Seminar: Special Holonomy

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Abstract

The focus of this seminar are Riemanian manifolds with special holonomy, and, in particular, Joyce's construction of compact G_2-manifolds.

Introduction

Our main goal is to understand Joyce's construction of the first compact manifold with G_2-holonomy.

We start by introducing the concept of holonomy associated to a connection, then talk about the holonomy of a Riemannian manifold as the holonomy of the Levi-Civita connection. For a generic Riemannian metric on an n-dimensional manifold, the holonomy is the full group $SO(n)$. A smaller holonomy group corresponds to extra-structure of the Riemannian metric. For example, $U(n)$-holonomy corresponds to a metric which is Kähler, $SU(n)$-holonomy to Ricci-flat Kähler metrics (also called Calabi-Yau metrics), $Sp(n)$-holonomy to hyperkähler metrics.

We then review a few "classical" things about $SU(n)$-holonomy: Yau's proof of the Calabi conjecture and examples of manifolds with $SU(2)$-holonomy, with focus on the Eguchi-Hanson space and the $K3$ surfaces. Then we dwelve into the definition of the group G_2, the properties of manifolds with G_2-holonomie, and Joyce's construction. This construction is a generalization of the Kummer construction of a $K3$ surface.

Talks:

1 Overview

Date: 20.10.2014 **Speaker: Anda Degeratu**

2 The concept of holonomy I

Date: 27.10.2014

1. Principal bundles and vector bundles
2. Example: The construction of the tangent bundle from the frame bundle
3. connection on a vector bundle, connection on a principal bundle; the case of the tangent bundle and the frame bundle.

Bibliography: [Joy00, Chapter 2], [KN63, II.1-4], [Sal89]

3 The concept of holonomy II

Date: 3.11.2014

1. connection on a manifold (i.e. on its tangent bundle), the torsion and the holonomy of a connection
2. G-structure on a manifold and its torsion
3. Riemannian manifold, the Levi-Civita connection, curvature and Ricci curvature
4. Riemannian holonomy; state Berger's Theorem (Theorem 3.4.1 in [Joy00])
5. Example of a symmetric space: either $\mathrm{SO}(2n)/U(n)$, or $\mathrm{SO}(k+l)/(\mathrm{SO}(k) \times \mathrm{SO}(l))$, or $\mathrm{F}_4/\mathrm{Spin}9$.

Bibliography: [Joy00, Chapter 2-3], [Sal89]

4 The meaning of holonomy

Date: 10.11.2014

1. $\mathrm{U}(n)$ holonomy: Kähler metrics. Talk about Kähler potentials.

2. SU(n) holonomy: Ricci-flat Kähler metrics (Calabi-Yau metrics)
3. Sp(n) holonomy: hyperkähler metrics
4. SU(2) = Sp(1); A metric has holonomy SU(2) if and only if it is hyperkähler. All hyperkähler metrics are Ricci-flat.

Bibliography: [Joy00, Chapter 3], [Sal89]

5 SU(n)-Holonomy and the Calabi conjecture

Date: 17.11.2014

1. Theorem: Let (M, g, ω, J) be a closed connected Kähler manifold. If $c_1(M) = 0$, then M has a unique Ricci-flat Kähler metric g' with Kähler class ω' satisfying $[\omega'] = [\omega]$.
2. Sketch the proof following [Bal06, Chapter 8]. In particular talk about elliptic regularity.

Additional bibliography: [Joy00, Chapter 5]

6 SU(2)-Holonomy I: the Eguchi-Hanson space

Date: 24.11.2014

1. $\mathbb{R}^4$, T^4 with the flat metric
2. the Eguchi-Hanson space: give the Kähler potential, describe the topology, show that it is a complete hyperkähler metric, and give its asymptotics at infinity.
3. generalize to higher dimension: $T^*\mathbb{C}P^n$.

Bibliography: [Joy00, Sections 7.1 and 7.2], [Cal79]

7 SU(2)-Holonomy II

Date: 1.12.2014

1. Define a $K3$ surface; Example 1: the Fermat quartic
2. $K3$ surfaces are hyperkähler 4-manifold.

3. the Kummer construction (talk about blowups and exceptional divisor of a blowup). Use Yau's proof of the Calabi conjecture to show that there exists a Ricci-flat Kähler metric.

Bibliography: [Joy00, Sections 7.3.1 and 7.3.3] and [Joy96, I, Section 1.3]

8 G_2-Holonomy I

Date: 8.12.2014

1. Define the group G_2.
2. Show that it is compact, connected, 14-dimensional Lie group.
3. G_2-structures on a 7-dimensional oriented manifold and their torsion.
4. the torus T^7 with the flat G_2-structure.

Bibliography: [Joy00, Section 10.1], [Joy96, I, Section 1.1], [Sal89]

9 G_2-Holonomy II

Date: 15.12.2014

1. Proposition 10.1.3 in [Joy00]: Characterization of G_2-manifolds
2. Lemma 1.1.3 in [Joy96, I]: If M simply-connected, compact, 7-dimensional manifold with a torsion-free G_2-structure, then the corresponding Riemannian metric has G_2-holonomy.
3. Proposition 10.1.5 in [Joy00]: If (M, g) is a Riemannian manifold with G_2-holonomy, then g is Ricci-flat.

10 Joyce's construction I

Date: 12.01.2015

1. the action of $\Gamma = \mathbb{Z}_2^3$ on T^7 preserving the flat G_2-structure
2. description of the singularities of T^7/Γ.
3. desingularization of T^7/Γ to a simply-connected manifold M using
4. a family ϕ_t of G_2-structures on M

Bibliography: [Joy96, I]

11 Joyce's construction II

Date: 19.01.2015

Theorem A: a G_2-structure with small torsion can be deformed to a torsion-free G_2-structure

Bibliography: [Joy96, I]

12 Joyce's construction III

Date: 26.01.2015

1. Theorem B: Theorem A applies to the family ϕ_t
2. Corollary: M has a G_2-metric

Bibliography: [Joy96, I]

13 Joyce's construction IV

Date: 2.02.2015

1. Theorem C: Deformations of a G_2-metric
2. Describe the topology of M: $b_1(M) = 0$, $b_2(M) = 12$, $b_3(M) = 43$.
3. Theorem: M admits a 43-dimensional family of metrics with G_2-holonomy.

Bibliography: [Joy96, I]

14 Optional: The topology of compact G_2-manifolds

Date: 9.02.2015

Section 10.2 in [Joy00].

References

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[Cal79] E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 2, 269–294.

[Joy00] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

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[Sal89] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.