# Geometrical McKay Correspondece

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A Calabi-Yau manifold is a complex Kähler manifold with trivial canonical bundle. In the attempt to construct such manifolds it is useful to take into consideration singular Calabi-Yaus. One of the simplest singularities which can arise is an orbifold singularity. An orbifold is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. Locally such an orbifold is modeled on $\mathbb{C}^n/G$, where $G$ is a finite subgroup of $SL(n,\mathbb{C})$.

From a geometrical perspective we can try to resolve the orbifold singularity. A resolution $(X,\pi)$ of $\mathbb{C}^n/G$ is a nonsingular complex manifold $X$ of dimension $n$ with a proper biholomorphic map $\pi : X \to \mathbb{C}^n/G$ that induces a biholomorphism between dense open sets. We call $X$ a *crepant resolution*[1] if the canonical bundles are isomorphic, $K_X \cong \pi^*(K_{\mathbb{C}^n/G})$. Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on $X$ one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/G$ depends dramatically on the dimension $n$ of the orbifold:

$n = 2$: A crepant resolution always exists and is unique. Its topology is entirely described in terms of the finite group $G$ (via the McKay Correspondence).

$n = 3$: A crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold [DHVW].

$n \geq 4$: In this case very little is known; crepant resolutions exist in rather special cases. Many singularities are terminal, which implies that they admit no crepant resolution.

We would like to completely understand the topology of crepant resolutions in the case $n = 3$. In this paper we are concerned with the study of the ring structure in cohomology. This is related to the generalization of the McKay Correspondence. In what follows we give a description of the problem by moving back and forth between the case $n = 2$ and $n = 3$.

**The case $n = 2$.** The quotient singularities $\mathbb{C}^2/G$, for $G$ a finite subgroup of $SL(2,\mathbb{C})$, were first classified by Klein in 1884 and are called *Kleinian singularities* (they are also known as *Du Val singularities* or *rational double points*). There are five families of finite subgroups of $SL(2,\mathbb{C})$: the cyclic subgroups $\mathcal{C}_k$, the binary dihedral groups $\mathcal{D}_k$ of order $4k$, the binary tetrahedral group $\mathcal{T}$ of order 24, the binary octahedral group $\mathcal{O}$ of order 48, and the binary icosahedral group $\mathcal{I}$ of order 120. A crepant resolution exists for each family and is unique. Moreover the finite group completely describes the topology of the resolution. This is encoded in the McKay Correspondence [McK1], which establishes a bijection between the set of irreducible representations of $G$ and the set of vertices of an extended Dynkin diagram of type $ADE$ (the Dynkin diagrams corresponding to the simple Lie algebras of the following five types: $A_{k-1}$, $D_{k+2}$, $E_6$, $E_7$ and $E_8$).

Concretely, let $\{R_0, R_1, \dots, R_r\}$ be the set of irreducible representations of $G$, where $R_0$ denotes the one-dimensional trivial representation. To $G$ and its irreducible representations we associate an $(r+1)\times(r+1)$

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[1]Etymology: For a resolution of singularities we can define a notion of *discrepancy* [R1]. A crepant resolution is a resolution without discrepancy.

adjacency matrix $A = [a_{ij}]$ with $i, j = 0, \ldots, r$. The entries $a_{ij}$ are positive integers; they are defined by the tensor product decompositions

$$R_i \otimes Q = \sum_{i=0}^{r} a_{ij} R_j,$$

where $Q$ denotes the natural two-dimensional representation of $G$ induced from the embedding $G \subset SL(2, \mathbb{C})$. McKay's insight was to realize that the matrix $A$ is related to the Cartan matrix $C$ of a Dynkin diagram of type $ADE$, via

$$A = 2I - \widetilde{C}. \tag{1}$$

(Here $\widetilde{C}$ is the Cartan matrix of the extended Dynkin diagram; the matrix $C$ is the $r \times r$-minor obtained by removing the first row and the first column from $\widetilde{C}$.)

Using McKay's correspondence it is easy now to describe the crepant resolution $\pi : X \to \mathbb{C}^2/G$. The exceptional divisor $\pi^{-1}(0)$ is the dual of the Dynkin diagram: the vertices of the Dynkin diagram correspond naturally to rational curves $C_i$ with self-intersection $-2$. Two curves intersect transversally at one point if and only if the corresponding vertices are joined by an edge in the Dynkin diagram, otherwise they do not intersect. The curves above form a basis for $H_2(X, \mathbb{Z})$. The intersection form with respect to this basis is the negative of the Cartan matrix. Concretely, let $\{R_0, R_1, \ldots, R_r\}$ be the set of irreducible representations of $G$, where $R_0$ denotes the one-dimensional trivial representation. To $G$ and its irreducible representations we associate an $(r+1) \times (r+1)$ adjacency matrix $A = [a_{ij}]$ with $i, j = 0, \ldots, r$. The entries $a_{ij}$ are positive integers; they are defined by the tensor product decompositions

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$$A = 2I - \widetilde{C}. \tag{3}$$

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The first geometrical interpretation of the McKay Correspondence was given by Gonzalez-Sprinberg and Verdier [GV]. To each of the irreducible representations $R_i$ they associated a locally free coherent sheaf $\mathcal{R}_i$. The set of all these coherent sheaves form a basis for $K(X)$, the $K$-theory of X. Moreover, the first Chern classes $c_1(\mathcal{R}_i)$ form a basis in $H^2(X, \mathbb{Q})$ and the product of two such classes in $H^*(X, \mathbb{Q})$ is given by the formula

$$\left[\int_X c_1(\mathcal{R}_i)c_1(\mathcal{R}_j)\right]_{i,j=1,\ldots,r} = -C^{-1}, \tag{4}$$

where $C^{-1}$ is the inverse of the Cartan matrix. The proof given by Gonzalez-Sprinberg and Verdier uses a case by case analysis and techniques from algebraic geometry. Kronheimer and Nakajima gave a proof of the formula using techniques from gauge theory [KroN].

To summarize, in the case of surface singularities, $\mathbb{C}^2/G$, the representation theory of the finite group $G$ completely determines the topology the crepant resolution. The Dynkin diagram and the Cartan matrix (and hence the simple Lie algebra $\mathfrak{g}$ associated to it) encode everything we want to know about the topology of the crepant resolution.

**The case $n = 3$.** The finite subgroups of $SL(3, \mathbb{C})$ were classified by Blichfeldt in 1917 [Bl]: there are ten families of such finite subgroups. In the early 1990's a case by case analysis was used to construct a crepant resolution of $\mathbb{C}^3/G$ with the given stringy Euler and Betti numbers (see [Ro] and the references therein). As a consequence of these constructions, we know that all the crepant resolutions of $\mathbb{C}^3/G$ have the Euler and Betti numbers given by the stringy Euler and Betti numbers of the orbifold (since these numbers are unchanged under flops). In 1995 Nakamura made the conjecture that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. In general, for $G$ a finite subgroup of $SL(n, \mathbb{C})$, the algebraic variety $\mathrm{Hilb}^G(\mathbb{C}^n)$ parametrizes the 0-dimensional $G$-invariant subschemes of $\mathbb{C}^n$ whose space of global sections is isomorphic to the regular representation of $G$. Nakamura made the conjecture based on his computations for the case $n = 2$ [INak]; then he proved it in dimension $n = 3$ for the case of abelian groups [Nak]. In 1999 Bridgeland, King and Reid gave a general proof of the conjecture in the case $n = 3$, relying heavily on derived category techniques [BKR]. In 2002 Craw and Ishii proved that (at least in the case $G$ abelian) all the crepant resolutions arrive as moduli spaces [CI].

In the case of surface singularities, an important feature of the McKay Correspondence is that it gives the ring structure in cohomology in terms of the finite group. For the case $n \geq 3$, nothing is known about the multiplicative structures in cohomology or $K$-theory.

Let $G \subset SL(3, \mathbb{C})$ be a finite subgroup acting with an isolated singularity on $\mathbb{C}^3/G$. Let $X$ be a crepant resolution of $\mathbb{C}^3/G$. On this resolution we associate a vector bundle $\mathcal{R}_i$ to each irreducible representation of $G$ – this is the extension of the Gonzalez-Sprinberg-Verdier sheaves. These bundles form a basis of the $K$-theory of $X$, and via the Chern character isomorphism, $\{\mathrm{ch}(\mathcal{R}_0), \mathrm{ch}(\mathcal{R}_1), \ldots, \mathrm{xch}(\mathcal{R}_r)\}$ basis of $H^*(X; \mathbb{Q})$.

The idea is to use the Atiyah-Patodi-Singer (APS) index theorem for studying multiplicative properties of the (Chern classes of the) bundles $\mathcal{R}_i$. In [De2] we show a that a generalization of Kronheimer and Nakajima's formula (4) holds in the compactly supported cohomology of $X$:

$$\left[\int_X \left(\mathrm{ch}(\mathcal{R}_i) - \mathrm{rk}(\mathcal{R}_i)\right)\left(\mathrm{ch}(R_j^*) - \mathrm{rk}(\mathcal{R}_i)\right)\right]_{i,j=1,\ldots,r} = C^{-1}. \tag{5}$$

Here $C$ is a matrix associated to the finite group $G$ and its embedding into $SL(3, \mathbb{C})$, generalizing the Cartan matrix of the case $n = 2$.

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