bigcap_{i \in \underline{i}} U_i,$$
where I_n is the set of tuples $(i_0, \dots, i_n)$. The boundary maps are the ones obtained by taking the alternating sum of the boundary maps of the simplicial scheme.

If $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ is a complex, $\tilde{U}_{X_*}$ a rigidified affine cover, let
$$C_\star(\tilde{U}_{X_*}) \in C_{-,b}(\mathbb{Z}[\mathrm{Aff}])$$
be the double complex $C_i(\tilde{U}_{X_j})$.

Note that all components of $C_\star(\tilde{U}_{X_*})$ are affine. The projective system of these complexes is filtered and functorial.

In the second step, we replace every affine X by a complex of very good pairs. This follows the key idea of Nori as follows: Using induction one gets from the Basic Lemma D.2:

Corollary D.11. *Every affine variety X has a filtration*
$$\emptyset = F_{-1}X \subset F_0 X \subset \cdots \subset F_{n-1}X \subset F_n X = X,$$
such that $(F_j X, F_{j-1}X, j)$ is very good.

Filtrations of the above type are called *very good filtrations*.

Proof. Let $\dim X = n$. Put $F_n X = X$. Choose a subvariety of dimension $n-1$ which contains all singular points of X. By the Basic Lemma there is a subvariety $F_{n-1}X$ of dimension $n-1$ such that $(F_n X, F_{n-1}X, n)$ is good. By construction $F_{n-1}X \smallsetminus F_{n-1}X$ is smooth and hence the pair is very good. We continue by induction. $\qquad\square$

Corollary D.12. *Let X be an affine variety. The inductive system of all very good filtrations of X is filtered and functorial.*

Proof. We follow Nori, [N]. Let F_*X and F'_*X be two very good filtrations of X. $F_{n-1}X \cup F'_{n-1}X$ has dimension $n-1$. By the Basic Lemma there is subvariety $G_{n-1}X \subset X$ of dimension $n-1$ such that $(X, G_{n-1}X, n)$ is a good pair. It is automatically very good. We continue by induction.

Consider a morphism $f : X \to X'$. Let F_*X be a very good filtration. Then $f(F_iX)$ has dimension at most i. As in the proof of Corollary D.11, we construct a very good filtration F_*X' with additional property $f(F_iX) \subset F_iX'$. $\qquad\square$

Definition D.13. Let X be a variety, $\{U_i\}_{i\in I}$ a rigidified affine cover of X. A *very good filtration* on $\tilde{U}_X$ is the choice of very good filtrations for

$$\bigcap_{i\in J} U_i$$

for all $J \subset I$ compatible with all inclusions between these.

Let $f : X \to Y$ be a morphism of varieties, $\phi : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ a morphism of rigidified affine covers above f . Fix very good filtrations on both covers. ϕ is called *filtered*, if for all $I' \subset I$ the induced map

$$\bigcap_{i\in J} U_i \to \bigcap_{i\in I} V_{\phi(i)}$$

is compatible with the filtrations.

Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ be bounded complex of varieties, $\tilde{U}_{X_*}$ an affine cover of X_*. A *very good filtration* on $\tilde{U}_{X_*}$ is a very good filtration on all $\tilde{U}_{X_n}$ compatible with all morphisms in the support of the boundary maps.

Note that the Čech complex associated to a rigidified affine cover with very good filtration is also filtered in the sense that there is a very good filtration on all $C_n(\tilde{U}_X)$ and all morphisms in the support of the differential are compatible with the filtrations.

Lemma D.14. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover. The inductive system of very good filtrations on $\tilde{U}_X$ is non-empty, filtered and functorial. The same statement also holds for a complex of varieties $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$.*

Proof. Let $\tilde{U}_X = \{U_i\}_{i\in I}$ be the affine cover. We chose recursively very good filtration on $\bigcap_{i\in J} U_i$ with decreasing order of J, compatible with the inclusions. We extend the construction inductively to complexes, starting with the highest term of the complex. $\qquad\square$

Definition D.15. Let $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. A *very good filtration* of X_* is given by a very good filtration $F_.X_n$ for all n which is compatible with all morphisms in the support of the differentials of X_*.

Lemma D.16. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$ an affine cover of X_* with a very good filtration. Then the total complex of $C_\star(\tilde{U}_{X_*})$ carries a very good filtration.*

Proof. Clear by construction. $\qquad\square$

Recall that $\tilde{D}^{\mathrm{eff}} \to \mathcal{A}$ be a representation is of the diagram of very good pairs.

Definition D.17. Let $F_.X$ be an affine variety with a very good filtration. We put $\tilde{R}(F_.X) \in C^b(\mathcal{A})$

$$\cdots \to T(F_jX_*, F_{j-1}X_*) \to T(F_{j+1}X_*, F_jX_*) \to \ldots$$

Let $F.X_*$ be a very good filtration of a complex $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. We put $\tilde{R}(F.X_*) \in C^+(\mathcal{A})$ the total complex of the double complex $\tilde{R}(F.X_n)_{n \in \mathbb{Z}}$.

Proof of Proposition D.3: We first define $R : C^b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$ on objects. Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Choose a rigidified affine cover $\tilde{U}_{X_*}$ of X_*. Choose a very good filtration on the cover. In induces a very good filtration on $\mathrm{Tot}C_\star(\tilde{U}_{X_*})$. Put

$$R(X_*) = \tilde{R}(\mathrm{Tot}C_\star(\tilde{U}_{X_*})).$$

Note that any other choice yields a complex isomorphic to this one in $D^+(\mathcal{A})$.

Let $f : X_* \to Y_*$ be a morphism. Choose a refinement $\tilde{U}'_{X_*}$ of $\tilde{U}_{X_*}$ which maps to $\tilde{U}_{Y_*}$ and a very good filtration on $\tilde{U}'_{X_*}$. Choose a refinement of the filtrations on $\tilde{U}_{X_*}$ and $\tilde{U}_{Y_*}$ compatible with the filtration on $\tilde{U}'_{X_*}$. This gives a little diagram of morphisms of complexes $\tilde{R}$ which defines $R(f)$ in $D^+(\mathcal{A})$. $\qquad\square$

Remark D.18. Nori suggests working with Ind-objects (or rather pro-object in our dual setting) in order to get functorial complexes attached to affine varieties. However, the mixing between inductive and projective systems in our construction does not make it obvious if this works out for the result we needed. In order to avoid this situation, one could however do the construction in two steps.

As a corollary of the construction in the proof, we also get:

Corollary D.19. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover with Čech complex $C_\star(\tilde{U}_X)$. Then*

$$R(X) \to R(C_\star(\tilde{U}_X))$$

is an isomorphism in $D^+(\mathcal{A})$.

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