ON $\mathfrak{sl}(2, K[x])$

MATTHIAS WENDT

Nagao's theorem [Nag59], cf. also [Ser80], states that for any infinite field k, there is an amalgamated free product decomposition

$$SL_2(k[t]) \cong SL_2(k) *_{B(k)} B(k[t]),$$

where $B(R)$ denotes the subgroup of upper triangular matrices over R. In some sense, this decomposition stems from the fact that $SL_2(k[t])$ acts on the corresponding Bruhat-Tits tree. We want to show that the Lie algebra analogue of the result is false.

Proposition 0.1. *Let k be a field of characteristic $\neq 2$. The obvious morphism*

$$\mathfrak{sl}_2 *_{\mathfrak{b}_2} (\mathfrak{b}_2 \otimes k[t]) \to \mathfrak{sl}_2 \otimes k[t]$$

is not an isomorphism.

The obvious morphism

$$\mathfrak{sl}_2 *_{\mathfrak{b}_2} (\mathfrak{b}_2 \otimes k[t]) \to \mathfrak{sl}_2 \otimes k[t]$$

is induced by the universal property of the free amalgamated product in the category of Lie algebras. For the definition and existence of free amalgamated products of Lie algebras, see [BK94, Definition 4.2.1 resp. Theorem 4.4.2].

(i) The morphism is surjective. For this, it suffices to show that each of the basis elements $e_i \otimes t^n$ lies in the image. But this follows from

$$\begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix} = \left[\begin{pmatrix} t^n & 0 \\ 0 & -t^n \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} \right]$$

(ii) The kernel is a free Lie algebra. This follows from a result of Kukin, cf. [Kuk72, Theorem 1] or [BK94, Theorem 4.9.2] because the intersection of the kernel with $\mathfrak{sl}_2$ and $\mathfrak{b}_2 \otimes k[t]$ is trivial.

(iii) For an amalgamated free product of Lie algebras, there is a Mayer-Vietoris sequence [Aya85]. The sequence for $\mathfrak{sl}_2 *_{\mathfrak{b}_2} (\mathfrak{b}_2 \otimes k[t])$ breaks up into short exact sequences

$$0 \to H_n(\mathfrak{b}_2) \to H_n(\mathfrak{b}_2 \otimes k[t]) \oplus H_n(\mathfrak{sl}_2) \to H_n(\mathfrak{sl}_2 *_{\mathfrak{b}_2} (\mathfrak{b}_2 \otimes k[t])) \to 0$$

since the inclusion $\mathfrak{b}_2 \hookrightarrow \mathfrak{b}_2 \otimes k[t]$ is split by evaluation at $t = 0$. Therefore, we have

$$H_1(\mathfrak{sl}_2 *_{\mathfrak{b}_2} (\mathfrak{b}_2 \otimes k[t]))/H_1(\mathfrak{sl}_2) \cong H_1(\mathfrak{b}_2 \otimes k[t])/H_1(\mathfrak{b}_2)$$

But $H_1(\mathfrak{b}_2 \otimes R) = R$ and we find

$$H_1(\mathfrak{b}_2 \otimes k[t])/H_1(\mathfrak{b}_2) \cong tk[t].$$

(iv) Using the Hochschild-Serre spectral sequence for the extension of $\mathfrak{sl}_2 \otimes k[t]$ by the kernel, we find

$H_1(\mathfrak{sl}_2 *_{\mathfrak{b}_2} \mathfrak{b}_2 \otimes k[t]) \cong (H_0(\mathfrak{sl}_2 \otimes k[t], H_1(\mathrm{ker}))/H_2(\mathfrak{sl}_2 \otimes k[t], k)) \oplus H_1(\mathfrak{sl}_2[t], k)$.

But $H_1(\mathfrak{sl}_2 \otimes k[t], k) = 0$ since $\mathfrak{sl}_2 \otimes k[t]$ is perfect, here we use the assumption that $\operatorname{char} k \neq 2$. Therefore, we have

$$H_0(\mathfrak{sl}_2 \otimes k[t], H_1(\mathrm{ker}))/H_2(\mathfrak{sl}_2 \otimes k[t], k) \cong tk[t]$$

implying that the kernel is quite big.

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Matthias Wendt, Mathematisches Institut, Universität Freiburg, Eckerstrasse 1, 79104, Freiburg im Breisgau, Germany

E-mail address: matthias.wendt@math.uni-freiburg.de