HOMOTOPY INVARIANCE FOR HOMOLOGY OF RANK TWO GROUPS

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Abstract. We show that homotopy invariance fails for homology of elementary groups of rank two over integral domains which are not fields. The proof is an adaptation of the argument used by Behr to show that rank two groups are not finitely presentable. We also show that homotopy invariance works for the Steinberg groups of rank two groups over integral domains with many units.

Contents

1. Introduction

In this paper, we discuss homotopy invariance for homology of linear groups of rank two. Here, homotopy invariance refers to the question if the canonical inclusion $G(R) \hookrightarrow G(R[t])$ induces an isomorphism on group homology. The groups we consider here are the Chevalley group $G(\Phi, R)$ resp. the elementary subgroup $E(\Phi, R)$ for a root system Φ of rank two.

For SL_2, homotopy invariance is known to fail because for any integral domain R which is not a field there are matrices in $SL_2(R[t])$ which are not elementary, cf. [KM97]. This in particular implies that $SL_2(R[t])$ typically has a much bigger abelianization than $SL_2(R)$. On the other hand, it follows from the theory of trees that the elementary subgroup E_2 has homotopy invariance, i.e. for any integral domain R, the inclusion $E_2(R) \to E_2(R[t])$ induces an isomorphism in group homology, cf. [Knu01, Theorem 4.6.7].

In this paper, we show that – similar to the case SL_2 – homotopy invariance fails for groups of rank two. Whereas the problem with SL_2 lies in the generators, the problem shifts to the relations and is exhibited in the second homology group $H_2(G(\Phi, R[t]), \mathbb{Z})$. Unlike the case SL_2, this problem can not be avoided by passing to the elementary subgroup. More precisely, we have the following, cf. Theorem 6.3.

Theorem 1. *Let R be an integral domain which is not a field. Let Φ be a reduced and irreducible root system of rank 2. If $\Phi = B_2$ assume that -1 is not a square in*

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R. Then the kernel of the reduction map

$$H_2(E(\Phi, R[t]), \mathbb{Z}) \to H_2(E(\Phi, R), \mathbb{Z})$$

surjects onto an abelian group of infinite rank. In particular, homotopy invariance for elementary groups of rank 2 fails.

This provides counterexamples to [Knu01, Theorem 4.6.8]. The condition in case $\Phi = B_2$ is the same appearing in [Beh79]. I expect it not to be necessary, but that would require a lot more computations with Bruhat decompositions.

It is actually possible to explicitly describe relations which span an infinite rank submodule of this kernel: any matrix h in $SL_2(R[t])$ which is not in $E_2(R[t])$ but becomes elementary in $G(\Phi, R[t])$ via a suitable embedding $SL_2(R[t]) \hookrightarrow G(\Phi, R[t])$ produces a non-trivial relation. These relations have the simple form $\tilde{h}\sigma(\tilde{h})^{-1}$ where $\tilde{h}$ is a chosen lift of h to the corresponding Steinberg group $\mathrm{St}(\Phi, R[t])$ and σ is a suitable automorphism of $\mathrm{St}(\Phi, R[t])$.

The argument we use is an adaptation of the technique used by Behr [Beh79] to show that rank two groups over $\mathbb{F}_q[t]$ are not finitely presentable. Informally, the structure of the argument is the following. First we recall from [KM97] that the subcomplex $SL_2(R[t]) \cdot \mathcal{Q}$ of the Bruhat-Tits tree has infinitely many distinct connected components. Then we embed SL_2 into the rank two group $G(\Phi)$ such that there is an automorphism σ of $G(\Phi)$ fixing the image of the embedding. This induces an automorphism of the rank two Bruhat-Tits building whose fixed point set contains an isomorphic copy of the Bruhat-Tits tree for SL_2, in particular many simplices of this fixed point set are not in the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$. The elementary factorization of a Krstić-McCool matrix $h_{p,k}$ produces a path from P_0 to $h_{p,k}P_0$. The automorphism of the building produces another path which compose to a loop, corresponding to the relation that there are two elementary factorizations of $h_{p,k}$ related by the automorphism σ. The resulting loop is obviously non-contractible because any contraction would produce a path *in the intersection of the fixed set of σ with the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$*.

I would like to point out the analogy between the behaviour of homotopy invariance for Chevalley groups over smooth rings and finite presentability of Chevalley groups over $\mathbb{F}_q[t]$. The group $SL_2(\mathbb{F}_q[t])$ is not finitely generated. The groups of rank two over $\mathbb{F}_q[t]$ are finitely generated, but are not finitely presentable. For homotopy invariance, $SL_2(R[t])$ has too many (non-constant) generators. For rank two groups over smooth rings this problem disappears because of the Suslin-Abe factorization ([Sus77], [GMV91, Theorem 1.2], and [Abe83]) but these groups do have too many (non-constant) relations. In view of the finite presentability of groups of rank at least three, cf. [RS76], it seems natural to expect that homotopy invariance for H_2 holds for all groups of rank at least three over rings which are smooth and essentially of finite type.

The second main result of the present paper is a reformulation of [Knu01, Theorem 4.6.8] which still holds. Whereas for SL_2 one has to pass to E_2, in the rank two case one has to pass to the Steinberg group to obtain homotopy invariance. Due to the previous theorem, it is not possible to argue with subcomplexes of the building. Instead, one has to pass to the universal covering of the subcomplex of the building. The proof of the following result is given in Theorem 6.4.

Theorem 2. *Let R be an integral domain with many units and let Φ be an irreducible and reduced root system of rank 2. Then the canonical inclusion $R \hookrightarrow R[t]$ induces isomorphisms*

$$H_\bullet(\mathrm{St}(\Phi, R), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(\mathrm{St}(\Phi, R[t]), \mathbb{Z}).$$

Remark on the non-split case: Theorem 1 is here formulated for the Chevalley groups. However, the arguments given for Theorem 1 work the same way for non-split groups of rank two. A non-trivial relation in $G(R[t])$ can be constructed from an automorphism σ of G which fixes a rank 1 subgroup G^σ and non-elementary elements of $G^\sigma \cap E(R[t])$. Such elements can be constructed for non-split groups of type A_1 by the same method employed in [KM97].

Structure of the paper: In Section 2 we recall preliminaries and notation for linear groups and buildings. In Section 3 we recall the work of Krstić-McCool on the SL_2 case. In Section 4, we discuss embeddings of the Bruhat-Tits tree in buildings of rank two. These preliminaries are used in Section 5 to construct many loops in the subcomplex of the building, and in Section 6 we deduce the consequences for group homology.

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2. Preliminaries and notation

The rings in this paper are commutative integral domains with multiplicative unit. For such an integral domain R we denote by $Q(R)$ the field of fractions of R.

For a background on linear algebraic groups, we refer to [Bor91]. For a reduced and irreducible root system Φ and a commutative ring R, we denote by $G(\Phi, R)$ the R-points of the (simply-connected) Chevalley group $G(\Phi)$ associated to Φ. By construction this comes with a natural choice of maximal torus T. The elements of the corresponding root subgroups of $G(\Phi)$ will be denoted by $x_\alpha(u)$ for $\alpha \in \Phi$ and $u \in R$. By $E(\Phi, R)$ we denote the elementary subgroup of $G(\Phi, R)$ which is generated by $x_\alpha(u)$ for $\alpha \in \Phi$ and $u \in R$. We denote by $\mathrm{St}(\Phi, R)$ the Steinberg group associated to Φ and R which is generated by $x_\alpha(u)$ for $\alpha \in \Phi$ and $u \in R$ subject to the usual commutator relations:

$$x_\alpha(u+v) = x_\alpha(u)x_\alpha(v), \text{ and}$$

$$[x_\alpha(u), x_\beta(v)] = \prod_{i,j>0, i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}u^iv^j) \text{ if } \alpha+\beta \neq 0.$$

We denote by $B(R)$ the R-points of the (fixed choice of) Borel subgroup B of $G(\Phi)$ containing the maximal torus T, and by $N(R)$ we denote the R-points of the normalizer of the maximal torus T. Canonical representatives of the Weyl group elements σ_α are given by $w_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$.

For an integral domain R, the field $Q(R)(t)$ has a valuation whose uniformizer is t^{-1} and we denote by $\mathcal{O}$ the corresponding discrete valuation ring. The group $G(\Phi, Q(R)(t))$ has a BN-pair $(B, N(Q(R)(t)))$, where the group B is given by the elements $G(\Phi, Q(R)(t))$ which lie in $G(\Phi, \mathcal{O})$ and whose reduction modulo t^{-1} lies in $B(Q(R))$. The Weyl group of the BN-pair is denoted by $W(\Phi)$. There is an associated affine (or euclidean) building, for background on the theory of Bruhat-Tits buildings we refer to [AB08]. The building will typically be denoted by $\mathcal{B}$, the corresponding group with BN-pair will be clear from the context. The cases we consider in this paper are root systems of rank ≤ 2. In the case of rank 1, this is the Bruhat-Tits tree which we will usually denote by $\mathcal{T}$. In the case of rank 2 root systems, the building is a two-dimensional simplicial complex obtained by gluing copies of the corresponding Coxeter complex of type $\tilde{A}_2$, $\tilde{B}_2$ resp. $\tilde{G}_2$. These Coxeter complexes are tilings of the euclidean plane by suitable triangles.

We use the following notation: the standard apartment is denoted by $\mathcal{A}$. The fundamental chamber $\mathcal{C}$ is a 2-simplex, its vertices are called P_0, P_1 and P_2. The notation P_i follows [Beh79], so P_0 is the 0-simplex whose stabilizer is $G(\Phi, \mathcal{O})$, and P_0P_1 is the long edge in case $\Phi = B_2$ and the short edge in case $\Phi = G_2$. The edge connecting P_i and P_j will be denoted by P_iP_j. Soulé's fundamental domain, cf. [Sou79], is denoted by $\mathcal{Q}$. It is the cone generated by the fundamental chamber $\mathcal{C}$. We do not distinguish in our notation which building $\mathcal{Q}$ lies in, this will always be clear from the context.

Recall also from [AB08] that the simplices of the building can be identified with cosets of standard parahoric subgroups in $G(\Phi, Q(R)(t))$. The action is then given by multiplication and the stabilizers are the corresponding conjugates of the standard parahoric subgroups.

3. Recalling the case SL_2

Let R be an integral domain. The following matrices in which k is a positive integer and $p \in R$ is a non-zero non-unit appear in the paper [KM97]:

$$h_{p,k} = \begin{pmatrix} 1 + pt^k & t^{3k} \\ p^3 & 1 - pt^k + p^2t^{2k} \end{pmatrix} \in SL_2(R[t]).$$

It is shown in [KM97] that if R is an integral domain which is not a field, then for a maximal subset P of non-associate non-invertible elements, the matrices $h_{p,k}$ for $p \in P$ span a free subgroup of $SL_2(R[t])$ which maps isomorphically to a free quotient of $SL_2(R[t])/U_2(R[t])$.

From the above it follows that these matrices span an infinite rank submodule of $H_1(SL_2(R[t]), \mathbb{Z}) = SL_2(R[t])^{\mathrm{ab}}$. For R a local integral domain we have $SL_2(R) = E_2(R)$, and the elementary group $E_2(R)$ is perfect if the residue field of R has at least 5 elements. Therefore $H_1(SL_2(R), \mathbb{Z}) = 0$, and the above matrices provide counterexamples to homotopy invariance for H_1 of SL_2. Note that homotopy invariance for H_1 of the elementary group E_2 is known for any integral domain R with many units, cf. [Knu01, Theorem 4.6.7].

The following proposition shows that for an integral domain which is not a field the subcomplex $SL_2(R[t]) \cdot \mathcal{Q}$ of the Bruhat-Tits tree $\mathcal{T}$ associated to $SL_2(Q(R)(t))$ has infinitely many distinct connected components. The arguments are reformulations of the proof of [KM97] adapted to the later application in the rank two case.

Proposition 3.1. *Let R be an integral domain.*

(i) If p is not invertible and $k > 0$, then the unique geodesic between P_0 and $h_{p,k}P_0$ is not contained in $SL_2(R[t]) \cdot \mathcal{Q}$.
(ii) If the unique geodesic between $h_{p,k}P_0$ and $h_{q,l}P_0$ is contained in $SL_2(R[t]) \cdot \mathcal{Q}$ then $k = l$ and p is associate to q.
(iii) Consider the filtration of the tree $\mathcal{T}(n) = SL_2(Q(R)[t]) \cdot \mathcal{Q}(n)$ obtained from $\mathcal{Q}(n) = \{x \in \mathcal{Q} \mid \alpha(x) \leq n\}$, i.e. the first n segments of $\mathcal{Q}$. Then P_0 and $h_{p,k}P_0$ can not be connected in $(SL_2(R[t]) \cdot \mathcal{Q}) \cup \mathcal{T}(k-1)$.

In particular, for R not a field and P a maximal subset of non-associate non-invertible elements, the matrices $h_{p,k}$ provide infinitely many distinct connected components of $SL_2(R[t]) \cdot \mathcal{Q}$.

Proof. First note that geodesics in trees are unique. In fact, the geodesic connecting two vertices is the unique path without backtracking between these two vertices. The geodesic between P_0 and $h_{p,k}P_0$ resp. between $h_{p,k}P_0$ and $h_{q,l}P_0$ can therefore be determined by producing a factorization of the matrices $h_{p,k}$ as iterated products of matrices in $SL_2(Q(R))$ and $B_2(Q(R)[t])$. Note that because $Q(R)[t]$ is a euclidean ring, we have an equality $SL_2(Q(R)[t]) = E_2(Q(R)[t])$, so such a factorization always exists.

An explicit factorization of $h_{p,k}$ into elementary matrices can be given as follows:

$$h_{p,k} = e_{12}(p^{-2}t^k)e_{21}(p^3)e_{12}(p^{-1}t^{2k} - p^{-2}t^k).$$

(i) From the above factorization, we can explicitly see the path from P_0 to $h_{p,k}P_0$ – first k steps in $\mathcal{Q}$, then k steps back in $e_{12}(p^{-2}t^k)\mathcal{Q}$, then $2k$ steps in $e_{12}(p^{-2}t^k)e_{21}(p^3)\mathcal{Q}$ and finally $2k$ steps back in $h_{p,k}\mathcal{Q}$.

This path does not contain a backtracking if the edges $e_{12}(t^kp^{-2})P_{01}$ and $h_{p,k}P_{01}$ are different from P_{01}. Other backtrackings can not appear because only the points $e_{12}(t^kp^{-2})P_0$ and $h_{p,k}P_0$ can be conjugate to P_0. The previous conditions are equivalent to $e_{12}(t^kp^{-2}) \notin B(Q(R))$ and $h_{p,k} \notin B(Q(R))$, where $B(Q(R))$ is the upper triangular matrix group with entries in $Q(R)$. But this is obvious. Therefore, the above path is a geodesic in the tree, and the distance between P_0 and $h_{p,k}P_0$ is $6k$. The proof that the path between P_0 and $h_{p,k}P_0$ breaks, i.e. there is a segment of the path not contained in $SL_2(R[t]) \cdot \mathcal{Q}$, is deferred to (iii).

(iii) To establish the claim, it suffices to show that $e_{12}(t^kp^{-2})P_{k-1,k} \notin SL_2(R[t])\cdot \mathcal{Q}$ where $P_{k-1,k}$ denotes the k-th edge of $\mathcal{Q}$. Denoting by B the stabilizer of $P_{k-1,k}$, this is equivalent to

$$e_{12}(t^kp^{-2}) \cdot B(k-1) \cap SL_2(R[t]) \cdot B(k-1) = \emptyset.$$

A matrix in $e_{12}(t^kp^{-2})$ has the form

$$\begin{pmatrix} a_1 + c_1t^kp^{-2} & b_1 + d_1t^kp^{-2} \\ c_1 & d_1 \end{pmatrix}$$

with $\deg a_1 = \deg d_1 = 0$, $\deg c_1 \leq -k$ and $\deg b_1 \leq k-1$. This matrix is contained in $SL_2(R[t])B(k-1)$ if there exist a_2, b_2, c_2, d_2 with $\deg a_2 = \deg d_2 = 0$, $\deg c_2 \leq -k$ and $\deg b_2 \leq k-1$ such that

$$\begin{pmatrix} a_1 + c_1t^kp^{-2} & b_1 + d_1t^kp^{-2} \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} =$$

$$\begin{pmatrix} (c_2d_1 + a_2c_1)t^kp^{-2} + b_1c_2 + a_1a_2 & (d_1d_2 + b_2c_1)t^kp^{-2} + b_1d_2 + a_1b_2 \\ c_2d_1 + a_2c_1 & d_1d_2 + b_2c_1 \end{pmatrix}$$

is contained in $SL_2(R[t])$. But because of the degree bounds on b_i, a_1 and d_2, we have $\deg(b_1d_2 + a_1b_2) \leq k-1$, in particular $(d_1d_2 + b_2c_1)p^{-2}t^k \in R[t]$. Again for degree reasons, $\deg b_2c_1 \leq -1$, therefore the coefficient of t^k is in fact $d_1(0)d_2(0)p^{-2} \in R$ where by $d_i(0)$ we denote the corresponding constant coefficients. Reduction modulo t^{-1} shows that $d_1(0)$ and $d_2(0)$ have to be invertible elements of R, therefore p has to be invertible as well. This contradicts the assumption and shows the claim. Note that the same argument shows $e_{12}(t^kp^{-2})P_{i-1,i} \notin SL_2(R[t]) \cdot \mathcal{Q}$ for any $1 \leq i \leq k$.

(ii) Probably, a similar argument also proves that the path from $h_{p,k}P_0$ to $h_{q,l}P_0$ has to break unless $k = l$ and p and q are associate. However, we deduce this fact from the corresponding result of Krstić-McCool: the quotient of the Bruhat-Tits tree modulo the $SL_2(R[t])$-action is homotopy equivalent to the quotient T/H in [KM97] – a homotopy equivalence is given by contracting Soulé's fundamental domain to the fundamental chamber. Under this homotopy equivalence, it is clear that the path connecting $h_{p,k}P_0$ and $h_{q,l}P_0$ is mapped to the loop in T/H consisting of the edges $e_{q,l}$, $(e'_{q,l})^{-1}$, $e'_{p,k}$ and $e_{p,k}^{-1}$. By the assertions (1) and (2) in [KM97], this loop is not contractible in T/H. Therefore, the path connecting $h_{p,k}P_0$ and $h_{q,l}P_0$ can not be contained in $SL_2(R[t]) \cdot \mathcal{Q}$. In particular, we get infinitely many distinct connected components in $SL_2(R[t]) \cdot \mathcal{Q}$. □

Remark 3.2. *Alternatively, the path in $\mathcal{T}$ from P_0 to $h_{p,k}P_0$ can be obtained from the Bruhat-decomposition*

$$h_{p,k} = \begin{pmatrix} 1 & 0 \\ t^{-3k} - pt^{-2k} + p^2t^{-k} & 1 \end{pmatrix} \begin{pmatrix} 0 & t^{3k} \\ -t^{-3k} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-3k} + pt^{-2k} & 1 \end{pmatrix}.$$

In particular, we see that the path has length $6k$ and lives in the apartment $g\mathcal{A}$ where $g = e_{21}(t^{-3k} - pt^{-2k} + p^2t^{-k})$. However, the factorizations provided in the proof above are better adapted to showing that there is no path from P_0 to $h_{p,k}P_0$ defined over $R[t]$.

4. The Bruhat-Tits building for rank two groups

We recall several pieces of information on Bruhat-Tits buildings of rank two from [Beh79] which we will use in the proof. First, we need a suitable set of generators of $E(\Phi, R[t])$, cf. [Beh79]. We use the notation of Behr.

Proposition 4.1. *We define a set of generators of $E(\Phi, R[t])$ which will be denoted by Γ. We denote by Γ_i, $i = 0, 1, 2, 3$ the stabilizer of P_i in $E(\Phi, R[t])$. The point P_3 appears in the case G_2 and is the point $w_{\alpha_0}P_0$ for α_0 the longest root.*

- *(i) In the cases $\Phi = A_2$ and $\Phi = B_2$, the group $E(\Phi, R[t])$ is generated by $\Gamma = \Gamma_0 \cup (\Gamma_1 \cap \Gamma_2)$.*
- *(ii) In the case $\Phi = G_2$, the group $E(\Phi, R[t])$ is generated by $\Gamma = \Gamma_0 \cup \Gamma_3$.*

Definition 4.2. *We define automorphisms of $G(\Phi)$ which will be denoted by σ in the sequel.*

- *(i) In the case $\Phi = A_2$, an automorphism of SL_3 is given by $w_\alpha \mapsto w_\beta$, $w_\beta \mapsto w_\alpha^{-1}$ and $x_{\alpha+\beta}(u) \mapsto x_{\alpha+\beta}(u)$. This is the diagram automorphism (i.e. taking the transpose inverse) followed by conjugation with $w_{\alpha+\beta}$.*
- *(ii) In the case $\Phi = B_2$, we take σ to be the inner automorphism given by conjugation with w_β which fixes $x_{2\alpha+\beta}$ and maps $w_\alpha \mapsto w_{\alpha+\beta}^{-1}$ and $x_\alpha(1) \mapsto x_{\alpha+\beta}(-1)$.*
- *(iii) In the case $\Phi = G_2$, we take σ to be the inner automorphism given by conjugation with w_α which leaves $x_{3\alpha+2\beta}$ invariant.*

Definition 4.3. *We now define an embedding of SL_2 into $G(\Phi)$ which will be denoted by ι in the sequel.*

- *(i) In the case $\Phi = A_2$, we embed SL_2 as subgroup corresponding to the root $\alpha + \beta$.*
- *(ii) In the case $\Phi = B_2$, we embed SL_2 as long root subgroup corresponding to the root $2\alpha + \beta$.*
- *(iii) In the case $\Phi = G_2$, we embed SL_2 as long root subgroup corresponding to $3\alpha + 2\beta$.*

It is obvious from the above definitions that the automorphism σ fixes the image of the embedding ι.

Remark 4.4. *(i) In the case $\Phi = A_2$ the fixed set is exactly the image of ι, as can be seen by the Bruhat decomposition: the automorphism σ fixes the Borel B, not pointwise but as a subgroup. Therefore $\sigma(BwB) = B\sigma(w)B$, hence any element fixed by σ must lie in a double coset of the form BwB with $w = \sigma(w)$. The only double cosets with this property are B and $Bw_{\alpha+\beta}B$ which form the Bruhat decomposition of the image of ι.*

(ii) In the other two cases B_2 and G_2, the fixed point group is strictly larger: obviously, conjugation with w_α fixes w_α. In fact, it fixes a larger subgroup of the SL_2 copy generated by x_α and $x_{-\alpha}$. Moreover, the matrix w_α is diagonalizable if and only if -1 is a square in R. In this case, for any matrix g

diagonalizing w_α, the whole sector $g\mathcal{Q}$ is fixed by w_α because w_α is contained in the stabilizer gBg^{-1} of $\mathcal{Q}$.

The automorphism σ induces an automorphism of the building. On the standard apartment, this automorphism induces reflection along a line. The intersection of this line with $\mathcal{Q}$ is a half-line $\mathcal{Q}^\sigma$. In case A_2, this half-line is the symmetry axis of the cone $\mathcal{Q}$, in the other two cases B_2 and G_2 the half-line is the ray generated by the short edge of the fundamental chamber $\mathcal{C}$.

Proposition 4.5. *The fixed point set of the automorphism of the building contains an isomorphic copy of the tree $\mathcal{T}$, and*

$$(E(\Phi,R[t])\cdot\mathcal{Q})\cap\mathcal{T}=(SL_2(R[t])\cap E(\Phi,R[t]))\cdot(\mathcal{Q}\cap\mathcal{T}).$$

Proof. There is a morphism from the tree $\mathcal{T}$ to the two-dimensional building $\mathcal{B}$ associated to $G(\Phi,Q(R)(t))$ given as follows. We denote the fundamental domain of $SL_2(Q(R)[t])$ on the tree $\mathcal{T}$ by $\mathcal{Q}'$ and the fundamental domain for $G(\Phi,Q(R)[t])$ on $\mathcal{B}$ by $\mathcal{Q}$. Then we identify $\mathcal{Q}'$ with the half-line $\mathcal{Q}^\sigma$ via an isomorphism also denoted ι. Then the inclusion $\iota: SL_2(Q(R)[t])\hookrightarrow G(\Phi,Q(R)[t])$ induces a map from $\mathcal{T}$ to $\mathcal{B}$ by mapping the point $gy\in\mathcal{T}$ to the point $\iota(g)\iota(y)$.

The inclusion $\supseteq$ is clear: a point in $(SL_2(R[t])\cap E(\Phi,R[t]))\cdot(\mathcal{Q}\cap\mathcal{T})$ has the form gy for $y\in\mathcal{Q}\cap\mathcal{T}$ and $g\in SL_2(R[t])\cap E(\Phi,R[t])$. In particular g and y are fixed by σ, so is gy, hence the image of $\mathcal{T}$ is contained in the fixed set.

Now we need to show that the map $\mathcal{T}\to\mathcal{B}$ is injective. Assume there exist vertices P and Q which are distinct in $\mathcal{T}$ and are identified in $\mathcal{B}$. Because the embedding is equivariant for the inclusion of $SL_2(Q(R)(t))$ into $E(\Phi,Q(R)(t))$ we can assume without loss of generality that $P=P_0$. Also, there exists $g\in SL_2(Q(R)(t))$ such that $Q=gP_0$ or $Q=gP_1$. Because the actions preserve types, only the first case is possible. From a Bruhat decomposition of g we can determine the distance between P_0 and gP_0 in the tree. But a Bruhat decomposition for g in $SL_2(Q(R)(t))$ also provides a Bruhat decomposition for $\iota(g)$ in $E(\Phi,Q(R)(t))$ with the corresponding Weyl group element. Therefore, if the distance between P and Q is non-zero in the tree $\mathcal{T}$, then it remains non-zero in $\mathcal{B}$. In fact, we obtain an isometric embedding of $\mathcal{T}$ into $\mathcal{B}$ if we metrize $\mathcal{Q}'$ via the identification with $\mathcal{Q}^\sigma$.

Finally, the nontrivial inclusion $\subseteq$ in the statement. Let $x\in(E(\Phi,R[t])\cdot\mathcal{Q})\cap\mathcal{T}$. From $x\in E(\Phi,R[t])\cdot\mathcal{Q}$ we conclude that $x=gy$ for $g\in E(\Phi,R[t])$ and $y\in\mathcal{Q}$, and from $x\in\mathcal{T}$ we conclude $x=g'y'$ with $g'\in SL_2(Q(R)[t])$ and $y'\in\mathcal{Q}\cap\mathcal{T}$. Since $\mathcal{Q}$ is a fundamental domain, it follows that $y=y'$. So if we denote by P the stabilizer of y, then $g'P$ contains g. But there exists a 2-simplex $z\in E(\Phi,R[t])$ containing y. In particular, we can assume that P is the stabilizer of a 2-simplex in $\mathcal{Q}$ and $g'P$ contains a matrix $g\in E(\Phi,R[t])$. Since g and g' are both in $E(\Phi,Q(R)[t])$, we can even replace P by $P\cap E(\Phi,Q(R)[t])$ which is an extension of the Borel of $E(\Phi,Q(R))$ by a unipotent group with entries in $Q(R)[t]$. But the finitely many denominators can be cleared using the torus action, so we can write an element $p\in P\cap E(\Phi,Q(R)[t])$ as product $p=p_1p_2$ with $p_1\in B(\Phi,Q(R))$ and $p_2\in U^+(\Phi,R[t])$, in the latter U^+ denotes the unipotent radical of the Borel. Summing up, we can assume that $g'B(Q(R))$ contains a matrix $g\in E(\Phi,R[t])$, i.e.

$$g\in E(\Phi,R[t])\cap SL_2(Q(R)[t])\cdot B(Q(R))=\iota(SL_2(R[t]))\cdot B(R).$$

Therefore, we can change g up to a matrix in $B(R)$ to a matrix in $\iota(SL_2(R[t]))$. This establishes the claim. □

5. Nontrivial loops and relations

We now define loops in $E(\Phi,R[t])\cdot\mathcal{Q}$ associated to Krstić-McCool matrices. As in Behr's argument, the relation is given by two different elementary factorizations of

$$h_{p,k} = \begin{pmatrix} 1+pt^k & t^{3k} \\ p^3 & 1-pt^k+p^2t^{2k} \end{pmatrix} \in SL_2(R[t]).$$

We use the homomorphism $\iota : SL_2(R[t]) \to G(\Phi, R[t])$ from Definition 4.3 and denote the image of $h_{p,k}$ in $G(\Phi, R[t])$ again by $h_{p,k}$. The next proposition provides elementary factorizations of $h_{p,k}$ showing that $h_{p,k} \in E(\Phi, R[t])$.

Proposition 5.1. *(i) Let $\Phi = A_2$. Then*

$$h_{p,k} = x_{-\beta}(p^2)x_\alpha(t^k)x_{-\alpha-\beta}(-p)x_{-\alpha}(p+t^k)x_\alpha(p)x_{\alpha+\beta}(t^k)$$
$$x_{-\alpha-\beta}(p)x_{-\alpha}(-t^k)x_\alpha(-p-t^k)x_{\alpha+\beta}(-t^k)x_\beta(-t^{2k})x_{-\alpha}(-p).$$

(ii) Let $\Phi = B_2$. Then

$$h_{p,k} = x_{-\alpha-\beta}(-p^2)x_{-\beta}(-p^2t^k-p)x_\alpha(-t^k)x_{-\alpha}(p)x_\beta(t^k)x_{-\beta}(p)$$
$$x_\alpha(t^k)x_{-\alpha}(-p)x_{\alpha+\beta}(t^{2k})x_\beta(-pt^{2k}-t^k).$$

(iii) Let $\Phi = G_2$. Then

$$h_{p,k} = x_{-\beta}(p^2)x_{3\alpha+\beta}(t^k)x_{-3\alpha-2\beta}(-p)x_{-3\alpha-\beta}(p+t^k)x_{3\alpha+\beta}(p)x_{3\alpha+2\beta}(t^k)$$
$$x_{-3\alpha-2\beta}(p)x_{-3\alpha-\beta}(-t^k)x_{3\alpha+\beta}(-p-t^k)x_{3\alpha+2\beta}(-t^k)x_\beta(-t^{2k})x_{-3\alpha-\beta}(-p).$$

Proof. Just do the matrix multiplication to verify (i) and (ii). The assertion (i) can be obtained by the factorization algorithm of Park and Woodburn, cf. [PW95]. The assertion (ii) is basically the proof from [BMS67, Section 13] of the Mennicke-symbol equality

$$\begin{bmatrix} t^k \\ 1+pt^k \end{bmatrix} \begin{Bmatrix} t^k \\ 1+pt^k \end{Bmatrix} = \begin{Bmatrix} t^{3k} \\ 1+pt^k \end{Bmatrix}$$

For (iii) we only rewrite (i) by replacing $\alpha \mapsto 3\alpha+\beta$ (and hence $\alpha+\beta \mapsto 3\alpha+2\beta$). □

For a local ring R which is smooth and essentially finite type over a field, the existence of such matrix factorizations is clear from the Suslin-Abe type factorization $G(\Phi, R[t]) = G(\Phi, R)E(\Phi, R[t])$. However, we chose to write some down explicitly.

Now we rewrite these elementary factorizations as follows:

(i) In case $\Phi = A_2$, we replace occurrences of $x_\alpha(t^k)$ by $w_\beta x_{\alpha+\beta}(t^k)w_\beta^{-1}$ and occurrences of $x_\beta(t^k)$ by $w_\alpha^{-1}x_{\alpha+\beta}(t^k)w_\alpha$, respectively. Then we apply Behr's method from [Beh79] to write $x_{\alpha+\beta}(t^k)$ as product of elements $w_\alpha^{\pm 1}$, $w_\beta^{\pm 1}$ and $x_{\alpha+\beta}(t)^{\pm 1}$ using inductively the commutator formula

$$x_{\alpha+\beta}(t^m) = [w_\beta^{-1}x_{\alpha+\beta}(-t)w_\beta, w_\alpha^{-1}x_{\alpha+\beta}(t^{m-1})w_\alpha].$$

This produces a new factorization of $h_{p,k}$ which only uses constant matrices and $x_{\alpha+\beta}(\pm t)$.

(ii) In case $\Phi = B_2$, we use

$$x_{\alpha+\beta}(t^n) = [x_\alpha(1), x_\beta(t^n)]x_{2\alpha+\beta}(t^n)^{-1}$$
$$x_\alpha(t^n) = w_{2\alpha+\beta}x_{\alpha+\beta}(t^n)w_{2\alpha+\beta}^{-1}$$

to replace any occurrences of short root elements with t-powers by constants or long-root elements. We can restrict to use the long root element $x_{2\alpha+\beta}$ by the formula

$$x_\beta(t^n) = w_{\alpha+\beta}x_{-2\alpha-\beta}(-t^n)w_{\alpha+\beta}^{-1}.$$

Then we can inductively use Behr's formula

$$x_{2\alpha+\beta}(t^m) = [w_\beta x_{2\alpha+\beta}(t)[w_\alpha x_{2\alpha+\beta}(-t)w_\alpha^{-1}, x_\alpha(1)]w_\beta^{-1}, w_\alpha x_{2\alpha+\beta}(-t^{m-2})w_\alpha^{-1}]$$
$$x_{2\alpha+\beta}(t^{m-1})[w_\alpha x_{2\alpha+\beta}(-t^{m-1})w_\alpha^{-1}, x_\alpha(1)]$$

to obtain an elementary factorization of $h_{p,k}$ which only uses constant matrices and $x_{2\alpha+\beta}(\pm t)$.

(iii) In case $\Phi = G_2$, we first replace occurrences of $x_{3\alpha+\beta}(t^k)$ by $w_\beta x_{3\alpha+2\beta}(t^k)w_\beta^{-1}$ and occurrences of $x_\beta(t^k)$ by $w_{3\alpha+\beta}^{-1}x_{3\alpha+2\beta}(t^k)w_{3\alpha+\beta}$. Then we use the commutator formula

$$x_{3\alpha+2\beta}(t^m) = [w_\beta^{-1}x_{3\alpha+2\beta}(-t)w_\beta, w_{3\alpha+\beta}^{-1}x_{3\alpha+2\beta}(t^{m-1})w_{3\alpha+\beta}]$$

to produce a factorization of $h_{p,k}$ which only uses constant matrices and $x_{3\alpha+2\beta}(\pm t)$.

Now we associate paths to these factorizations, this is the construction from [Beh79]. In each case, we have a factorization $h_{p,k} = e_1 \cdots e_n$ where e_i is in the set of generators exhibited in Proposition 4.1. Since these generators are in stabilizer subgroups, each e_i stabilizes an end-point of the edge P_0P in the building, where $P = P_1$ or $P = P_2$ in the case A_2, $P = P_2$ in case B_2, and $P = P_1$ in case G_2. Therefore, to the word $e_1 \cdots e_n$ we associate the path

$$P_0P, e_1(P_0P), e_1e_2(P_0P), \ldots, e_1 \cdots e_n(P_0P).$$

For the above elementary factorizations of $h_{p,k}$, we denote the path obtained by this construction by $H_{p,k}$. This path connects the two vertices P_0 and $h_{p,k}P_0$. Looking at Proposition 5.1 and the fact that the Behr factorizations of $x_\alpha(t^m)$ are defined over $\mathbb{Z}$, we have the following obvious proposition.

Proposition 5.2. *The factorizations constructed above are defined over $E(\Phi, R[t])$, therefore the paths $H_{p,k}$ are contained in $E(\Phi, R[t]) \cdot \mathcal{Q}$.*

Now we are ready to construct loops associated to the Krstić-McCool matrices $h_{p,k}$. We consider the automorphism σ of the group $G(\Phi, R[t])$ resp. the building defined in Definition 4.2. This automorphism fixes the embedded copy of $SL_2(R[t])$, so it fixes $h_{p,k}$. Note also that this automorphism preserves the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$. If we denote by $h_{p,k} = e_1 \cdots e_n$ the above elementary factorizations, the application of σ produces a new factorization $\sigma(h_{p,k}) = h_{p,k} = \sigma(e_1) \cdots \sigma(e_n)$. Applying the automorphism to the path

$$H_{p,k} = P_0P, e_1(P_0P), e_1e_2(P_0P), \ldots, e_1 \cdots e_n(P_0P).$$

yields a new path, where we denote $P_\sigma = \sigma(P)$:

$$\sigma(H_{p,k}) = P_0P_\sigma, \sigma(e_1)(P_0P_\sigma), \sigma(e_1)\sigma(e_2)(P_0P_\sigma), \ldots, \sigma(e_1) \cdots \sigma(e_n)(P_0P_\sigma).$$

We compose the two paths $H_{p,k}$ and $\sigma(H_{p,k})$ and obtain a loop denoted by $L_{p,k}$ this is a path associated to the relation $e_1 \cdots e_n = \sigma(e_1) \cdots \sigma(e_n)$ which could also be written as $\tilde{h}_{p,k}\sigma(\tilde{h}_{p,k})^{-1} = 1$.

We now use these loops $L_{p,k}$ to show that the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$ has a quite big fundamental group. First of all, we show that the fundamental group of this complex is free. Its non-triviality will be established in the next proposition.

Proposition 5.3. *We denote $X = E(\Phi, R[t]) \cdot \mathcal{Q}$ and $\pi = \pi_1(X, P_0)$. The fundamental group π of X is free and X has the weak homotopy type of $B\pi$.*

Proof. First note that by definition $E(\Phi, R[t])$ is generated by $x_\alpha(u)$ for $\alpha \in \Phi$ and $u \in R[t]$. Using $w_\alpha x_{-\alpha}(u)w_\alpha^{-1} = x_\alpha(-u)$ the group $E(\Phi, R[t])$ can be generated by constant matrices and $x_\alpha(u)$ for $\alpha \in \Phi^+$ and $u \in R[t]$. In particular, $E(\Phi, R[t])$ is generated by stabilizers of vertices of $\mathcal{Q}$. This implies that $X = E(\Phi, R[t]) \cdot \mathcal{Q}$ is connected. In particular, $\pi = \pi_1(X, P_0)$ does not depend on the choice of base point.

We next show that X is aspherical, i.e. $\pi_n(X) = 0$ for $n \geq 2$. Note that the simplicial complex X considered as a simplicial set is obviously not fibrant: there are lots of Λ_i^2-configurations which can not be extended to triangles. However, any homotopy class of a map $S^n = \Delta^n/\partial\Delta^n \to \mathrm{Ex}^\infty(X)$ is already represented by a morphism $S \to X$ where S is a suitable subdivision of S^n. The Bruhat-Tits building is two-dimensional and contractible, so the composition of $S \to X$ with the inclusion of X as subcomplex of the building factors through $D \to X$ where D is a suitable subdivision of the two-simplex Δ^2. Therefore, any homotopy class $S^n \to \mathrm{Ex}^\infty(X)$ is null-homotopic for $n \geq 2$. This shows that X is aspherical, in particular X is weakly equivalent to $B\pi$.

It remains to show that π is free. For this, it suffices to show that $H_i(X, \mathbb{Z}) = H_i(\pi, \mathbb{Z}) = 0$ for $n \geq 2$, by a theorem of Swan [Swa69]. This is done as in [Knu01, Theorem 4.6.8]: we consider the inclusion of X into the Bruhat-Tits building $\mathcal{B}$ and the associated long exact sequence for relative homology

$$\cdots \to H_{n+1}(\mathcal{B}, X) \to H_n(X) \to H_n(\mathcal{B}) \to H_n(\mathcal{B}, X) \to \cdots$$

Since all complexes involved are two-dimensional, $H_i(X) = 0$ for $i \geq 3$ and $H_3(\mathcal{B}, X) = 0$. Contractibility of the building implies $H_2(\mathcal{B})$ and hence $H_2(X) = 0$. Therefore, π is a free group. □

Proposition 5.4. *Let R be an integral domain, let Φ be an irreducible and reduced root system of rank two, and denote $X = E(\Phi, R[t]) \cdot \mathcal{Q}$ and $\pi = \pi_1(X)$. If $\Phi = B_2$ assume that -1 is not a square in R.*

Assume R is not a field. For p not invertible and $k \geq 1$, the loops $L_{p,k}$ are not contractible in X, in particular the complex X is not simply-connected. Moreover, the abelianization of π has infinite $\mathbb{Z}$-rank.

Proof. First some preparatory remarks: note that Proposition 3.1, part (i), implies that $h_{p,k}P_0$ and P_0 are in different connected components of $SL_2(R[t]) \cdot \mathcal{Q}$. By the elementary factorizations in Proposition 5.1, we have

$$h_{p,k}P_0 \in (SL_2(R[t]) \cap E(\Phi, R[t])) \cdot (\mathcal{Q} \cap \mathcal{T}),$$

and by Proposition 4.5, $h_{p,k}P_0$ and P_0 lie in different connected components of $(E(\Phi, R[t]) \cdot \mathcal{Q}) \cap \mathcal{T}$. Note also that the automorphism σ fixes not only the points P_0 and $h_{p,k}P_0$ but also the geodesic line joining them. Here we take the geodesic line in the building, which lies entirely inside the embedded tree $\mathcal{T}$. We conclude that there is a segment S of this geodesic line which is not contained in the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$.

Now we recall the non-contractibility arguments from [Beh79].

(i) In the case $\Phi = A_2$, there is a whole triangle Δ containing this geodesic segment S as fixed set of a reflection. The building can be contracted along geodesic lines to the barycentre of Δ. We use this retraction to retract the loop $L_{p,k}$ onto $\partial\Delta$ inside the building. We already know the geodesic between P_0 and $h_{p,k}P_0$ runs through the barycentre of Δ. Therefore, the above retraction maps P_0 and $h_{p,k}P_0$ to the opposite ends of the segment S. The path $H_{p,k}$ retracts to a path on the boundary of the triangle joining the opposite ends of the segment, and $\sigma(H_{p,k})$ is the corresponding symmetric path. So the image of $L_{p,k}$ is not contractible in $\partial\Delta$, so $L_{p,k}$ is not contractible in $E(\Phi, R[t]) \cdot \mathcal{Q}$.

(ii) In the case $\Phi = B_2$, any triangle in the building containing the segment S as a side is not contained in the subcomplex $E(\Phi, R[t]) \cdot \mathcal{Q}$. The building can be contracted along geodesics to the midpoint of the segment S and this contraction induces a retraction from the complement of the open star of S in the building onto the link of S. We know the geodesic between P_0 and $h_{p,k}P_0$ runs through the midpoint of S, therefore the above retraction maps P_0 and $h_{p,k}P_0$ to opposite ends of S. It then suffices to show that the image of the path $H_{p,k}$ is not fixed under the action of w_β. Assume it is fixed. Then w_β must fix a whole two-simplex in the building, which is equivalent to w_β being diagonalizable. This is the case if and only if -1 is a square in R. If

-1 is not a square, there is no two-simplex fixed by w_β, so $H_{p,k}$ retracts onto a non-contractible loop in the link of S.

(iii) In the case $\Phi = G_2$, we use the embedding of $E(A_2, R[t])$ into $E(G_2, R[t])$ given by the inclusion of the root system A_2 into G_2 as long roots. This induces a morphism of the corresponding buildings. Note that the construction of the loop $L_{p,k}$ in the case G_2 given in Proposition 5.1 was exactly induced from the A_2 situation. Therefore, the loop $L_{p,k}$ lies in the image of the building for A_2. The image of Soulé's fundamental domain $\mathcal{Q}$ for A_2 is then the union of the fundamental domain $\mathcal{Q}'$ for G_2 and $w_\alpha \mathcal{Q}'$. In particular the outer automorphism of A_2 becomes conjugation with w_α in G_2. By Proposition 4.5, the triangle which was used for establishing non-contractibility in case A_2 is also not contained in the subcomplex $E(G_2, R[t])\cdot\mathcal{Q}'$. Therefore, the argument in (i) shows that the loop $L_{p,k}$ is not contractible. Note that the case G_2 differs from Behr's argument and uses a reduction to A_2 in order to avoid the arithmetic assumption that -1 is not a square.

The same argument shows that $L_{p,k}$ and $L_{q,l}$ are not homotopic unless p and q are associate and $k = l$ using part (ii) of Proposition 3.1. Instead of the loop $L_{p,k}$, we now use $L_{p,k}^{-1}L_{q,l}$. Then we apply the above retract argument to any segment of the geodesic connecting $h_{p,k}P_0$ and $h_{q,l}P_0$ and not lying in $E(\Phi, R[t])\cdot\mathcal{Q}$, which shows that $L_{p,k}^{-1}L_{q,l}$ is not contractible, or equivalently, $L_{p,k}$ and $L_{q,l}$ are not homotopic.

Now we discuss the abelianization H_1 of π. By Proposition 5.3, π is a free group, so H_1 is a free abelian group. We use the retract argument again – if L is a commutator, its class in $\pi_1(\partial\Delta)$ (i.e. the winding number of L around the barycentre of Δ) is trivial for any triangle Δ. Assume that H_1 has a finite basis $L_1, \ldots, L_n$. For all but finitely many triangles Δ the winding numbers of L_i around Δ are zero, the same then holds for linear combinations of L_i. This follows since in a building of dimension two, a loop has (up to a suitable two-dimensional notion of backtracking) a unique contraction which moreover is compact. But from part (iii) of Proposition 3.1 we obtain infinitely many distinct triangles $\Delta_{p,k}$ and corresponding loops $L_{p,k}$ such that the winding number of $L_{p,k}$ around $\Delta_{p,k}$ is non-trivial. This contradicts the existence of a finite basis. □

Remark 5.5. *It seems quite likely that the assumption -1 not a square in case B_2 is unnecessary. From the proof we see, however, that this would require more information on the retraction of the path $H_{p,k}$ onto the link of the segment S. This could be obtained by computing geodesics from the vertices of the path to the midpoint of S, which in turn can be read off from a Bruhat-decomposition of the partial products of the elementary factorization of $h_{p,k}$.*

We denote by $\widetilde{E}(\Phi, R[t])$ the amalgam of the stabilizers of vertices of $\mathcal{Q}$. Recall from [Sou79, Theorem 2] that there is an exact sequence

$$\pi_1(X, x_0) \to \widetilde{E}(\Phi, R[t]) \to E(\Phi, R[t]) \to \pi_0(X) \to 0.$$

As in the proof of Proposition 5.3, $E(\Phi, R[t])$ is generated by stabilizers, so X is connected. Moreover, the fundamental domain $\mathcal{Q}$ is simply-connected and each complex $g\mathcal{Q} \cap \mathcal{Q}$, $g \in E(\Phi, R[t])$, is connected or empty, because $\mathcal{Q}$ is convex and $E(\Phi, R[t])$ acts by isometries. Therefore, [Sou79, Theorem 2] implies that we have an extension of groups

$$1 \to \pi_1(X) \to \widetilde{E}(\Phi, R[t]) \to E(\Phi, R[t]) \to 1.$$

Proposition 5.6. *With the above notation, the assignment*

$$\tilde{x}_\alpha(u) \mapsto \begin{cases} x_\alpha(u) & \alpha \in \Phi^+ \\ w_\alpha x_\alpha(-u) w_\alpha^{-1} & \alpha \in \Phi^- \end{cases}$$

extends to a surjective group homomorphism $\phi : \mathrm{St}(\Phi, R[t]) \to \widetilde{E}(\Phi, R[t])$.

Proof. It suffices to show that the commutator formulas hold in $\widetilde{E}(\Phi, R[t])$. Surjectivity is then clear since all generators of $\widetilde{E}(\Phi, R[t])$ are in the image of ϕ.

To establish the commutator formula, recall that a presentation for the amalgam $\widetilde{E}(\Phi, R[t])$ can be obtained as the union of suitable presentations of the stabilizer subgroups. In particular, it is generated by constant elementary matrices and $x_\alpha(u)$ for $\alpha \in \Phi^+$ and $u \in R[t]$ subject to the relations defining the stabilizer subgroups. For the positive roots, it is clear that the commutator formula holds because it holds in some stabilizer. For the other cases, we can conjugate the corresponding commutator formula between positive roots and obtain the desired relation between positive roots and $w_\alpha x_\alpha(u) w_\alpha^{-1}$ replacing the negative roots. □

Corollary 5.7. *Let R be an integral domain and let Φ be an irreducible and reduced root system of rank two. If $\Phi = B_2$ assume that -1 is not a square in R. Denote by*

$$K_2(\Phi, R[t]) = \ker\left(\mathrm{St}(\Phi, R[t]) \to E(\Phi, R[t])\right)$$

the unstable K_2 associated to the root system Φ and the ring $R[t]$. If R is not a field, then $K_2(\Phi, R[t])$ surjects onto a free group of infinite rank. In particular, homotopy invariance fails for unstable $K_2(\Phi)$ if Φ is of rank two.

Proof. The homomorphism $\phi : \mathrm{St}(\Phi, R[t]) \to \widetilde{E}(\Phi, R[t])$ from Proposition 5.6 restricts to a homomorphism $\psi : K_2(\Phi, R[t]) \to \pi_1(X)$. Let $g \in \pi_1(X)$. Since the homomorphism ϕ is surjective, there exists $\tilde{g} \in \mathrm{St}(\Phi, R[t])$ such that $g = \phi(\tilde{g})$. But g maps to 1 in $E(\Phi, R[t])$, so does $\tilde{g}$, hence $\tilde{g} \in K_2(\Phi, R[t])$. So ψ is surjective and the claim follows from Proposition 5.3 and Proposition 5.4.

For the consequence on homotopy invariance for unstable K_2, note that $K_2(\Phi, R)$ lies in the kernel of ϕ: the relations in $K_2(\Phi, R)$ are constant and therefore already satisfied in the stabilizer of P_0. Therefore, ψ factors through $K_2(\Phi, R[t])/K_2(\Phi, R)$. □

Remark 5.8.
- (i) *I want to remark that the above result has an important consequence for the realization algorithm of Park and Woodburn, cf.* [PW95]. *The above proposition shows that there are infinitely many different realizations of any matrix in $SL_3(k[x_1, \ldots, x_n])$, $n \geq 2$. These different realizations are pairwise distinct elements of the Steinberg group, so it is not possible to rewrite them using only the commutator formula. Also, there is no upper bound on the size of the realization. This is however special for the rank two case and does not appear in the case SL_4.*
- (ii) *Note also that the above result implies that the Steinberg group* $\mathrm{St}(\Phi, R[t])$ *is not a central extension of $E(\Phi, R[t])$ if R is not a field. Again, this phenomenon disappears stably. I only bring this up because I am not aware of any explicit examples of non-centrality of Steinberg groups in the literature.*

6. Consequences for group homology

In this section, we can now draw the consequences for homology of linear groups of rank two. We have already seen that $K_2(\Phi, R[t])$ is quite big if R is not a field, and in particular homotopy invariance fails for unstable K_2. Obviously, if $\mathrm{St}(\Phi, R[t])$ was the universal central extension of $E(\Phi, R[t])$ we could also conclude failure of homotopy invariance for group homology. But the very same results above imply that the Steinberg group is not even a central extension. Some more work needs to be done.

Recall from the previous section that the amalgam $\widetilde{E}(\Phi, R[t])$ sits in an extension

$$1 \to \pi = \pi_1(X) \to \widetilde{E}(\Phi, R[t]) \to E(\Phi, R[t]) \to 1.$$

The next proposition provides a relation between H_2 and a quotient of π.

Proposition 6.1. *Let Φ be an irreducible and reduced root system of rank two, and let R be an integral domain with at least five elements. Then there is a surjective homomorphism*

$$H_2(E(\Phi, R[t]), \mathbb{Z})/H_2(E(\Phi, R), \mathbb{Z}) \twoheadrightarrow \pi/[\pi, \widetilde{E}(\Phi, R[t])].$$

Proof. First note that $\widetilde{E}(\Phi, R[t])$ is perfect because by Proposition 5.6 it is a quotient of $\mathrm{St}(\Phi, R[t])$ and the latter is perfect if R has at least five elements, cf. e.g. [Ste71, Corollary 4.4]. We form the quotient $\widetilde{E}(\Phi, R[t])/[\pi, \widetilde{E}(\Phi, R[t])]$ which is still perfect. Moreover, the extension

$$1 \to \pi/[\pi, \widetilde{E}(\Phi, R[t])] \to \widetilde{E}(\Phi, R[t])/[\pi, \widetilde{E}(\Phi, R[t])] \to E(\Phi, R[t]) \to 1$$

is now central. Denote by $\widetilde{St}$ the universal central extension of the quotient $\widetilde{E}(\Phi, R[t])/[\pi, \widetilde{E}(\Phi, R[t])]$. By uniqueness, this must be the universal central extension of $E(\Phi, R[t])$ as well. We identify

$$H_2(E(\Phi, R[t]), \mathbb{Z}) = \ker\left(\widetilde{St} \to E(\Phi, R[t])\right)$$

and this group obviously surjects onto

$$\pi/[\pi, \widetilde{E}(\Phi, R[t])] = \ker\left(\widetilde{E}(\Phi, R[t])/[\pi, \widetilde{E}(\Phi, R[t])] \to E(\Phi, R[t])\right).$$

In fact, the constant elements in $H_2(E(\Phi, R), \mathbb{Z})$ come from $K_2(\Phi, R)$ which maps to 1 in $\widetilde{E}(\Phi, R[t])$ as remarked in the proof of Corollary 5.7. □

Now it suffices to show that there are infinitely many linearly independent elements in $\pi/[\pi, \widetilde{E}(\Phi, R[t])]$.

Proposition 6.2. *Let Φ be an irreducible and reduced root system of rank two, and let R be an integral domain which is not a field. If $\Phi = B_2$ assume that -1 is a not square in R. Then the abelian group $\pi/[\pi, \widetilde{E}(\Phi, R[t])]$ has infinite rank.*

Proof. An element of the extension $\widetilde{E}(\Phi, R[t])$ is an iterated product of elements from π and $E(\Phi, R[t])$. The multiplication in π is composition of loops and the action of $E(\Phi, R[t])$ is conjugation – equivalently, it is induced from the action of $E(\Phi, R[t])$ on the building.

At this point, we use the size argument of Behr. Recall from [Beh79] that there is a filtration of the fundamental domain

$$\mathcal{Q}(n) = \{x \in \mathcal{Q} \mid \alpha_0(x) \leq n\} \subseteq \mathcal{Q}$$

where α_0 is the highest root of Φ. This induces a filtration F_n of the complex $E(\Phi, R[t]) \cdot \mathcal{Q}$ by setting $F_n = E(\Phi, R[t]) \cdot \mathcal{Q}(n)$. Since $\mathcal{Q}$ is a fundamental domain, the subcomplexes F_n are invariant under the $E(\Phi, R[t])$-action.

By Proposition 3.1, part (iii), together with Proposition 4.5 and Proposition 5.4, we find that for each n there exists a triangle Δ_n not contained in F_n and a loop $L_{p,k}$ which has nontrivial winding number around Δ_n. In particular the action of $E(\Phi, R[t])$ on π has infinitely many orbits.

For a triangle Δ in $\mathcal{B} \setminus E(\Phi, R[t]) \cdot \mathcal{Q}$ such that there exists a loop $L_{p,k}$ with nontrivial winding number around it, the connected component of Δ in $\mathcal{B} \setminus E(\Phi, R[t]) \cdot \mathcal{Q}$ consists only of finitely many triangles, and the boundary of this connected component is a loop in π. We call this *the loop associated to* Δ. Now let Δ_i be an infinite set of triangles in $\mathcal{B} \setminus E(\Phi, R[t]) \cdot \mathcal{Q}$ satisfying the following:

(i) For each i, there exists a loop $L_{p,k}$ with nontrivial winding number around Δ_i.
(ii) No two Δ_i have associated loops which are conjugate under the $E(\Phi, R[t])$-action.
(iii) For each n there are only finitely many Δ_i contained in F_n.

Such an infinite set exists by the previous remarks. We obtain a well-defined map $w : \pi \to \bigoplus_i \mathbb{Z}\Delta_i$ by associating to each loop L the formal sum $w(L, \Delta_i)\Delta_i$ where $w(L, \Delta_i)$ denotes the *sum of the winding numbers of* L *around triangles in the* $E(\Phi, R[t])$*-orbit of* Δ_i. This sum is finite since there are only finitely many triangles for which this winding number is nonzero. Obviously, the image of w is a free abelian group of infinite rank. More precisely, the loops associated to the Δ_i provide a generating set for an infinite rank abelian subgroup of $\bigoplus_i \mathbb{Z}\Delta_i$.

We now investigate the map w on $[\pi, \widetilde{E}(\Phi, R[t])]$. For this, consider the action of $\widetilde{E}(\Phi, R[t])$ on π. Conjugating a loop L in π by an element of π does not change any of the winding numbers $w(L, \Delta_i)$. Conjugating with an element of $E(\Phi, R[t])$ is equivalent to the action of $E(\Phi, R[t])$ on the building, and because we are taking the sum of winding numbers in the $E(\Phi, R[t])$-orbit, the corresponding $w(L, \Delta_i)$ also do not change. In particular, w maps $[\pi, \widetilde{E}(\Phi, R[t])]$ to 0, so w factors through

$$\overline{w} : \pi/[\pi, \widetilde{E}(\Phi, R[t])] \to \bigoplus_i \mathbb{Z}\Delta_i.$$

which still surjects onto an infinite rank submodule. □

Proposition 6.1 and Proposition 6.2 now immediately imply the following theorem.

Theorem 6.3. *Let R be an integral domain which is not a field. Let Φ be a reduced and irreducible root system of rank 2. If $\Phi = B_2$ assume that -1 is not a square in R. Then the kernel of the reduction map*

$$H_2(E(\Phi, R[t]), \mathbb{Z}) \to H_2(E(\Phi, R), \mathbb{Z})$$

surjects onto an abelian group of infinite rank.

Now we give a correction of [Knu01, Theorem 4.6.8]. Using the Steinberg group $\mathrm{St}(\Phi, R[t])$ avoids the problem with the fundamental group.

Theorem 6.4. *Let R be an integral domain with many units and let Φ be an irreducible and reduced root system of rank 2. Then the canonical inclusion $R \hookrightarrow R[t]$ induces isomorphisms*

$$H_\bullet(\mathrm{St}(\Phi, R), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(\mathrm{St}(\Phi, R[t]), \mathbb{Z}).$$

Proof. As in Proposition 5.3, we denote $X = E(\Phi, R[t]) \cdot \mathcal{Q}$ and $\pi = \pi_1(X, P_0)$ and recall that X is weakly equivalent to the classifying space of π. In particular, the universal covering $\widetilde{X}$ of X is contractible.

Recall from Proposition 5.6 that there is a surjective homomorphism

$$\phi : \mathrm{St}(\Phi, R[t]) \to \widetilde{E}(\Phi, R[t]),$$

where $\widetilde{E}(\Phi, R[t])$ denotes the amalgam of the stabilizers of vertices of $\mathcal{Q}$. Recall from [Sou79] that $\widetilde{X}$ can be constructed by gluing together the set $g\mathcal{Q}$ with $g \in E(\Phi, R[t])$. It follows from this that $\widetilde{E}(\Phi, R[t])$ acts on $\widetilde{X}$, with fundamental domain $\mathcal{Q}$ and the same stabilizers. Using the homomorphism ϕ, the Steinberg group $\mathrm{St}(\Phi, R[t])$ also acts on the contractible space $\widetilde{X}$.

The homomorphism ϕ is surjective, so the fundamental domain for the action of $\mathrm{St}(\Phi, R[t])$ on X is also isomorphic to $\mathcal{Q}$. The stabilizers of vertices are the preimages of stabilizers in $E(\Phi, R[t])$ under the canonical projection $\mathrm{St}(\Phi, R[t]) \to E(\Phi, R[t])$.

Now for a simplex $\sigma \subseteq \mathcal{Q}$, we denote the stabilizer of σ in $E(\Phi, R[t])$ by P_σ, and choose a Levi subgroup L_σ. Then the inclusion $\phi^{-1}(L) \hookrightarrow \mathrm{St}(\Phi, R[t])_\sigma$ induces an isomorphism in homology. The argument for this is exactly Knudson's argument in [Knu01, Theorem 4.6.2], which was generalized to arbitrary Chevalley groups in [Wen11, Theorem 4.5]. One only needs to know that the standard projection $\mathrm{St}(\Phi, R[t]) \to E(\Phi, R[t])$ induces isomorphisms on the corresponding unipotent radicals. Then for a ring with many units, the extension of the torus in $\mathrm{St}(\Phi, R[t])$ acts non-trivially on these unipotent subgroups and the spectral sequence argument of Knudson [Knu01, Theorem 4.6.2] then yields the result. □

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