REPORT ON THE STRUCTURE OF PERIOD SPACES

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Abstract. The vector space generated by the periods of a single motive is finite-dimensional. In this expository article we review what is known about this dimension and the structure of the period space. We concentrate on the case of 1-motives, where these are known facts rather than conjectures.

Contents

1. Introduction

Periods are complex numbers of the form

$$\int_\sigma \omega$$

where ω is an algebraic differential d-form on some algebraic variety over $\overline{\mathbb{Q}}$ and σ is a chain in the sense of singular homology: a formal linear combinations of smooth maps from the d-simplex Δ_d to X^{an}. This set (in fact algebra) contains numbers like $2\pi i$ or $\log(2)$, which are of long-standing interest in transcendence theory. In fact, Grothendieck's *period conjecture* (see Conjecture 6.1 below) makes a precise prediction on the transcendence degree of the algebra generated by a set of periods.

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We are interested in a variant of the question. Rather than the transcendence degree of an algebra we study the *dimension* of the *vector space* generated by a set of periods.

In our report, we concentrate on 1-periods (the case $d = 1$ in the above definition). In this case the set has a very good conceptual description as *periods of* 1*-motives*. Via Wüstholz's Analytic Subgroup Theorem, the conjectures can actually be verified. This has been explored in detail in the monograph [HW22]. In particular, unconditional dimension formulas are available that describe the dimension of a period space of a 1-motive in terms of its data.

The report follows closely the contents of the survey talk at the Simon's Symposium. We explain the results of [HW22] and the main results of Nesa's thesis [Nes22]. He gives a more conceptual reinterpretation of the terms in the dimension formulas and investigates the connection to the Tannakian version of the picture and the older literature.

Notably, Brown [Bro17] gives an explicit description of the coradical filtration of the period algebra under a technical assumption. However, as we will explain in Section 6 this does not translate directly into formulas for period spaces.

Under the period conjecture, describing the period algebra amounts to describing a motivic Galois group or a Mumford-Tate group. The case of 1-motives has been studied by Bertrand [Ber98] and Bertolin [Ber02, Ber20]. They introduce and study the notion of a *deficient* 1-motive based on an explicit construction by Jacquinot and Ribet in [JR87]. The (conjectural) transcendence degree of their period algebra is smaller than in the generic case. Betrand asked me why this phenomenon does not appear in the study of period spaces. Nesa discusses in detail the period space of such deficient 1-motive, answering this question. We explain this in Section 7

As the report will make clear, the theory is not yet in final shape, see Remark 5.9.

Acknowledgments. This note discusses results obtained jointly with Gisbert Wüstholz in [HW22]. I thank him for a wonderful and productive collaboration.

The discussion of the interplay between the period space and the period algebra owes a lot to the input of Peter Jossen to myself and Nicola Nesa. The type of example to look for in Section 6 were pointed out by Fritz Hörmann, who was also a rich source of advice on everything to do with Tannaka duality.

Finally, I am thankful to Francis Brown for pointing me to [Bro17] and his patient explanations during the Simons Symposium. I am optimistic that the approach suggested by his formulas will settle the remaining open questions that I still see.

2. Periods and the weight filtration

We work over the field of algebraic numbers $\overline{\mathbb{Q}}$ and fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. All motives are over $\overline{\mathbb{Q}}$ with $\mathbb{Q}$-coefficients.

We generally use the notation of [HW22]. We denote by $1\text{-}\mathrm{Mot}_{\overline{\mathbb{Q}}}$ the category of *iso-1-motives* over $\overline{\mathbb{Q}}$ studied by Deligne in [Del74]. Its objects have the form $[L \to G]$ where L is a free abelian group of finite rank, and G a semi-abelian group over $\overline{\mathbb{Q}}$, i.e., an algebraic group which is an extension

$$0 \to T \to G \to A \to 0$$

of an abelian variety by a torus, and $L \to G$ is a group homomorphism. Morphisms in the category are morphisms of complexes tensored with $\mathbb{Q}$. In particular, the category $1\text{-}\mathrm{Mot}_{\overline{\mathbb{Q}}}$ contains the isogeny category of the category of abelian varieties as a full subcategory. The category is abelian.

The category of Nori motives is equipped with exact weight filtration functors. They have an explicit description in the case of 1-motives.

Definition 2.1 (Deligne)**.** The *weight filtration* on $M = [L \to G]$ is given by

$$W_n M = \begin{cases} 0 & n < -2, \\ [0 \to T] & n = -2, \\ [0 \to G] & n = -1, \\ M & n \geq 0, \end{cases}$$

where T is the torus part of G.

Moreover, there are (at least) two natural functors on the category of motives that we call *realisations*. Again they have a very explicit description in the case of $1\text{-}\mathrm{Mot}_{\overline{\mathbb{Q}}}$. We refer to Deligne's original article [Del74] or [HW22, Chapter 8] for details.

Definition 2.2 (Deligne)**.** Let $M \in 1\text{-}\mathrm{Mot}_{\overline{\mathbb{Q}}}$. We denote $V_{\mathrm{sing}}(M)$ the *singular realisation* of M, a finite dimensional $\mathbb{Q}$-vector space, and by $V_{\mathrm{dR}}(M)$ the *de Rham realisation* of M, a finite dimensional $\overline{\mathbb{Q}}$-vector space. We denote by

$$\phi_M : V_{\mathrm{sing}}(M)_{\mathbb{C}} \to V_{\mathrm{dR}}(M)_{\mathbb{C}}$$

the *period isomorphism*.

In particular, the two realisations are vector spaces of the same dimension, actually of dimension

$$\mathrm{rk}(L) + 2 \dim A + \dim T,$$

where $M = [L \to G]$ and A, T are the abelian and torus parts of G, respectively.

This allows us to give the central definition of this note:

Definition 2.3. Let $M \in 1\text{–Mot}_{\overline{\mathbb{Q}}}$ be a one-motive. The *space of periods of* M is the $\overline{\mathbb{Q}}$-subspace $\mathcal{P}\langle M\rangle$ of $\mathbb{C}$ generated by the entries of a matrix for the period isomorphism ϕ_M in a $\mathbb{Q}$-basis of $V_{\mathrm{sing}}(M)$ and a $\overline{\mathbb{Q}}$-basis of $V_{\mathrm{dR}}(M)$.

While the entries of the period matrix depend on a choice of basis, the vector space generated by it does not. Alternatively, we can consider the *period pairing*

$$V_{\mathrm{dR}}^{\vee}(M) \times V_{\mathrm{sing}}(M) \to \mathbb{C}, \quad (\omega, \sigma) \mapsto \omega(\phi_M(\sigma))$$

where $V_{\mathrm{dR}}^{\vee}(M)$ is the space of $\overline{\mathbb{Q}}$-linear forms on $V_{\mathrm{dR}}(M)$. From this point of view, $\mathcal{P}\langle M\rangle$ is the subgroup generated by the image of period pairing or the image of the linear map

$$V_{\mathrm{dR}}^{\vee}(M) \otimes_{\mathbb{Q}} V_{\mathrm{sing}}(M) \to \mathbb{C}.$$

. Note that $\mathcal{P}\langle M\rangle$ is finite dimensional over $\overline{\mathbb{Q}}$.

Question 2.4. What is the dimension of $\mathcal{P}(M)$?

This is what we want to address here. Note that the weight filtration on M induces filtrations on $V_{\mathrm{sing}}(M)$ and $V_{\mathrm{dR}}^{\vee}(M)$ and hence a bifiltration on $\mathcal{P}\langle M\rangle$. In fact, in bases adapted to the filtrations, the period matrix has a structure

$$\begin{pmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & P_{22} \end{pmatrix}.$$

We will analyse the blocks one by one. The entries P_{00}, P_{11}, P_{22} correspond to pure 1-motives (see Section 3, and the entries P_{01}, P_{12} to 1-extensions (see Section 4. It is the contribution of P_{02} which is most complex and somewhat mysterious(see Section 5.

3. Pure motives

The category $1\text{–Mot}_{\overline{\mathbb{Q}}}$ is Artinian: every object is a finite extension of simple ones. There are three types of simple 1-motives:

- $M = [\mathbb{Z} \to 0]$ with space of periods equal to $\overline{\mathbb{Q}}$;
- $M = [0 \to \mathbb{G}_m]$ with space of periods equal to $2\pi i\overline{\mathbb{Q}}$;
- $M = [0 \to A]$ where A is a simple abelian variety.

Their period spaces have trivial intersection.

Proposition 3.1. *Let* M_1, M_2 *be non-isomorphic simple* 1*-motives. Then*

$$\mathcal{P}\langle M_1\rangle \cap \mathcal{P}\langle M_2\rangle = 0.$$

This is a special case of [HW22, Theorem 16.2].

Moreover, the space of periods of a simple abelian variety is well-understood. If A is simple of dimension g, then its singular and de Rham realisation have dimension $2g$. Endomorphisms of A give rise to relations between periods.

Proposition 3.2. *Let A be a simple abelian variety of dimension g with endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(A)$ of dimension e. Then*

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}\langle A\rangle = \frac{4g^2}{e}.$$

This is a special case of [HW22, Proposition 15.14], but also appears in earlier work Wüstholz. Translated to the language of periods of curves, these periods correspond to periods obtained by integrating differential forms of the first or second kind (differential forms without poles, or without residues, respectively) on smooth projective curves over $\overline{\mathbb{Q}}$ over closed paths.

Remark 3.3. Equivalently, we can say that $V_{\mathrm{sing}}(A)$ and $V_{\mathrm{dR}}^{\vee}(A)$ are $d = 2g/e$-dimensional vector spaces over the division algebra $\mathrm{End}_{\mathbb{Q}}(A)$ and then

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}\langle M\rangle = d^2 e.$$

It turns out that this second point of view is the one that generalises well.

Together these propositions give a dimension formula for periods of pure motives.

Corollary 3.4. *Let $M = \bigoplus_{i=1}^N M_i^{n_i}$ be a direct sum with non-isomorphic simple motives M_i. Then*

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}\langle M\rangle = \sum_{i=1}^N \dim_{\overline{\mathbb{Q}}} \mathcal{P}\langle M_i\rangle$$

where $\dim \mathcal{P}\langle M_i\rangle = 1$ if $M_i = [\mathbb{Z} \to 0], [0 \to \mathbb{G}_m]$ and given by the formula in Proposition 3.2 if $M = [0 \to A]$ with a simple abelian variety.

4. 1-extensions

A general 1-motive $[L \to G]$ contains two subquotients which are 1-extensions of pure motives:

$$[L \to A], \quad [0 \to G]$$

where $0 \to T \to G \to A$ is the decomposition of G into its torus and abelian part. Sometimes, there is also a subquotient of the form

$$[L \to T]$$

for a torus T. These are the motives whose periods we consider in this section.

Remark 4.1. In terms of periods of curves, a 1-motive of the form $[L \to A]$ appears for differential forms of the first or second kind on a curve of positive genus, integrated over formal linear combinations of paths with end points defined over $\overline{\mathbb{Q}}$. A 1-motive of the form $[0 \to G]$ corresponds to an arbitrary algebraic differential form (no assumption on the poles or residues) on a curve of positive genus, integrated over a closed path. A 1-motive of the form $[L \to T]$ is found when describing differential forms of the third kind on a curve of genus 0 integrated over a formal linear combination of paths with points defined over $\overline{\mathbb{Q}}$.

This list is not exclusive: they can also appear for non-trivial reasons inside more complicated motives.

Their periods only intersect, if there is a trivial reason.

Proposition 4.2 ([HW22, Theorem 16.2]). *Assume that G is a semi-abelian variety with abelian part A. Then:*

$$\mathcal{P}\langle[L \to A]\rangle \cap \mathcal{P}\langle[0 \to G]\rangle = \mathcal{P}\langle A\rangle.$$

We now concentrate on the case $M = [L \to A]$ and assume $L \neq 0$, $A \neq 0$. We call them *motives of the second kind.* As in the pure case, periods for non-isogenous simple abelian varieties do not interact.

Proposition 4.3. *Let A_1, A_2 be non-isogenous simple abelian varieties, $[L_1 \to A_1]$ and $[L_2 \to A_2]$ two 1-motives with $L_1, L_2 \neq 0$. Then*

$$\mathcal{P}\langle[L_1 \to A_1]\rangle \cap \mathcal{P}\langle[L_2 \to A_2]\rangle = \mathcal{P}\langle[\mathbb{Z} \to 0]\rangle = 2\pi i\overline{\mathbb{Q}}.$$

The statement follows directly from the dimension formula [HW22, Proposition 15.18]. This allows us to reduce dimension computations to the case of a single simple abelian variety.

Definition 4.4 ([HW22, Chapter 16]). Let $M = [L \to A]$ with $L \neq 0$. We call

$$\mathcal{P}_{\mathrm{inc2}}(M) = \mathcal{P}\langle M\rangle/(\overline{\mathbb{Q}} + \mathcal{P}\langle A\rangle)$$

the *space of incomplete periods of the second kind.*

By the results formulated in the previous section, the sum is direct, so that we have

$$\dim \mathcal{P}\langle[L \to A]\rangle = \dim \mathcal{P}_{\mathrm{inc2}}(M) + \dim \mathcal{P}\langle A\rangle + 1.$$

Proposition 4.2 can be reformulated as

$$\mathcal{P}_{\mathrm{inc2}}(M) = \bigoplus_B \mathcal{P}_{\mathrm{inc2}}([L \to B])$$

where B runs through the simple factors of A (without multiplicities) and $L \to B$ is the composition $L \to A \to B$.

Proposition 4.5 ([HW22, Proposition 15.18]). *Let $[L \xrightarrow{s} A]$ be a 1-motive where A is a simple abelian variety of dimension g with endomorphism algebra $E = \mathrm{End}_{\mathbb{Q}}(A)$ of $\mathbb{Q}$-dimension e and $L \neq 0$. Then*

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}_{\mathrm{inc2}}\langle[L \to A]\rangle = 2g \cdot \mathrm{rk}_A(L, M)$$

where $\mathrm{rk}_A(L, M)$ is the E-dimension of $E \cdot s(L)_{\mathbb{Q}} \subset A(\overline{\mathbb{Q}})_{\mathbb{Q}}$.

The same number has a more conceptual interpretation. We need a bit of notation:

Definition 4.6. Let $\mathcal{C}$ be an abelian category, $X \in \mathcal{C}$ an object. We denote by $\langle X\rangle$ the smallest full abelian category closed under subquotients containing X.

The objects of X are the subquotients of the objects X^n for $n \in \mathbb{N}$.

Proposition 4.7 (Nesa [Nes22, Proposition 3.3.12])**.** *Let $M = [L \to A]$ be a 1-motive with A a simple abelian variety with endomorphism algebra $E = \mathrm{End}_{\mathbb{Q}}(A)$, $L \neq 0$. Then*

$$\mathrm{rk}_A(L, M) = \dim_E \left(\mathrm{Ext}^1_{\langle M\rangle}([\mathbb{Z} \to 0], A)\right).$$

This allows us to give a better reformulation of the dimension formula.

Corollary 4.8. *Let $M = [L \to A]$ be a 1-motive with A a simple abelian variety with endomorphism algebra $E = \mathrm{End}_{\mathbb{Q}}(A)$, $L \neq 0$. Put d the E-dimension of $V_{\mathrm{sing}}(A)$. Then*

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}_{\mathrm{inc2}}([L \to A]) = d \cdot \dim_{\mathbb{Q}} \left(\mathrm{Ext}^1_{\langle M\rangle}([\mathbb{Z} \to 0], A)\right).$$

Proof. We have $d = 2g/e$. □

Remark 4.9. Note that $\langle M\rangle$ is *not* closed under extensions in the category $1\text{–}\mathrm{Mot}_{\overline{\mathbb{Q}}}$. In fact,

$$\mathrm{Ext}^1_{1\text{–}\mathrm{Mot}_{\overline{\mathbb{Q}}}}([\mathbb{Z} \to 0], A) \cong A(\overline{\mathbb{Q}})_{\mathbb{Q}}$$

is not finite-dimensional. Recall that $1\text{–}\mathrm{Mot}_{\overline{\mathbb{Q}}}$ is a *hereditary* category, i.e., it has homological dimension 1. This is still true for $\langle M\rangle$ of the shape considered right now, but *not* for general M, see the example after Lemma 5.7. It is becoming increasingly clear that this is a source for the problems that will be discussed later.

The case of a motive of the form $[0 \to G]$ is completely analogous. Indeed, its period space is in bijection with the periods of the Cartier dual $[X(T) \to A^\vee]$ where A is the dual abelian variety and $X(T)$ the character group of the torus.

Periods of motives of the form $[L \to T]$ are linear combinations of logarithms of algebraic numbers. The dimension formula is nothing but Baker's Theorem. Again it has a reinterpretation in terms of $\mathrm{Ext}^1_{\langle M\rangle}(\mathbb{G}_m, [\mathbb{Z} \to 0])$.

For details we refer to [HW22, Chapter 15.3.1] and [Nes22, Chapter 3].

5. Incomplete periods of the third kind

We now turn to the general case: $M = [L \to G]$ with $0 \to T \to G \to A \to 0$ the decomposition into torus and and abelian part. We are interested in the periods not coming from the subquotients $[L \to A]$ or $[0 \to G]$.

Definition 5.1. We call

$$\mathcal{P}_{\mathrm{inc3}}\langle M\rangle = \mathcal{P}\langle M\rangle / (\mathcal{P}\langle [L \to A]\rangle + \mathcal{P}\langle G\rangle)$$

the *space of incomplete periods of the third kind.*

In the setting of curves, these periods arise as integrals of arbitrary algebraic differential forms with respect to (formal linear combinations of) paths with end point in algebraic points.

As $V_{\mathrm{sing}}(L) \cong L_{\mathbb{Q}}$ and $V_{\mathrm{dR}}^{\vee}(T) \cong X(T)_{\mathbb{Q}}$, the weight filtration induces a surjection

$$\Phi : L_{\mathbb{Q}} \otimes X(T)_{\mathbb{Q}} \twoheadrightarrow \mathcal{P}_{\mathrm{inc3}}(M).$$

In [HW22, Chapter 17], we give an explicit description of its kernel. It is easier in the special case $L = \mathbb{Z}$ and, in fact, the general case can easily be reduced to it.

Lemma 5.2. *Let $M = [\mathbb{Z}^r \xrightarrow{s} G]$ be a 1-motive. Then the periods of M agree with the periods of*

$$[\mathbb{Z} \to G^r]$$

with structure map $1 \mapsto (s(e_1), \ldots, s(e_r))$ (where $e_1, \ldots, e_r$ is the standard basis of $\mathbb{Z}^r$).

We concentrate on the special case.

Proposition 5.3. *Let $M = [L \to G]$ with L of rank 1. Then*

$$\mathcal{P}_{\mathrm{inc3}}\langle M \rangle \cong L_{\mathbb{Q}} \otimes X(T)_{\mathbb{Q}} / R_1(M)$$

where $R_1(M)$ is generated by the subspaces

$$\alpha_*(L_{1\mathbb{Q}}) \otimes \beta^*(X(T_2)_{\mathbb{Q}}) \subset L_{\mathbb{Q}} \otimes X(T)_{\mathbb{Q}}$$

with respect to all exact sequences

$$M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2.$$

Example 5.4. Let E be an elliptic curve, G an extension of E by $\mathbb{G}_m$ which is non-split in the isogeny category, $g_0 \in G$ a non-torsion point. Let $M = [\mathbb{Z} \to G]$ be given by $1 \mapsto g_0$. In order to compute $R_1(M)$, we have to consider all non-trivial surjections

$$\beta : M \to M_2.$$

We claim that the torus part of M_2 is trivial. Assume it is not, hence isomorphic to $\mathbb{G}_m$. Then the group part of β is of the form $G \to G_2$ with $G_2 = \mathbb{G}_m$ or $G_2 \cong G$. In the first case, we would have a splitting of G, contradicting our assumption. This leaves $G_2 \cong G$. As β is assumed non-trivial, we must have $M_2 \cong [0 \to G]$. The kernel of β is isomorphic to $[\mathbb{Z} \to 0]$. This defines a splitting

$$M \cong [\mathbb{Z} \to 0] \oplus [0 \to G],$$

contradicting the choice of g_0.

We have thus established that the torus part of M_2 is trivial. This implies $\beta^*(X(T_2)_{\mathbb{Q}}) = 0$ and hence $R_1(M) = 0$. In other words,

$$\dim \mathcal{P}_{\mathrm{inc3}}\langle M \rangle = 1.$$

Remark 5.5. The argument is independent of the endomorphism algebra of E, so it applies equally in the CM and non-CM case. There are also no conditions on the relation of the points in E and $E^\vee$ involved in the construction as they appear for deficient 1-motives, see Section 7. In particular, the result applies to Ribet's example of a deficient motive in Example 7.3.

This is much less satisfactory than the formulas in the other cases. Under an additional condition, there is indeed a clean formula:

Proposition 5.6 ([HW22, Corollary 15.7])**.** *Let $M = [L \to G]$ such that the abelian part of M is simple with endomorphism algebra $E = \mathrm{End}_{\mathbb{Q}}(A)$ of dimension e.*

If M is saturated, then

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}_{\mathrm{inc3}}\langle M\rangle = e \cdot \mathrm{rk}_A(L, M)\mathrm{rk}_A(T, M)$$

Instead of going through the definition of saturatedness (see [HW22, Definition 15.1]), we point out its conceptual meaning:

Lemma 5.7 (Appendix A)**.** *If M is saturated, then $\langle M\rangle$ is hereditary.*

Note that this is not always the case. For the motive M of Example 5.4 with a CM-elliptic curve, the category *fails* to be hereditary. Another example was worked out by Nesa in [Nes22, Example 3.4.13]:

$$M = [\mathbb{Z} \xrightarrow{x} A] \oplus [G]$$

where G is an extension of A by $\mathbb{G}_m$ which is non-split up to isogeny, $x(1)$ non-torsion.

Using Nesa's identification in Proposition 4.7 of the ranks, we get a more conceptual formulation. Note that $\mathrm{Ext}^1_{\langle M\rangle}(A, [\mathbb{Z} \to 0])$ has an operation of E on the right via the first argument, whereas $\mathrm{Ext}^1_{\langle M\rangle}(\mathbb{G}_m, A)$ has an operation of E on the left via the second argument.

Corollary 5.8. *Let $M = [L \to G]$ such that the abelian part of M is simple with endomorphism algebra $E = \mathrm{End}_{\mathbb{Q}}(A)$.*

If M is saturated, then

$$\dim_{\overline{\mathbb{Q}}} \mathcal{P}_{\mathrm{inc3}}\langle M\rangle = \dim_{\mathbb{Q}} \left(\mathrm{Ext}^1_{\langle M\rangle}(A, [\mathbb{Z} \to 0]) \otimes_E \mathrm{Ext}^1_{\langle M\rangle}(\mathbb{G}_m, A)\right)$$

Proof. We have non-canonically

$$\mathrm{Ext}^1_{\langle M\rangle}(A, [\mathbb{Z} \to 0]) \cong E^{\mathrm{rk}_A(L,M)}$$
$$\mathrm{Ext}^1_{\langle M\rangle}(\mathbb{G}_m, A) \cong E^{\mathrm{rk}_A(T,M)}$$

and hence their tensor product is isomorphic to

$$E^{\mathrm{rk}_A(L,M)\mathrm{rk}_A(T,M)}.$$

Its $\mathbb{Q}$-dimension is

$$e \cdot \mathrm{rk}_A(L, M)\mathrm{rk}_A(T, M).$$

□

Remark 5.9. This gives us a good interpretation of the formula completely in terms of the category. Note that the same assumption is also used by Brown in [Bro17] in the Tannakian setting. In fact, I was led to consider this notion in the present context after a discussion with him at the Simons Symposium.

The fact that not all $\langle M\rangle$ are hereditary is the key reason why the formulas for $\mathcal{P}_{\mathrm{inc3}}(M)$ are complicated.

6. The Tannakian point of view

The presentation here stresses the *space* of period numbers and $\overline{\mathbb{Q}}$-linear relations between them. A lot of the existing literature considers the *algebra* generated by the period numbers and its transcendence degree. In particular, Brown gave a very good description of the latter in the hereditary case, see [Bro17]. This makes it worthwhile to relate the two approaches.

Let $M \in 1\text{–}\mathrm{Mot}_k$. We denote by $\langle M\rangle^{\otimes}$ the smallest full abelian rigid tensor subcategory of the category of Nori motives closed under subquotients and containing M. It becomes Tannakian via the fibre functor $H := H_{\mathrm{sing}}$ (singular homology) on the category or Nori motives:

$$H : \langle M\rangle^{\otimes} \to \mathbb{Q}\text{–Vect}.$$

Restricting to the subcategory $\langle M\rangle$, we get back the singular realisation V_{sing} considered above. Let $G(M)$, the *motivic Galois group of* M, be the Tannakian dual. It is a linear algebraic group over $\mathbb{Q}$ operating on $H(N)$ for all objects $N \in \langle M\rangle^{\otimes}$ such that the category is equivalent is the category of $G(M)$-representations.

We formulate the (mixed motive) case of the classical period conjecture.

Conjecture 6.1 (Grothendieck, André)**.**

$$\mathrm{trdeg}\overline{\mathbb{Q}}(\mathcal{P}\langle M\rangle) = \dim G(M).$$

The 1-motive case was studied by Bertrand and Bertolin, see also the next section. It is wide open: even the case of a single non-CM elliptic curve is not known.

It is now tempting to extract results from the structure theory of algebraic groups. Recall that every linear algebraic group G sits in a canonical short exact sequence

$$0 \to G^u \to G \to G^r \to 0$$

with G^r reductive and G^u connected and unipotent. In the case $G = G(M)$, the group G^r is nothing but the Tannaka dual of $\langle M^{ss}\rangle^{\otimes}$, where M^{ss} is the semi-simplification of M.

Example 6.2. The Krull dimension is additive in the short exact sequence. In [Bro17], Brown gives an algorithmic description of the coradical filtration on $\mathcal{O}(G(M)^u)$. Because of the Grothendieck period conjecture, this can be read as a computation of the period spaces.

In fact, we have even finer information from the weight filtration. The weight filtration on $\mathcal{C} = \langle M\rangle^{\otimes}$ induces, as in [SR72] by Saavedra, a filtration of $G(M)$.

Definition 6.3.

$$W_iG(M) = \Big\{g \in G(M) \mid \\ \forall N \in \mathcal{C}, k \in \mathbb{Z}, v \in W_k(H(N)) : (g \cdot v) - v \in W_{k+i}(H(N))\Big\}.$$

In fact, in suffices to check the condition for $N = M$. This gives the alternative description

$$W_iG(M) = \ker\left(G \to \mathrm{GL}\left(\bigoplus_k W_k(H(M))/W_{i+i}(H(M))\right)\right).$$

In our case the filtration is necessarily concentrated in degrees $-2, -1, 0$ as it is on M itself.

Corollary 6.4. *$W_{-1}G(M)$ agrees with the unipotent radical of $G(M)$ and the reductive quotient $\mathrm{gr}_0 G(M) = G(M)^r$ is $G(M^{ss})$ where M^{ss} is the semi-simplification of M.*

We are interested in the relation with the linear space of periods. Note that

$$\langle M\rangle \subset \langle M\rangle^{\otimes}$$

and the restriction of the fibre functor gives back V_{sing}. By construction of Tannaka duality, $G(M) \subset \mathrm{GL}(V_{\mathrm{sing}}(M))$.

Definition 6.5. Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. The *space of matrix coefficients* $\mathcal{O}^V(G)$ of G with respect to V is the image of

$$\mathrm{End}(V)^\vee \to \mathcal{O}(\mathrm{GL}(V)) \to \mathcal{O}(G)$$

which maps each linear functional to an algebraic function on G.

Example 6.6.
(1) Let $G \subset \mathrm{GL}_n(\mathbb{Q})$ be the group of upper triangular matrices. Then the space of matrix coefficients is the subspace of $\mathbb{Q}[G]$ generated by the coordinate functions X_{ij} for $1 \le i, j \le n$. They vanish for $i > j$.
(2) Let $G \subset \mathrm{GL}_2(\mathbb{Q})$ be the subgroup of matrices of the form
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$
Then the space of matrix coefficients is spanned by the constant function 1 and the coordinate function X_{11}. It has dimension 2.

Note that the space of matrix coefficients depends on the choice of a faithful representation of G. In the application to $G(M)$ it depends on the choice of generator for $\langle M\rangle^{\otimes}$, not only on the category. More precisely: on the choice of $\langle M\rangle \subset \langle M\rangle^{\otimes}$.

This notion is actually very analogous to the definition of the period space. We can reinterpret

$$\mathrm{End}(V_{\mathrm{sing}}(M))^\vee \cong V_{\mathrm{sing}}(M)^\vee \otimes V_{\mathrm{sing}}(M).$$

On the other hand, $\mathcal{P}\langle M\rangle$ is the image of

$$V_{\mathrm{sing}}(M)^\vee_{\overline{\mathbb{Q}}} \otimes V_{\mathrm{sing}}(M) \to V^\vee_{\mathrm{dR}}(M) \otimes V_{\mathrm{sing}}(M)$$

under the period pairing. Indeed:

Proposition 6.7 ([Nes22, Proposition 4.3.2])**.** *Let M be a 1-motive, $V = V_{\mathrm{sing}}(M)$ as representation of $G(M)$. Then:*

$$\dim_{\mathbb{Q}} \mathcal{O}^V(G(M)) = \dim_{\overline{\mathbb{Q}}} \mathcal{P}\langle M\rangle.$$

We learned about this fact from Jossen, but have not found a reference in the literature. There is one subtlety worth stressing: The proof depends on the validity of the period conjecture for $\langle M\rangle$ (which is known) rather that $\langle M\rangle^\otimes$ (which is open).

Examples. We want to understand how the space of matrix coefficients interacts with the structure theory of an algebraic group G: the decomposition into unipotent and reductive part or even the weight filtration on motivic Galois groups.

Let G be a linear algebraic group. Every faithful representation V of G restricts to a faithful representation of G^u and induces a surjection

$$\mathcal{O}^V(G) \to \mathcal{O}^V(G^u).$$

We fix a cocharacter of G, or, equivalently a splitting of $G \to G^r$. This induces another surjection

$$\mathcal{O}^V(G) \to \mathcal{O}^V(G^r).$$

Remark 6.8. In the setting of motives such a splitting is induced by the functor $N \mapsto N^{ss}$ together with the choice of splitting of the weight filtration of the fibre functor.

Question 6.9. Does the map

$$\mathcal{O}^V(G) \to \mathcal{O}^V(G^u) \times \mathcal{O}^V(G^r)$$

on spaces of matrix coefficients have good properties?

It is an isomorphism in the case of upper triangular matrices considered in Example 6.6 (1). However, we argue that this is misleading and there is no good relation in general. This is in contrast to the case of coordinate rings where we have

$$\mathcal{O}(G) \cong \mathcal{O}(G^r) \otimes \mathcal{O}(G^r).$$

We begin with a couple of examples of abstract linear algebraic groups (not necessarily linked to motives) in order to illustrate the problems.

Example 6.10 ([Nes22, Section 6.4.1]). Let $G = \{e\}$, $V = \mathbb{Q}$. Then $G^u = G^r = \{e\}$. We have $\mathcal{O}^V(G) = \mathbb{Q}$ (the constant functions) in each case. In particular,

$$1 = \dim \mathcal{O}^V(G) < \dim \mathcal{O}^V(G^r) + \dim \mathcal{O}^V(G^u) = 2.$$

This trivial example suggests that it is the constant functions causing the problem. However, this is not the only issue. Examples like the following were first suggested by Hörmann.

Example 6.11 ([Nes22, Example 4.1.17]). Let $G \subset \mathrm{GL}_4$ be the subgroup of matrices of the form

$$\begin{pmatrix} \lambda & \lambda r_1 & \lambda r_2 & \frac{\lambda}{2}(r_1^2 + r_2^2) \\ 0 & \sigma & -\tau & \sigma r_1 - \tau r_2 \\ 0 & \tau & \sigma & \tau r_1 + \sigma r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\lambda = \sigma^2 + \tau^2 \neq 0$. It has dimension 4. The space of matrix coefficients has dimension 9. On the other hand, G^u is the subgroup of matrices of the form

$$\begin{pmatrix} 1 & r_1 & r_2 & \frac{1}{2}(r_1^2 + r_2^2) \\ 0 & 1 & 0 & r_1 \\ 0 & 0 & 1 & r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and G^r is the subgroup of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \sigma & -\tau & 0 \\ 0 & \tau & \sigma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In both cases the dimension is 2 and the space of matrix coefficients has dimension 4:

	G	G^u	G^r
$\dim(\cdot)$	4	2	2
$\dim_{\mathbb{Q}} \mathcal{O}^V(\cdot)$	9	4	4

Note that the coefficient λr_2 is linearly independent of the other entries for G, but the corresponding entry in G^u and G^r no longer is.

Remark 6.12. The example can be blown up to arbitrary size, destroying any hope of a general dimension formula for the spaces of matrix coefficients in terms of G^r and G^u.

In the next section, we will see an explicit example of a 1-motive where the same phenomenon occurs, see Proposition 7.5.

Remark 6.13. In these examples, there are many quadratic relations between matrix coefficients. Understanding the linear relations between matrix coefficients for G requires not only knowledge of the matrix coefficients of G^u and G^r, but also of the quadratic relations between them.

7. Deficient motives

Recall from Definition 6.3 the weight filtration on $G(M)$.

Definition 7.1. A 1-motive is called *deficient*, if $W_{-2}G(M) = 0$.

This happens for example if the lattice or torus part of M vanishes or if

$$M = [0 \to G] \oplus [L \to 0]$$

decomposes. However, there are also non-trivial examples. A whole class appears in the work of Jaquinot and Ribet [JR87], [Rib87]. In fact, this was the starting point for Bertolin's investigation [Ber02].

Remark 7.2. Note that being deficient is a property of the category $\langle M\rangle^{\otimes}$ rather than of M itself. As Nesa points out in [Nes22, Proposition 6.3.4] this is not the case for *trivially deficient motives*. Being trivially deficient is a property of the motive itself. Theorems 3.5 and 3.7. in [Ber02] are misstated. If M is deficient, it is *not* necessarily trivially deficient. Rather the category $\langle M\rangle^{\otimes}$ has another generator which is trivially deficient. We refer to [Nes22, Section 6.3] for more details.

We follow Nesa's description of Ribet's construction in a special case.

Example 7.3 ([Nes22, Example 6.3.2, Section 6.4.6])**.** Let E be an elliptic curve with CM by $J = i$ (the complex unit). Let $\rho : E \to E^\vee$ the polarisation. Then $f = \rho \circ J : E \to E^\vee$ is an anti-symmetric isogeny (see [Nes22, Ex. 1.4.20]). Let $a \in E$ be non-torsion. Ribet defines an object

$$M = [\mathbb{Z} \to G] \in \mathrm{Ext}^1([\mathbb{Z} \xrightarrow{1\mapsto a} E], [\mathbb{G}_m])$$

which is well-behaved with respect to Cartier duality.

For this let G' be the semi-abelian variety defined by $f(a) \in E^\vee$ and G the semi-abelian variety defined by $(f - f^\vee)(a)$. We choose $b' \in G'$ above $a \in A$. This defines

$$M' = [\mathbb{Z} \xrightarrow{1\mapsto b'} G'].$$

Its Cartier dual is in

$$\mathrm{Ext}^1([\mathbb{Z} \xrightarrow{f(a)} A^\vee, \mathbb{G}_m).$$

Let M'' be the pull-back of the Cartier dual motive of M' along

$$[\mathbb{Z} \xrightarrow{a} A] \to [\mathbb{Z} \xrightarrow{f(a)} A^\vee].$$

Let

$$M = M' - M'' \in \mathrm{Ext}^1([\mathbb{Z} \xrightarrow{a} A, \mathbb{G}_m)$$

where $-$ refers to the Baer sum. In [Nes22, Section 6.3.3], Nesa explains how the deficiency of M follows from Ribet's [Rib87, Theorem §2].

Remark 7.4. The motive of the example M is of the form considered in Example 5.4. Its space of incomplete periods of the third kind has dimension 1. The vanishing of $W_{-2}G(M)$ does *not* imply the vanishing of this matrix coefficient. This answers a question of Bertrand. The explicit computation below shows how this phenomenon arises.

Nesa computes the motivic Galois group explicitly as a subgroup of $\mathrm{GL}(V)$ where $V = H(M) = V_{\mathrm{sing}}(M)$ in a suitable basis compatible with the weight filtration.

Proposition 7.5 ([Nes22, Prop. 6.4.6–6.4.12])**.** *The group $G(M) \subset \mathrm{GL}(V)$ consists of matrices of the form*

$$\begin{pmatrix} \lambda & \lambda r_1 & \lambda r_2 & \lambda s \\ 0 & \sigma & -\tau & \sigma r_1 - \tau r_2 \\ 0 & \tau & \sigma & \tau r_1 + \sigma r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\lambda = \sigma^2 + \tau^2$, $s = (r_1^2 + r_2^2)/2$.

The space of matrix coefficients has dimension 9. In particular, the entry λs is linearly independent from the others, confirming the computation of $\mathcal{P}_{\mathrm{inc3}}\langle M\rangle$ also in this case.

On the other hand, the entry λs is *algebraically dependent* on the others. The group $G(M)^u = W_{-1}G(M)$ consists of matrices of the form

$$\begin{pmatrix} 1 & r_1 & r_2 & (r_1^2 + r_2^2)/2 \\ 0 & 1 & 0 & r_1 \\ 0 & 0 & 1 & r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It has dimension 2, whereas the space of matrix coefficients has dimension 4. We see that indeed $W_{-2}G(M) = \{e\}$.

Moreover, $G(M)^r$ is a form of $\mathbb{G}_m^2$. It consists of matrices of the form

$$\begin{pmatrix} \sigma^2 + \tau^2 & 0 & 0 & 0 \\ 0 & \sigma & -\tau & 0 \\ 0 & \tau & \sigma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dimension is 2, whereas the space of matrix coefficients has dimension 4.

	G	G^u	G^r
$\dim(\cdot)$	4	2	2
$\dim_{\mathbb{Q}} \mathcal{O}^V(\cdot)$	9	4	4

Again there is no good relation between the spaces of matrix coefficients of $G(M)$, $G(M)^u$ and $G(M))^r$ as in the abstract examples of the last chapter.

Appendix A. An Ext-computation

We verify Lemma 5.7:

Lemma A.1. *If $\tilde{M}$ is saturated, then $\langle \tilde{M} \rangle$ is hereditary.*

Proof. Let $\tilde{M} = [\tilde{L} \xrightarrow{\tilde{s}} \tilde{G}]$ with torus part $\tilde{T}$ and abelian part $\tilde{A}$. The assumption means that $\tilde{L} \to \tilde{A}(\overline{\mathbb{Q}})_{\mathbb{Q}}$ and $X(\tilde{T}) \to \tilde{A}^\vee(\overline{\mathbb{Q}})_{\mathbb{Q}}$ are injective and $\mathrm{End}_{\mathbb{Q}}(\tilde{A}) = \mathrm{End}_{\mathbb{Q}}(\tilde{M})$.

We abbreviate Ext^i for $\mathrm{Ext}^i_{\langle \tilde{M}\rangle}$. Let $S \in \langle \tilde{M}\rangle$. We claim that

$$\mathrm{Ext}^2(S, \cdot) = 0.$$

This equivalent to right-exactness of $\mathrm{Ext}^1(S, \cdot)$. If we have a short exact sequence

$$0 \to S_1 \to S_2 \to S_3 \to 0$$

and the claim holds for S_1 and S_3, then it holds for S_2. Hence it suffices to consider simple objects S.

For $S = [\mathbb{G}_m]$, every short exact sequence in $\mathrm{Ext}^1([\mathbb{G}_m], M)$ is split by the weight-filtration. We have $\mathrm{Ext}^1([\mathbb{G}_m], \cdot) = 0$ and there is nothing to show.

For $S = [A]$ with a simple abelian variety A, we have (again by the weight filtration)

$$\mathrm{Ext}^1([A], M) = \mathrm{Ext}^1([A], [T])$$

where T is the torus part of M. Given a surjection $M_1 \to M_2$, we also have a surjection $T_1 \to T_2$ on the torus parts. The latter is split up to isogeny, providing a splitting on the level of Ext^1.

Finally, consider $S = [\mathbb{Z} \to 0]$. With the same argument as in the last case, we have

$$\mathrm{Ext}^1([\mathbb{Z} \to 0], M) = \mathrm{Ext}^1([\mathbb{Z} \to 0], [G])$$

where G is the group part of M. Given a surjection $T_1 \to T_2$ of tori or a surjection of abelian varieties $A_1 \to A_2$, we get again a splitting on the level of Ext^1. A short diagram chase shows that is suffices to establish surjectivity of

$$\mathrm{Ext}^1([\mathbb{Z} \to 0], G) \to \mathrm{Ext}^1([\mathbb{Z} \to 0], A)$$

where A is the abelian part of G.

Let $A_1, \dots, A_n$ be the simple constituents of A and $G_i = A_i \times_A G$. Then $\prod G_i \to \prod A_i \cong A$ factors up to isogeny via G. Hence surjectivity for all G_i implies surjectivity for G. Without loss of generality, A is simple. As $[A]$ is in $\langle \tilde{M}\rangle$, the abelian variety has to be a subquotient of $\tilde{A}^n$ for some n. As A is simple, it is even a direct factor of $\tilde{A}$. Let $E = \mathrm{End}_{\mathbb{Q}}(A)$ be its isomorphism algebra.

We will construct a morphism $\tilde{G} \to G$ lifting $\tilde{A} \to A$. Admitting this, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1([\mathbb{Z} \to 0], \tilde{G}) \arrow[r] \arrow[d] & \mathrm{Ext}^1([\mathbb{Z} \to 0], G) \arrow[d] \\
\mathrm{Ext}^1([\mathbb{Z} \to 0], \tilde{A}) \arrow[r, "\pi"] & \mathrm{Ext}^1([\mathbb{Z} \to 0], A)
\end{tikzcd}$$

By [Nes22, Proposition 3.2.1], every element in $\mathrm{Ext}^1([\mathbb{Z} \to 0], A)$ is of the form

$$\alpha\pi_*\tilde{s}(l) \in A(\overline{\mathbb{Q}})_{\mathbb{Q}}$$

for $l \in \tilde{L}$, the structure map $\tilde{s} : \tilde{L} \to \tilde{G} \to \tilde{A}$ and $\alpha \in E$. The motive $\tilde{M}$ is saturated. By definition this means that the endomorphism α lifts to $\tilde{M}$. By replacing l by $\alpha(l)$, we may assume that $\alpha = \mathrm{id}$. The 1-motive

$$[\mathbb{Z} \xrightarrow{s} \tilde{G}], \quad s : 1 \mapsto \tilde{s}(l) \in \tilde{G}$$

is in $\mathrm{Ext}^1([\mathbb{Z} \to 0], \tilde{G})$ and a preimage of our extension class. Its image in $\mathrm{Ext}^1([\mathbb{Z} \to 0, G)$ is the preimage we were looking for.

It remains to verify the claim. It is solely a statement about semi-abelian varieties, which we treat via their classifying maps. We want to find a dotted arrow fitting in the diagram

$$\begin{tikzcd} X(T) \arrow[r, dotted] \arrow[d, "{[G]}"'] & X(\tilde{T}) \arrow[d, "{[\tilde{G}]}"] \\ A^\vee(\overline{\mathbb{Q}})_{\mathbb{Q}} \arrow[r, "\pi^*"] & \tilde{A}^\vee(\overline{\mathbb{Q}})_{\mathbb{Q}}. \end{tikzcd}$$

Let $x_1, \dots, x_t$ be a basis of $X(T)$. Then x_i defines an object in $\mathrm{Ext}^1(\mathbb{G}_m, A)$. By [Nes22, Proposition 3.2.1] it can be identified with an element of the form

$$\beta^*(\chi(y_i)) \in \tilde{A}^\vee(\overline{\mathbb{Q}})_{\mathbb{Q}}$$

where χ is the classifying map of $\tilde{G}$ applied to $y_i \in X(\tilde{T})$ and β is an endomorphism of $\tilde{A}$. As $\tilde{M}$ is saturated, it lifts to an endomorphism of $\tilde{G}$. By replacing y_i by $\beta^*(y_i)$, we may assume that $\beta = \mathrm{id}$. We define the dotted arrow by sending x_i to y_i. □

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