RATIONALLY TRIVIAL TORSORS IN $\mathbb{A}^1$-HOMOTOPY THEORY

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Abstract. In this paper, we show that rationally trivial torsors under split smooth linear algebraic groups induce fibre sequences in $\mathbb{A}^1$-homotopy theory. The results are geometric proofs of stabilization results for unstable Karoubi-Villamayor K-theories and a computation of the second $\mathbb{A}^1$-homotopy group of the projective line.

Contents

1. Introduction

$\mathbb{A}^1$-homotopy theory is a relatively new approach to a homotopy theory for schemes, which was developed by Morel and Voevodsky [MV99]. Although some of the necessary requirements for such a theory are met, i.e. "classical" homology theories like intersection theory and algebraic K-theory are representable in the stable homotopy category, few concrete calculations of homotopy groups of schemes have been produced so far. Among these are the fundamental group of $\mathbb{P}^1$ resp. the first non-vanishing homotopy group of $\mathbb{A}^n \setminus \{0\}$, cf. [Mor06b], as well as the homotopy groups of Chevalley groups, cf. [Wen09] which builds essentially on [Mor07].

In classical algebraic topology, fibre sequences are a ubiquitous computational tool, because they connect the homotopy groups of spaces via long exact sequences. Although fibrations and fibre sequences are a part of the model category structure of $\mathbb{A}^1$-homotopy theory, they are hard to understand. The problem is to figure out conditions on a morphism $p: E \to B$ of schemes which make sure that the point-set fibre $p^{-1}(b)$ is weakly equivalent to the homotopy fibre. The general model categorical aspects of this question have been investigated in [Wen07]. The first examples one could look at are fibre bundles with smooth algebraic structure groups. Indeed, in the case of GL_n-bundles over a field, the results of [Mor07] show that any

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GL_n-bundle induces an $\mathbb{A}^1$-local fibre sequence, provided $n \geq 3$. In this paper, we explain how to extend this result to rationally trivial torsors under split smooth linear algebraic groups satisfying only rather weak conditions. Breaking down the model category machinery to geometrical assertions, the main step consists in establishing homotopy invariance for torsors. This has been done in a variety of cases, and the most powerful result of which we will make heavy use throughout this paper was given in [CTO92]. This result states that rationally trivial torsors over affine spaces over smooth local rings are always extended. In this paper, we use this result to establish the following, cf. Proposition 4.3:

Theorem 1. *Let k be an infinite field, and let G be a smooth connected split reductive group. Let $E \to B$ be a G-torsor, and let R be a local ring which is smooth and essentially of finite type over k. Then for any base point $p_0 : \operatorname{Spec} R \to B$ of B for which the induced G-torsor over $\operatorname{Spec} R$ is rationally trivial, there is a fibre sequence of pointed simplicial sets:*

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1} G(R) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1} E(R) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1} B(R).$$

This implies in particular the existence of many fibre sequences in the simplicial model structure, which indeed turn out to be $\mathbb{A}^1$-local. The corresponding long exact sequences for $\mathbb{A}^1$-homotopy groups can be used to show the following result, cf. Proposition 5.11:

Corollary 1.1. *The second homotopy group of the projective line sits in an exact sequence*

$$0 \to \operatorname{coker}\big(KV_4(C_\infty, k)) \to K_4^{MW}(k)\big) \to \pi_2^{\mathbb{A}^1}(\mathbb{P}^1)(k) \to KV_3(C_\infty, k) \to 0.$$

Here $KV_n(C_\infty, k)$ denotes the symplectic version of Karoubi-Villamayor K-theory of k, and $K_n^{MW}(k)$ is the Milnor-Witt K-theory of k.

Moreover, the results can be used to provide "topological" proofs of stabilization results in Karoubi-Villamayor type K-theories, cf. Theorem 5.8.

Structure of the paper: In Section 2, we repeat basic definitions and notations from $\mathbb{A}^1$-homotopy theory. In Section 3, we recall the fundamental results on rationally trivial torsors over affine spaces. These results are applied in Section 4 to show that torsors induce fibre sequences. Finally, in Section 5, some sample computations show how to apply the results.

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2. Preliminaries and Notation

2.1. **Algebraic Groups and Torsors:** One of the standard references for algebraic groups is [Bor91], and we will freely use terminology from the theory of algebraic groups.

We will be interested in torsors under algebraic groups. A G-torsor E over a base scheme B is then a B-scheme $E \to B$ equipped with a G-action such that the morphism

$$G \times_B E \to E \times_B E$$

is an isomorphism.

A torsor is called *locally isotrivial* if it is locally trivial in the étale topology. A result of Seshadri [Ses63] states that torsors over normal base schemes are always locally trivial in the étale topology.

A torsor is called *rationally trivial* if there is a Zariski open subset U of B such that $E \times_B U$ is trivial. We will be mostly interested in torsors over smooth schemes which are locally trivial in the Nisnevich topology. The result of Seshadri implies that all rationally trivial torsors over smooth schemes are locally trivial in the Nisnevich topology.

2.2. **$\mathbb{A}^1$-Homotopy Theory:** The general definition of $\mathbb{A}^1$-homotopy theory is due to Morel and Voevodsky [MV99], and we give a brief sketch of the construction. Note that the construction we survey below uses the model category of simplicial *presheaves* as opposed to the category of simplicial sheaves used in [MV99]. However, both constructions yield equivalent homotopy categories, which is proven in [Jar00, Theorem B.6].

Consider the category of simplicial presheaves $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$ on the category of smooth schemes Sm_S. This category has a model structure where the cofibrations are monomorphisms, the weak equivalences are those morphism which induce weak equivalences of simplicial sets on the stalks, and the fibrations are given by the right lifting property. The topologies which we will put on Sm_S are either the Zariski topology, whose coverings are surjective collections of open subsets, or the Nisnevich topology, which is generated by elementary distinguished squares of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \underset{i}{\hookrightarrow} & X, \end{array}$$

where p is an étale morphism and i is an open embedding such that p restricts to an isomorphism over $X \setminus U$. We denote the corresponding homotopy category by $\mathrm{Ho}_s(\mathrm{Sm}_S)$.

From this model structure on the category of simplicial presheaves, one proceeds to the $\mathbb{A}^1$-local model structure as follows, cf. [MV99, Section 2.2]. An object $X \in \mathrm{Ho}_s(\mathrm{Sm}_S)$ is said to be $\mathbb{A}^1$-*local* if for any object $Y \in \mathrm{Ho}_s(\mathrm{Sm}_S)$, the morphism

$$\mathrm{Hom}_{\mathrm{Ho}_s(\mathrm{Sm}_S)}(Y, X) \to \mathrm{Hom}_{\mathrm{Ho}_s(\mathrm{Sm}_S)}(Y \times \mathbb{A}^1, X)$$

induced by the projection $Y \times \mathbb{A}^1 \to Y$ is a bijection, where the affine line $\mathbb{A}^1$ is considered as a simplicially constant simplicial presheaf. A morphism $f : X_1 \to X_2$ of simplicial presheaves is then called an $\mathbb{A}^1$-*weak equivalence* if for any $\mathbb{A}^1$-local object Y the morphism

$$\mathrm{Hom}_{\mathrm{Ho}_s(\mathrm{Sm}_S)}(X_2, Y) \to \mathrm{Hom}_{\mathrm{Ho}_s(\mathrm{Sm}_S)}(X_1, Y)$$

induced by f is a bijection. This allows to define a new model structure on $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$ by taking cofibrations to be monomorphisms, weak equivalences to be $\mathbb{A}^1$-weak equivalences, and fibrations to be defined via the right lifting property. This is proved e.g. in [MV99, Theorem 2.2.5] for simplicial sheaves – the simplicial presheaf case can be proven analogously.

Our use of the term *fibre sequence* is the same as in Hovey's definition [Hov98, Definition 6.2.6], which basically states that a sequence of morphisms $F \to E \to B$ is a fibre sequence if after replacing $p : E \to B$ by a fibration of fibrant spaces $\tilde{p} : \tilde{E} \to \tilde{B}$, the (homotopy) fibre of $\tilde{p}$ is still weakly equivalent to F. Fibre sequences in the model structure on simplicial presheaves $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$ will be called *simplicial fibre sequences*, and fibre sequences in the $\mathbb{A}^1$-local model structure will be called $\mathbb{A}^1$*-local fibre sequences*. Equivalently, an $\mathbb{A}^1$-local fibre sequence is a simplicial fibre sequence which is preserved by the $\mathbb{A}^1$-localization functor, cf. [Wen07, Theorem 4.3.10].

3. Recollections on Rationally Trivial Torsors

In this section, we recall the main geometric input for the construction of fibre sequences in $\mathbb{A}^1$-homotopy theory: the statement that torsors over affine spaces (over smooth affine schemes) are always extended. I will provide an overview of the relevant results of Raghunathan [Rag78, Rag89], Colliot-Thélène and Ojanguren [CTO92], and a local-global principle from the work Bass, Connell and Wright [BCW76] originating in work of Quillen [Qui76].

Theorem 3.1. *Let k be an infinite field, let R be a ring which is smooth and essentially of finite type over k, and let G be a split smooth linear algebraic group. Then any rationally trivial G-torsor over* $\operatorname{Spec} R \times \mathbb{A}^n$ *is extended from* $\operatorname{Spec} R$.

Remark 3.2. *Bootstrapping the proof, the reader can easily imagine that this result holds for suitably isotropic k-groups G as well.*

Moreover, the results can be strengthened for the special groups SL_n resp. Sp_{2n}. In these cases, the result holds for all torsors over schemes which are smooth and essentially of finite type over an excellent Dedekind ring*.*

The following subsections contain the components of the proof, which proceeds in various steps: first, we recall the local-global principle which allows to reduce the assertion to the case where R is a *local* ring smooth and essentially of finite type over k. The proof structure in [CTO92] then allows to prove the result for such local rings.

3.1. **Local-Global Principle:** In this section, we discuss the local-global principle for torsors. The question we consider is the following: Let $\sigma : E \to \operatorname{Spec} A[t_1, \dots, t_n]$ be a torsor under a linear group G, which is locally in the Zariski topology extended from $\operatorname{Spec} A$. Is it extended globally?

This kind of result has been stated without proof for smooth linear algebraic groups over a field in [Rag78, Theorem 2]. By the work of Bass, Connell and Wright, cf. [BCW76], it suffices to check Axiom Q for a suitable group-valued functor on the category of k-algebras. The functor which we are interested in is induced by the automorphisms of a G-torsor on a k-scheme. Axiom Q for this torsor then allows to check if such a torsor is extended by localizing at the maximal ideals. In the case of GL_n, this is exactly the way Quillen proved the local-global principle for projective modules over polynomial rings, cf. [Qui76] and [Lam06].

We first need to introduce the relevant notation. Let R be a commutative ring, and let G be a functor from R-algebras to groups. For R-algebras A

and B, and an R-algebra homomorphisms $f : A[t] \to B : t \mapsto s$, we denote $g \in G(A[t])$ by $g(t)$ and its image under $G(f)$ in $G(B)$ by $g(s)$. Applied to $B = A$, $s \in A$ and $A[t] \to A[t] : t \mapsto st$, this defines $g(st)$. We denote

$$G(A[t],(t)) = \{g(t) \in G(A[t]) \mid g(0) = I\},$$

which is the kernel of the evaluation at zero morphism $G(A[t]) \to G(A)$.

Definition 3.3. *Let R be any commutative ring, and let G be a functor from R-algebras to groups. We say that G* satisfies Axiom Q *if for any given R-algebra A, any element $s \in A$ and any element $u(t)$ in $G(A_s[t],(t))$ there exists an integer $n \geq 0$ and an element $v(t) \in G(A[t],(t))$ such that $u(s^r t) = v(t)_s$.*

The key in the proof of the local-global principle for G-torsors is to check Axiom Q for automorphism groups of a torsor. The next result is a consequence of the work of Bass, Connell and Wright [BCW76].

Proposition 3.4. *Let k be an infinite field, and let G be a split smooth linear algebraic group over k. Let R be a ring which is smooth and essentially of finite type over k, and let E be a G-torsor over* $\operatorname{Spec} R \times \mathbb{A}^n$.

If for each maximal ideal $\mathfrak{m} \subseteq R$, the torsor $E_{\mathfrak{m}}$ obtained by restricting E to $\operatorname{Spec} R_{\mathfrak{m}} \times \mathbb{A}^n$ *is extended, then R is extended.*

Proof. The proof is sketched in [BCW76, Remark 4.15]. The category $\mathcal{C}(L)$ would be the category of rationally trivial (or étale locally trivial) G-torsors over L. The conditions for Quillen induction are met using basic properties of torsors. Only the sheaf condition needs proof, but this follows from Axiom Q for automorphism groups of torsors, cf. [BCW76, Lemma 1.12]. □

To prove Theorem 3.1 for all rings R which are smooth and essentially of finite type over k, it thus suffices to prove it for *local rings* which are smooth and essentially of finite type over k.

3.2. **Case of Local Rings:** The basic idea in settling the case of local rings is the following structure theorem for local rings which are smooth and essentially of finite type. This result is due to Lindel [Lin81].

Theorem 3.5. *Let A be a local ring which is smooth and essentially of finite type over a field k. Then there exists a subring B of A and an element $h \in B$ such that:*

(i) B is the localization of a polynomial ring $k[x_1, \ldots, x_d]$ at a prime ideal, and
(ii) $Ah + B = A$ and $Ah \cap B = Bh$.

Roughly speaking, this result states that any local ring which is smooth and essentially of finite type is a Nisnevich neighbourhood of a localization of a polynomial ring.

We now recall the main results of [CTO92], which we will need in the sequel. The object of interest in [CTO92] are functors F from algebras over an infinite field to pointed sets satisfying the following properties.

(P1) The functor F commutes with filtered direct limits with flat transition morphisms.

(P2) For any extension field L of k, and all $n \geq 0$, the following morphism is an injection:
$$F(L[t_1,\dots,t_n]) \to F(L(t_1,\dots,t_n)).$$
(P3) For all elementary affine Nisnevich squares of the form
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & B_f, \end{array}$$
the induced morphism
$$\ker\big(F(A) \to F(A_f)\big) \to \ker\big(F(B) \to F(B_f)\big)$$
is surjective.

The main example of such functors are $H^1_{\text{ét}}(-,G)$ with G a smooth algebraic group satisfying the isotropy hypothesis. The main results [CTO92, Theorem 1.1] states that for such functors F, the morphism
$$H^1_{\text{ét}}(R,G) \to H^1_{\text{ét}}(Q(R),G)$$
is injective for all local rings R which are smooth and essentially of finite type over an infinite field. This means that rationally trivial torsors are locally trivial.

The method of proof employed in [CTO92] also yields the the following statement, which is the homotopy invariance we will need in the sequel.

Proposition 3.6. *Let k be an infinite field, and let F a functor satisfying the above conditions (P1), (P2) and (P3). Let R be a local ring which is smooth and essentially of finite type over k. Then for all $n \geq 0$ the morphism*
$$F(R[t_1,\dots,t_n]) \to F(Q(R)(t_1,\dots,t_n))$$
is injective.

Proof. The case of a local ring of the form $L[t_1,\dots,t_n]_{\mathfrak{m}}$, where L is an extension field of k and $\mathfrak{m}$ is a maximal ideal of the polynomial ring $L[t_1,\dots,t_n]$ is exactly [CTO92, Proposition 1.5]. Then the first two reduction steps in the proof of [CTO92, Theorem 1.1] go through to show the claim. One uses the structure result of Lindel, cf. Theorem 3.5, which states that a smooth local ring R is a Nisnevich neighbourhood of a smooth local ring of the special form above, i.e. there is an elementary distinguished square for the Nisnevich topology
$$\begin{array}{ccc} L[t_1,\dots,t_n]_{\mathfrak{m}} & \longrightarrow & R \\ \downarrow & & \downarrow \\ L[t_1,\dots,t_n]_{(\mathfrak{m},f)} & \longrightarrow & R_f, \end{array}$$
where $L[t_1,\dots,t_n]_{\mathfrak{m}} \to R$ is an étale morphism, the element f is a non-zerodivisor in $L[t_1,\dots,t_n]_{\mathfrak{m}}$, and the morphism
$$L[t_1,\dots,t_n]_{\mathfrak{m}}/(f) \to R/(f)$$
is an isomorphism. But then the following diagram is also a distinguished square for the Nisnevich topology:

$$\begin{array}{ccc} L[t_1,\dots,t_n]_{\mathfrak{m}}[t_{n+1},\dots,t_d] & \longrightarrow & R[t_{n+1},\dots,t_d] \\ \downarrow & & \downarrow \\ L[t_1,\dots,t_n]_{(\mathfrak{m},f)}[t_{n+1},\dots,t_d] & \longrightarrow & R_f[t_{n+1},\dots,t_d]. \end{array}$$

Property (P3) then implies that the homotopy invariance for $L[t_1,\dots,t_n]_{\mathfrak{m}}$ implies the homotopy invariance for R. □

To prove Theorem 3.1 for local rings smooth and essentially of finite type over k, it suffices by Proposition 3.6 to prove the properties (P1), (P2), and (P3) for the functor $H^1_{\text{ét}}(-,G)$ for G a split smooth algebraic group. A rationally trivial torsor over $R[t_1,\dots,t_n]$ is an element in $H^1(R[t_1,\dots,t_n],G)$, which is in the kernel of

$$H^1_{\text{ét}}(R[t_1,\dots,t_n],G) \to H^1_{\text{ét}}(Q(R)(t_1,\dots,t_n),G).$$

By the above, such a torsor has to be trivial, and in particular extended from R. The only nontrivial property is (P2), which is the subject of the next subsection.

3.3. **Base Case:** The base case is provided by the work of Raghunathan [Rag78], cf. also [CTO92, Section 2]. Most of what is known about torsors over polynomial rings over fields seems to be due to the work of Raghunathan. We first recall the definition of an acceptable group, cf. [Rag78].

Definition 3.7. *An algebraic group G over a field k is called* acceptable *if for every extension field L/k, any principal $(G\otimes_k L)$-torsor over* $\operatorname{Spec} L[t]$ *is extended from* $\operatorname{Spec} L$.

Remark 3.8. *As stated in Raghunathan* [Rag78]*, the following groups are known to be acceptable:*

(i) all groups over a field of characteristic zero,
(ii) the groups $O(n)$ and $SO(n)$ over a base field of characteristic not 2*,*
(iii) simply-connected groups of classical type over a field of characteristic $p>5$.
(iv) tori, semisimple groups of inner type A_n, or spinor groups.

It is also conjectured in [Rag78] *that any smooth simply-connected reductive group is acceptable, where simply-connected means that the derived subgroup is a simply-connected semisimple group.*

The following result is stated as [Rag78, Theorem C].

Proposition 3.9. *Let k be a separably closed field, and let G be a connected, reductive and acceptable group over k. Then all torsors over $\mathbb{A}^n_k$ are extended.*

The following isotropy hypothesis is taken from [CTO92] resp. [Rag89, Theorem A]:

> **(I)** Each of the k-simple components of the derived group of G is isotropic over k.

Then it is possible to prove the following result, cf. [CTO92, Proposition 2.4 and Theorem 2.5]:

Proposition 3.10. *Let k be an infinite field and let G be a smooth, reductive and connected linear group satisfying the isotropy hypothesis. Denote by k_s a separable closure of k.*

Then any torsor over $\mathbb{A}^n_k$ which becomes trivial over $\mathbb{A}^n_{k_s}$ and which is trivial at a k-rational point of $\mathbb{A}^n_k$ is trivial. In particular, every rationally trivial G-torsor on $\operatorname{Spec} k[t_1, \dots, t_n]$ is trivial.

4. Fibre Sequences From Torsors

Let $E \to B$ be a torsor under a group G, and let R be a ring which is smooth and essentially of finite type. We want to show that

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G)(R) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$$

is a fibre sequence of simplicial sets. We have to consider the pointed situation, and we have to take care which base point to choose.

In this section we will use the notation

$$R[\Delta^n] = R[t_0, \dots, t_n]/(\sum t_i = 1).$$

Note that this ring is isomorphic to a polynomial ring in n variables.

4.1. **Choice of Basepoint:** Let $\operatorname{Spec} R \to B$ be a morphism such that the induced torsor $E \times_B \operatorname{Spec} R \to \operatorname{Spec} R$ is rationally trivial. Then for any morphism $\operatorname{Spec} R[\Delta^n] \to B$ which preserves the base point, the induced G-torsor over $\operatorname{Spec} R[\Delta^n]$ is rationally trivial. The following is a simple consequence of [CTO92, Theorem 2.5] making precise the above statement, but see also [Rag78].

Proposition 4.1. *Let G be a smoooth connected and reductive group scheme which is acceptable in the sense of* [Rag78]*, and let $E \to B$ be a G-torsor. Let R be any ring, and let $\phi : \operatorname{Spec} R \to B$ be a morphism such that the induced torsor $\tilde{E} \to \operatorname{Spec} R$ is rationally trivial. Then for any morphism $\psi : \operatorname{Spec} R[\Delta^n] \to B$, such that the diagram*

$$\begin{array}{ccc} \operatorname{Spec} R[\Delta^n] & \xrightarrow{\psi} & B \\ \uparrow & \nearrow_{\phi} & \\ \operatorname{Spec} R & & \end{array}$$

is commutative, the torsor over $R[\Delta^n]$ which is induced by ψ is rationally trivial.

Proof. Since G is acceptable, it follows that for $\overline{k}$ an algebraically closed field any G-torsor over $\overline{k}[\Delta^n]$ is trivial. Moreover, [CTO92, Theorem 2.5] states that for any field k, any torsor over $k[\Delta^n]$ which is trivial at a k-rational point and which becomes trivial over $\overline{k}[\Delta^n]$ is already trivial. Applying the conjunction of these statements to the quotient field $Q(R)$ of R, we obtain the result. □

Remark 4.2. *Why do we have to take such care to choose the base points? The reason is that homotopy invariance can only be obtained for rationally trivial torsors. There are plenty examples, cf. for example* [Rag89]*, of étale torsors over affine spaces which are not extended, and therefore we can not* *use the arguments in this paper for all base points, only for those for which the induced torsor is rationally trivial.*

4.2. **Simplicial Fibre Sequences:** In this section, we show that for any G-torsor $E \to B$ and any commutative ring R which is smooth and essentially of finite type over an infinite field, the sequence $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is a fibre sequence of simplicial sets, provided one chooses the base point appropriately. Recall that the group G has to satisfy the isotropy hypothesis (I). This implies that we also obtain a fibre sequence of simplicial presheaves $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)$

Proposition 4.3. *Let k be an infinite field, let G be a smooth linear group scheme satisfying the isotropy hypothesis (I), and let R be a ring which is smooth and essentially of finite type. Furthermore, let $p : E \to B$ be a G-torsor, and let $b : \mathrm{Spec}\, R \to B$ be a morphism such that the induced G-torsor over R is trivial. Then the corresponding sequence*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$$

is a fibre sequence of simplicial sets. We choose b to be the base point of B, choose an isomorphism $G \times \mathrm{Spec}\, R \to p^{-1}(b)$, and choose the base point of E as the image of $1 \in G$.

Proof. The proof is structured as follows: in Step (i), we show that the morphism $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is a fibration for any ring R which is smooth and essentially of finite type over k. We also identify the fibre as $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R)$. Then we prove in Step (iii) that the space $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is the quotient of a free action of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R)$ on $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R)$. Putting together these statements, we obtain the result, see Step (iv).

(i) We first show that $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is a fibration of simplicial sets. Therefore, consider the following lifting problem:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \\ \big\downarrow & & \big\downarrow \\ \Delta^n & \xrightarrow[\sigma]{} & \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R). \end{array}$$

The morphism σ corresponds to a morphism $\sigma : \mathrm{Spec}\, R[\Delta^n] \to B$, which again corresponds to a $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$-torsor $\sigma : \tilde{E} \to \mathrm{Spec}\, R[\Delta^n]$. Note that this $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$-torsor is in fact induced from a G-torsor via change of structure group along $G \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$. The existence of the commutative diagram implies that the restriction of this torsor to $\mathrm{Spec}\, R[\Lambda^n_k] \hookrightarrow \mathrm{Spec}\, R[\Delta^n]$ has a section, and is therefore trivial.

By Proposition 4.1, the induced G-torsor over $R[\Delta^n]$ is induced from is rationally trivial, and by Theorem 3.1, it is induced from a G-torsor over R. Therefore, the pullback of σ to $R[\Lambda^n_k]$ is also extended from R, and by the above remark is in fact trivial. This can be visualized by the following commutative triangle of isomorphisms:

$$\begin{array}{ccc} H^1_{Nis}(R[\Lambda^n_k], G) & \xrightarrow{\cong} & H^1_{Nis}(R[\Delta^n], G) \\ & \searrow^{\cong} & \downarrow \cong \\ & & H^1_{Nis}(R, G). \end{array}$$

The two statements above imply that the torsor $\sigma : \tilde{E} \to \operatorname{Spec} R[\Delta^n]$ is already trivial. Therefore, it has a section, i.e. we have a morphism $\operatorname{Spec} R[\Delta^n] \to \operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)$ lifting σ.

The existence of the section implies that the lifting problem can be rewritten, cf. [GJ99, Corollary V.2.7]:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \operatorname{Sing}^{\mathbb{A}^1}_\bullet(G)(R) \times \Delta^n \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[=]{} & \Delta^n. \end{array}$$

Note that this uses the fact that the preimage of the base point $b \in B$ is isomorphic to G.

The preimage of $\sigma : \operatorname{Spec} R[\Delta^n] \to B$ in the set of morphisms $\tau : \operatorname{Spec} R[\Delta^n] \to E$ is isomorphic to $G(R[\Delta^n])$, since the set of commutative triangles

$$\begin{array}{ccc} & & E \\ & \nearrow^{\tau} & \downarrow p \\ \operatorname{Spec} R[\Delta^n] & \xrightarrow[\sigma]{} & B \end{array}$$

coincides with the set of sections of $E' \to \operatorname{Spec} R[\Delta^n]$, which is isomorphic to $G(R[\Delta^n])$.

In particular, the section can be chosen such that it agrees with the given morphism $\Lambda^n_k \to \operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R)$. This proves that the morphism

$$\operatorname{Sing}^{\mathbb{A}^1}_\bullet(p) : \operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R) \to \operatorname{Sing}^{\mathbb{A}^1}_\bullet(B)(R)$$

is in fact a fibration with fibre $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(G)$.

(ii) It follows now from the previous step that the image of $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R)$ in $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(B)(R)$ is the quotient of $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R)$ by the action of the simplicial group $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(G)(R)$. The simplices which are not in the image are points of B over which the G-torsor is not trivial, and which consequently do not lift to points of E.

We only need to verify that this action is degreewise free, i.e. for any $\sigma \in G(R[\Delta^n])$, if $\sigma x = x$ for some $x \in E(\operatorname{Spec} R[\Delta^n])$, then $\sigma = I$. We can again consider the trivialization over the simplex $p \circ \sigma : \operatorname{Spec} R[\Delta^n] \to B$, since p is equivariant for the trivial action on B, i.e. $p \circ \sigma(x) = p(x)$. Because we can equivariantly identify the n-simplices in E with $G(R[\Delta^n])$ equipped with the multiplication action, the action of $G(R[\Delta^n])$ on $E(\operatorname{Spec} R[\Delta^n])$ is free.

(iii) We have shown that the image of $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R)$ in $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(B)(R)$ is the quotient of the simplicial group action of $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(G)(R)$ on $\operatorname{Sing}^{\mathbb{A}^1}_\bullet(E)(R)$, and therefore we have the following fibre sequence of simplicial presheaves:

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)$$

This follows from [GJ99, Corollary V.2.7] and is an assertion similar to the statements proved in [Mor07, Theorem 12(2)]. In fact, if $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is connected or $H^1_{\text{ét}}(R, G) = 0$, we have

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R)/\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R) = \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R).$$

□

Corollary 4.4. *Under the conditions of Proposition 4.3, the following sequence is a fibre sequence of simplicial presheaves on the category of smooth schemes with any topology finer than the Zariski topology:*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B).$$

Proof. The following result shows that the property of being a fibre sequence can be checked on points, i.e. is a local one. This can be found in [Wen07, Proposition 3.1.11].

Let T be a site with enough points, and let the following commutative diagram $\mathcal{X}$ of simplicial presheaves in $\Delta^{op}P\mathcal{S}hv(T)$ be given:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \longrightarrow & D. \end{array}$$

This is a homotopy pullback diagram iff for each point p of T, the diagram $p^*(\mathcal{X})$ of simplicial sets is a homotopy pullback diagram.

Using this result, it suffices to check that the sequence

$$x^* \mathrm{Sing}_\bullet^{\mathbb{A}^1} G \to x^* \mathrm{Sing}_\bullet^{\mathbb{A}^1} E \to x^* \mathrm{Sing}_\bullet^{\mathbb{A}^1} B$$

is a fibre sequence of simplicial sets for any point x of the topos Sm_S with the Zariski topology. The stalk of a simplicial presheaf at such a point is given by the sections over local rings of smooth schemes. □

From a principal bundle E with structure group G, we can also construct further locally trivial morphisms, by glueing in a fibre F which is equipped with an action $\rho : F \times G \to G$ of G, yielding a fibre bundle $E \times_\rho F$. We can then show that the resulting morphisms induce fibre sequences. This applies e.g. to homogeneous space bundles like projective bundles.

Proposition 4.5. *Let k be an infinite field, and let $p : E \to B$ be a smooth morphism which is obtained from a G-torsor $E' \to B$ by change-of-fibre along an action $G \times F \to F$. Assume that the group G satisfies the isotropy hypothesis (I). Let $b \in B$ be a rational point in B. Then the following is a fibre sequence:*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B).$$

The base points are chosen as in Proposition 4.3.

Proof. It again suffices via [Wen07, Proposition 3.1.11] to prove that for any smooth local ring R,

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$$

is a fibre sequence of simplicial sets.

This can be proved via the same argument as [GJ99, Corollary V.2.7]. Replacing the space $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)$ with a fibrant model of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)$ does not change the assertion about the fibre sequence above, therefore we can assume that $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)(R)$ is a fibrant simplicial set for any R.

Now consider a lifting problem:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[\sigma]{} & \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R). \end{array}$$

By the assumption on the group, the G-torsor is trivial over the simplex σ, and the sequence $F \to E \to B$ is obtained by change-of-fibre along the group action $G \times F \to F$. Therefore also $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(E)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B)(R)$ is trivial over σ, and the lifting problem above is equivalent to the following lifting problem:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)(R) \times \Delta^n \\ \downarrow & & \downarrow \pi \\ \Delta^n & \xrightarrow[=]{} & \Delta^n, \end{array}$$

where π is the projection away from $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)(R)$. Since $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)(R)$ is fibrant, the lifting problem has a solution. □

4.3. **$\mathbb{A}^1$-Locality.** In the previous subsection, we showed that G-torsors induce fibre sequences in the simplicial model structure. These fibre sequences are indeed $\mathbb{A}^1$-local, which is also a consequence of the fact that rationally trivial torsors over affine spaces are extended, cf. Theorem 3.1. In this subsection, we explain this. The main steps are due to Morel in the papers [Mor06b, Mor07, Mor09].

We first recall the relevant definitions from the theory of classifying spaces. Let $G_\bullet$ be a simplicial sheaf of groups. One can define a simplicial sheaf $EG_\bullet$ on which $G_\bullet$ acts freely: the n-simplices are given by

$$(EG_\bullet)_n = G_\bullet^{n+1},$$

and there are the obvious face and degeneracy maps. The classifying space is obtained taking the quotient $BG_\bullet = EG_\bullet/G_\bullet$. There is a fibre sequence of simplicial sheaves

$$G_\bullet \to EG_\bullet \to BG_\bullet,$$

which implies isomorphisms $\pi_n G_\bullet \cong \pi_{n+1} BG_\bullet$. For a further discussion of classifying spaces of simplicial sheaves of groups in $\mathbb{A}^1$-homotopy theory, cf. [MV99].

We now consider the particular case where we apply the above construction to the singular resolution $\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$ for a split smooth linear algebraic group G. In this situation, Morel has proved that the classifying space $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-local, cf. [Mor09, Theorem 1.17]. The basic idea of his result is as follows: by [Mor06b, Theorem 3.46], it suffices to prove $\mathbb{A}^1$-invariance of the homotopy group sheaves of $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$. The previous isomorphisms and the computations in [Wen09] imply this for the sheaves $\pi_n^{\mathbb{A}^1}$ for $n \geq 2$. The space $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-connected, and $\pi_1^{\mathbb{A}^1} B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$ has been computed by Morel [Mor09, Theorem 1.18].

Let us remark that it is also possible to prove $\mathbb{A}^1$-locality of the classifying space of rationally trivial torsors using the results from the previous section. In [Mor07], Morel showed that the classifying space of $\mathrm{Sing}_\bullet^{\mathbb{A}^1} GL_n$ is $\mathbb{A}^1$-local. The methods developed there can be extended to deal with isotropic reductive groups.

Morel's results on the locality of classifying spaces directly imply the following result:

Theorem 4.6. *Let k be an infinite field, let G be a smooth split linear group. Let $p : E \to B$ be a G-torsor, and let $b : \mathrm{Spec}\, k \to B$ be a morphism such that the induced G-torsor over $\mathrm{Spec}\, k$ is trivial. Then there is an $\mathbb{A}^1$-local fibre sequence*

$$G \to E \to B.$$

The base points are chosen as in Proposition 4.3.

Proof. By Morel's result, the classifying space $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-local, hence the fibre sequence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} G \to E\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G \to B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$$

is $\mathbb{A}^1$-local. By [Wen07, Proposition 4.3.14], any fibre sequence obtained by pullback from this universal one is $\mathbb{A}^1$-local. But since we started with a torsor, the fibre sequence $\mathrm{Sing}_\bullet^{\mathbb{A}^1} G \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} E \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} B$ is obtained by pulling back the universal fibre sequence along a morphism $\mathrm{Sing}_\bullet^{\mathbb{A}^1} B \to B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G$. This proves the result. □

5. Applications

5.1. **Projective Homogeneous Spaces:** We first state a simple consequence concerning the $\mathbb{A}^1$-homotopy groups of some projective homogeneous spaces. If G is a split reductive group, and B is a Borel subgroup, then

$$B \hookrightarrow G \to G/B$$

is an $\mathbb{A}^1$-fibre sequence. By [Wen09, Proposition 5.6], B has the homotopy type of a connected group T of multiplicative type. This readily implies the following statement, which generalizes Morel's computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$, cf. [Mor06b].

Proposition 5.1. *Let G be a split, semisimple, simply-connected group, and let B be a Borel subgroup. Then the following assertions hold for homotopy groups of G/B:*

(i) G/B is $\mathbb{A}^1$-connected.

(ii) *There is an exact sequence*

$$0 \to \pi_1^{\mathbb{A}^1}(G) \to \pi_1^{\mathbb{A}^1}(G/B) \to T \to 0,$$

where T is a group of multiplicative type which is $\mathbb{A}^1$-equivalent to B.

(iii) *The canonical morphism $G \to G/B$ induces isomorphisms of homotopy group sheaves*

$$\pi_n^{\mathbb{A}^1}(G) \xrightarrow{\cong} \pi_n^{\mathbb{A}^1}(G/B),$$

for $n \geq 2$.

5.2. **Stabilization for KV-theory:** In this section, we apply the previous results to obtain fibre sequences which relate different algebraic groups. The long exact homotopy sequences for these fibre sequences provide stabilization theorems for unstable Karoubi-Villamayor K-theories. It turns out that for the classical groups, the successive quotients are quadrics, hence have the $\mathbb{A}^1$-homotopy type of spheres – this is a motivic analogue of the stabilization sequences for orthogonal and unitary groups in classical topology.

The following result has been proved in [Mor07, Theorem 8].

Proposition 5.2. *For $n \geq 3$, there is an $\mathbb{A}^1$-local fibre sequence*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_n) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_{n+1}) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_{n+1}/SL_n)$$

and the morphism

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_{n+1}/SL_n) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus \{0\})$$

induced by the projection is a simplicial weak equivalence between $\mathbb{A}^1$-local spaces.

Using the results from Section 4, we can prove analoguous results for the other split classical groups. First, we can state a result concerning all "stabilization sequences", which is an easy consequence of Proposition 4.3 resp. Theorem 4.6:

Proposition 5.3. *Let $\Phi_1 \to \Phi_2$ be an inclusion of root systems, and let $G(\Phi_1) \to G(\Phi_2)$ be the corresponding homomorphism of Chevalley groups. Then there is a simplicial fibre sequence*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi_1) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi_2) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi_2)/G(\Phi_1)).$$

These fibre sequences are $\mathbb{A}^1$-local if the classifying space $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi_1)$ is $\mathbb{A}^1$-local.

Note that $G(\Phi_2)/G(\Phi_1)$ is a smooth affine variety. The base points are the images of the identity in $G(\Phi_1)$.

To obtain stabilization results, we need to show that the homogeneous space $G(\Phi_2)/G(\Phi_1)$ is highly $\mathbb{A}^1$-connected. In the case of the classical groups, we will see that the homogeneous spaces are quadrics. It was remarked in [AD08] that split smooth affine quadrics have the $\mathbb{A}^1$-homotopy types of motivic spheres. More precisely, let Q_n be the quadric defined by the equation

$$1 = q_n(\mathbf{x}) = \begin{cases} \sum_i^k x_i x_{m+i} & n = 2k \\ \sum_i^k x_i x_{m+i} + x_{2k+1}^2 & n = 2k+1. \end{cases}$$

For the stabilization of the symplectic groups, we now have the following result:

Proposition 5.4. *Let k be a field, and let $Sp_{2n} \hookrightarrow Sp_{2n+2}$ be the canonical inclusion. Then there is an isomorphism*

$$Sp_{2n+2}/Sp_{2n} \to Q_{4n+4}.$$

Proof. We consider $\mathbb{A}^{2n+2}$ with the standard symplectic form

$$\omega : \mathbb{A}^{2n+2} \times \mathbb{A}^{2n+2} \to \mathbb{A}^1 : (v, w) \mapsto v^t \cdot J \cdot w,$$

where J is the standard $(2n+2)\times(2n+2)$ alternating matrix. The symplectic group Sp_{2n+2} is realized as

$$Sp_{2n+2}(k) = \left\{ A \in M_{2n+2,2n+2}(k) \mid A^\top \cdot J \cdot A = J \right\}.$$

It acts on $\mathbb{A}^{2n+2} \times \mathbb{A}^{2n+2}$ by left multiplication in each factor:

$$A \cdot (v, w) \mapsto (A \cdot v, A \cdot w).$$

It is easy to see that this action preserves the form ω:

$$(A \cdot v)^\top \cdot J \cdot (A \cdot w) = v^\top \cdot J \cdot w,$$

hence there is an action of Sp_{2n+2} on the hypersurface defined by $\omega = 1$. Fixing a point x in this hypersurface, we have a morphism

$$\pi : Sp_{2n+2} \to V(\omega - 1) : x \mapsto A \cdot x.$$

The isotropy group of the point x with coordinates $v_{2n+1} = w_{2n+2} = 1$ and $v_i = w_i = 0$ otherwise is Sp_{2n} via the standard inclusion. Incidentally, this point is in $V(\omega - 1)$.

We therefore obtain a morphism $Sp_{2n+2}/Sp_{2n} \to V(\omega - 1)$ which we want to show is an isomorphism. This morphism is surjective for dimension reasons: $V(\omega - 1)$ has dimension $4n + 3$, which is exactly the difference

$$\dim Sp_{2n+2} - \dim Sp_{2n} = 2(n + 1)^2 + n + 1 - 2n^2 - n.$$

By [Bor91, Proposition 6.7] this morphism is a quotient morphism if it is separable, i.e. the differential of π induces a surjection on tangent spaces. For Sp_{2n+2}, this is the Lie algebra $\mathfrak{sp}_{2n+2}$, the morphism $\mathfrak{sp}_{2n} \to T_xV(\omega-1)$ is basically taking the last two vectors of the matrix. A given pair of vectors can be obtained as the last two columns of a matrix in $\mathfrak{sp}_{2n+2}$ if the trace is zero. But this is precisely the condition one obtains by computing the tangent space of $V(\omega - 1)$ at the point x – and the differential is surjective.

Therefore, the morphism $\pi : Sp_{2n+2} \mapsto V(\omega - 1)$ is a quotient morphism, and we have an isomorphism $Sp_{2n+2}/Sp_{2n} \cong V(\omega - 1)$. Obviously, the hypersurface $V(\omega - 1)$ is isomorphic to the quadric Q_{4n+4} by a suitable change of signs isomorphism, which proves the claim. □

The case for the spin groups is similar. Note that over an algebraically closed field k, we have $k^\times/(k^\times)^2 = 0$ and therefore the morphism $\mathrm{Spin}_n \to SO_n$ is surjective with kernel $\mathbb{Z}/2\mathbb{Z}$.

Proposition 5.5. *Let k be a field of characteristic $\neq 2$. Let q_n be the standard quadratic form on $\mathbb{A}^n$, i.e.*

$$q_n(\mathbf{x}) = \begin{cases} \sum_{i=1}^k x_i x_{m+i} & n = 2k \\ \sum_{i=1}^k x_i x_{m+i} + x_{2k+1}^2 & n = 2k+1. \end{cases}$$

Let $\mathrm{Spin}(q_n) \hookrightarrow \mathrm{Spin}(q_{n+1})$ *be the standard inclusion.*

The quotient $\mathrm{Spin}(q_n)/\mathrm{Spin}(q_{n-1})$ *is isomorphic to the quadric Q_n which is the hypersurface defined by $q_n(\mathbf{x}) = 1$.*

Proof. The standard quadratic forms in the statement of the proposition are morphisms

$$q_n : \mathbb{A}^n \to \mathbb{A}^1.$$

Then we have the quadric $Q_n \subseteq \mathbb{A}^n$ which is the hypersurface defined by $q_n(x) = 1$. Fixing a vector $x \in Q_n$, we can again define a morphism

$$\pi : SO_n \to Q_n : A \mapsto A \cdot x.$$

For $SO(q_{2n+1})$ we fix the vector $(0, \ldots, 0, 1)$, the isotropy group is $SO(q_{2n})$ acting on the plane $x_{2n+1} = 0$. For $SO(q_{2n})$ we fix the vector $(0, \ldots, 0, 1, 1)$, the isotropy group is $SO(q_{2n+1})$ acting on the hyperplane $x_n = x_{2n}$ where the induced form is $\sum x_i x_{m+i} + x_n^2$.

The corresponding morphism are then surjective again for dimension reasons:

$$\dim SO(q_{2n+1}) - \dim SO(q_{2n}) = 2n^2 + n - 2n^2 + n = 2n$$

$$\dim SO(q_{2n+2}) - \dim SO(q_{2n+1}) = (n+1)(2n+1) - 2n^2 - n = 2n+1$$

These are exactly the dimensions of the quadrics considered above.

By [Bor91, Proposition 6.7] this morphism is a quotient morphism if it is separable, i.e. the differential of π induces a surjection on tangent spaces. As in the case of the symplectic groups, one can see surjectivity of the tangent map by using the standard generators of the Lie algebra $\mathfrak{o}_{2n}$ resp. $\mathfrak{o}_{2n+1}$: in the first case, the differential of π is given by the sum of the last two columns, in the latter case simply by the last column. Any vector with last entry zero is the last column of a matrix in $\mathfrak{o}_{2n+1}$ and any vector with the condition $v_{2n-1} + v_{2n} = 0$ is the sum of the last two columns of a matrix in $\mathfrak{o}_{2n}$. Therefore, π is a quotient morphism, and therefore we have an isomorphism

$$SO_n / SO_{n-1} \cong Q_n.$$

The same statement holds for the groups Spin_n acting on $\mathbb{A}^n$ via the quotient map $\mathrm{Spin}_n \to SO_n$. The center acts trivially, and therefore is also contained in the isotropy groups, which therefore are conjugates of Spin_{n-1}. □

The following propositions describe the low-dimensional homotopy of the simplicial sets $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(R)$ resp. $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(Q_n)(R)$, where Q_n is a odd-dimensional split quadric. This follows from computations of $\mathbb{A}^1$-homotopy groups for these spaces, together with the affine Brown-Gersten property for split semisimple groups.

Proposition 5.6. *Let k be an infinite field. For any regular local ring R over k, the simplicial set $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(A^n\setminus\{0\})(R)$ is $(n-2)$-connected. Moreover,*

$$\pi_{n-1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n\setminus\{0\})(R))\cong K_n^{MW}(R),$$

where $K_n^{MW}(R)$ is the unramified Milnor-Witt K-theory of the regular local ring R, cf. [Mor06b].

Moreover, there is a simplicial weak equivalence

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1} Q_{2n}\to \operatorname{Sing}_\bullet^{\mathbb{A}^1}\mathbb{A}^n\setminus\{0\},$$

hence the above assertions also hold for $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(Q_{2n})$.

Proof. The first claim is a result of Morel: in [Mor06b], it was shown that $\mathbb{A}^n\setminus\{0\}$ is $\mathbb{A}^1$-$(n-2)$-connected and that

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n\setminus\{0\})\cong K_n^{MW}(R).$$

By [Mor07], the morphism

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(SL_n)/\operatorname{Sing}_\bullet^{\mathbb{A}^1}(SL_{n-1})\to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n\setminus\{0\}$$

is a weak equivalence of simplicial presheaves, and the simplicial presheaf

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(SL_n)/\operatorname{Sing}_\bullet^{\mathbb{A}^1}(SL_{n-1})$$

has the affine Brown-Gersten property. This implies

$$\pi_n(\operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n\setminus\{0\})(R))\cong \pi_n^{\mathbb{A}^1}(\mathbb{A}^n\setminus\{0\})(R)$$

for any regular local ring R over k.

For the second claim: as explained in [AD08, Proposition 2.4, Example 2.7], the quadric Q_{2n} is a vector bundle torsor over $\mathbb{A}^n\setminus\{0\}$. In particular, the projection morphism is Zariski locally trivial with fibres affine spaces. By Proposition 4.5, we obtain a simplicial fibre sequence

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}\mathbb{A}^{n-1}\to \operatorname{Sing}_\bullet^{\mathbb{A}^1} Q_{2n}\to \operatorname{Sing}_\bullet^{\mathbb{A}^1}\mathbb{A}^n\setminus\{0\}$$

whose fibre is simplicially contractible. □

The $\mathbb{A}^1$-homotopy groups of the Chevalley groups have been described in [Wen09], building on work in [Mor07]. It turns out that these groups are unstable versions of Karoubi-Villamayor K-theory. The following result can be found in [Wen09, Theorem 1]. Note that the second isomorphism is well-known by the work of Rector, cf. also [Jar83].

Proposition 5.7. *Let Φ be a root system not equal to A_1, let k be an infinite field, and let R be a ring which is smooth and essentially of finite type over k. Then there are isomorphisms*

$$\pi_n^{\mathbb{A}^1}(G(\Phi),I)(R)\cong \pi_n\operatorname{Sing}_\bullet^{\mathbb{A}^1} G(\Phi)(R)\cong KV_{n+1}(\Phi,R).$$

Now we have all the information to prove the stabilization result for unstable Karoubi-Villamayor K-theories.

Theorem 5.8. *Let R be a regular local ring over an infinite field k.*

(i) The homomorphisms

$$KV_m(SL_{n-1},R)\to KV_m(SL_n,R)$$

are surjections for $m\leq n-1$ and isomorphisms for $m\leq n-2$.

(ii) *Let* $n = 2k$ *and assume that* $\operatorname{char} k \neq 2$. *The homomorphisms*

$$KV_m(\mathrm{Spin}_{n-1}, R) \to KV_m(\mathrm{Spin}_n, R)$$

are surjections for $m \leq k-1$ *and isomorphisms for* $m \leq k-2$.

(iii) *The homomorphisms*

$$KV_m(Sp_{2n-2}, R) \to KV_m(Sp_{2n}, R)$$

are surjections for $m \leq 2n-1$ *and isomorphisms for* $m \leq 2n-2$.

Proof. (i) From Proposition 5.2 and Proposition 4.3, we obtain a fibre sequence of simplicial sets

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_{n-1})(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_n)(R) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(R).$$

By Proposition 5.7 and Proposition 5.6, the long exact sequence looks as follows:

$$\cdots \to KV_{m+1}(SL_{n-1}, R) \to KV_{m+1}(SL_n, R) \to \pi_m(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(R)) \to$$

$$\to KV_m(SL_{n-1}, R) \to KV_m(SL_n, R) \to \cdots$$

But by Proposition 5.6, the first $n-2$ homotopy groups of $\mathrm{Sing}_\bullet^{\mathbb{A}^1} \mathbb{A}^n \setminus \{0\}$ vanish, inducing isomorphisms $KV_m(SL_{n-1}, R) \to KV_m(SL_n, R)$ for $m \leq n-2$. Moreover, we have a surjection $KV_{n-1}(SL_{n-1}, R) \to KV_{n-1}(SL_n, R)$.

In a similar way, (ii) follows from Proposition 5.5 and Proposition 5.6, and (iii) follows from Proposition 5.4 and Proposition 5.6. □

Remark 5.9. *Using the argument in* [Jar83, Theorem 3.14], *another way to prove the above stabilization result for all* KV_n *would be to obtain stabilization results for* KV_1. *As far as I know, this has not been done, except in the case of regular rings via the comparison with classical* K_1. *This is however not sufficient for Jardine's argument to go through.*

Finally, I would like to note that the fibre sequences obtained above can be used to give a geometric proof of the $\mathbb{A}^1$-locality of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_2)$ originally due to Moser.

Corollary 5.10. *The simplicial presheaf* $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(SL_2)$ *is* $\mathbb{A}^1$*-local.*

Proof. From Proposition 4.3 and Proposition 5.4, we obtain a fibre sequence of simplicial presheaves

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} Sp_2 \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} Sp_4 \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} Q_7.$$

By Proposition 5.6, we have a simplicial weak equivalence $\mathrm{Sing}_\bullet^{\mathbb{A}^1} Q_7 \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} \mathbb{A}^4 \setminus \{0\}$, and the latter space is indeed $\mathbb{A}^1$-local, by Proposition 5.2. The space $\mathrm{Sing}_\bullet^{\mathbb{A}^1} Sp_4$ is $\mathbb{A}^1$-local by [Wen09, Theorem 4.10]. Since in the above fibre sequence, base space and total space are $\mathbb{A}^1$-local, the fibre is also $\mathbb{A}^1$-local. □

5.3. **The second homotopy group of the projective line:** The results above yield a description of $\pi_2^{\mathbb{A}^1}(\mathbb{P}^1)$. Recall that $\mathbb{P}^1$ is $\mathbb{A}^1$-connected, and its fundamental group is described in Proposition 5.1.

Proposition 5.11. *Let k be an infinite field. For each $n \geq 1$, there is an exact sequence*

$$KV_{2n+2}(C_\infty, k) \to K^{MW}_{2n+2}(k) \to \pi^{\mathbb{A}^1}_{2n}(Sp_{2n})(k) \to KV_{2n+1}(C_\infty, k) \to 0.$$

In particular, the second homotopy group of the projective line sits in an exact sequence

$$0 \to \operatorname{coker}\big(KV_4(C_\infty, k)) \to K^{MW}_4(k)\big) \to \pi_2^{\mathbb{A}^1}(\mathbb{P}^1)(k) \to KV_3(C_\infty, k) \to 0.$$

Proof. The following fibre sequence can be obtained from Proposition 5.4 and Proposition 4.3:

$$\mathrm{Sing}^{\mathbb{A}^1}_\bullet(Sp_{2n}) \to \mathrm{Sing}^{\mathbb{A}^1}_\bullet(Sp_{2n+2}) \to \mathrm{Sing}^{\mathbb{A}^1}_\bullet(\mathbb{A}^{2n+2} \setminus \{0\}).$$

This implies the existence of a long exact homotopy group sequence

$$\to \pi^{\mathbb{A}^1}_{2n+1}(Sp_{2n+2})(k) \to \pi^{\mathbb{A}^1}_{2n+1}(\mathbb{A}^{2n+2} \setminus \{0\})(k) \to$$
$$\to \pi^{\mathbb{A}^1}_{2n}(Sp_{2n})(k) \to \pi^{\mathbb{A}^1}_{2n}(Sp_{2n+2})(k) \to \pi^{\mathbb{A}^1}_{2n}(\mathbb{A}^{2n+2} \setminus \{0\})(k)$$

By Proposition 5.6, the last term is 0, and we have

$$\pi^{\mathbb{A}^1}_{2n+1}(\mathbb{A}^{2n+2} \setminus \{0\})(k) \cong K^{MW}_{2n+2}(k)$$

By Theorem 5.8 and Proposition 5.7, the first and fourth terms are isomorphic to the respective homotopy groups of the infinite symplectic group. These groups can be identified with the corresponding Karoubi-Villamayor K-groups $KV_\bullet(C_\infty, k)$, cf. [Wen09, Theorem 1]. This yields the exact sequence claimed in the statement.

The remark about the second homotopy group of the projective line follows from this using Proposition 5.1. □

Remark 5.12. *Homotopy invariance for the (infinite) symplectic group over an infinite field implies an isomorphism $KV_n(C_\infty, k) \cong \pi_n BSp_\infty(k)^+$, hence the symplectic K-theory defined via a plus construction and the Karoubi-Villamayor K-theory for the symplectic groups agree.*

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