On the Positive Mass Conjecture in Higher Dimensions

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(joint work with Mark Stern)

In the context of general relativity, it has been proved [SY79, Wi81, PT82] that the total mass of an isolated system is never negative, provided that the sources of the gravitational field consist of matter with positive mass density moving no faster than light and that spacetime is asymptotically flat.

In 1961 Arnowitt, Desser and Misner introduced the mass of an asymptotically flat hypersurface in spacetime [ADM61]; their definition extends to higher dimension. A non-compact Riemannian manifold (M^n, g) is *asymptotically flat* if, outside a compact set, the metric asymptotically approaches the Euclidean metric on $\mathbb{R}^n$. This means that at infinity $g_{ij} = \delta_{ij} + \mathcal{O}(r^{-n+2})$, with appropriate decay for the derivatives of g. The mass of M is defined to be

$$m(M, g) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) d\Omega^i, \tag{1}$$

where S_r denotes the sphere of radius r in the coordinate system at infinity.

Positive Mass Conjecture: If (M^n, g) is an asymptotically flat manifold of dimension $n \geq 3$ and the scalar curvature is positive, then the mass is positive. Moreover, the mass vanishes if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \mathrm{eucl})$.

In the case $n = 3$ this conjecture was proved by Schoen and Yau in 1979 using minimal surfaces techniques. Their proof generalizes inductively up to dimension $n \leq 7$. The reason for which it cannot be pushed further is that for manifolds of dimension 8, minimal representatives of classes in $H_{n-1}(M)$ have singularities in codimension 7. Recently Lohkamp announced a strategy to deal with this kind of singularities.

In 1981 Witten gave another proof of the conjecture in the case $n = 3$ using a spinorial approach, [Wi81]. A rigorous mathematical interpretation was given by Parker and Taubes [PT82]. This proof generalizes to all asymptotically flat spin manifolds with positive scalar curvature [Ba86].

In this talk I reported on a possible approach towards solving the conjecture in higher dimension, based on Witten's spinorial proof. Unlike the case of 3-manifolds, a higher dimensional manifold need not be spin. The obstruction to having a spin structure on an oriented Riemannian manifold is given by the second Stiefel-Whitney class. Our idea is to cut the manifold M, replace it with an incomplete manifold $M \setminus V$ with a spin structure, and consider the corresponding Dirac operator. The challenge is to find the right way to do the analytical manipulations near V so that the positivity of the mass be obtained through a similar argument as in Witten's proof.

References

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