JProf. Abhishek Oswal
Dr. Andreas Demleitner WS 2024/25

Exercises "Algebraic Number Theory"

Sheet 1

Problem 1: (5 points) An integral domain A is called *Euclidean*, if there is a map

$$N\colon A \setminus \{0\} \to \mathbb{Z}_{>0} \qquad (\text{“Euclidean norm”})$$

such that for every $a, b \in A \setminus \{0\}$ such that $b \neq 0$, there are $q, r \in A$ such that $a = qb + r$ and either $r = 0$ or $N(r) < N(b)$.
Show that if A is Euclidean, then A is a PID (= principal ideal domain).

HINT: Let $\mathfrak{a}$ be a non-zero ideal of A. Choose $b \in \mathfrak{a} \setminus \{0\}$ with $N(b)$ minimal.

Problem 2: (4+1+2+2 points) Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ the ring of Gaussian integers.

(1) Show that $\mathbb{Z}[i]$ is a Euclidean domain with respect to the norm $N\colon \mathbb{Z}[i] \setminus \{0\} \to \mathbb{Z}_{>0}$ defined by

$$N(a + bi) := a^2 + b^2.$$

HINT: For every $z \in \mathbb{C}$, there is $w \in \mathbb{Z}[i]$ such that $|z - w| \leq \frac{1}{\sqrt{2}}$. (Why?)

(2) Show that N is multiplicative, i.e., $N(w_1 w_2) = N(w_1)N(w_2)$ for every $w_1, w_2 \in \mathbb{Z}[i] \setminus \{0\}$.

(3) Determine the units of $\mathbb{Z}[i]$.

(4) Let p be a prime number. Show that any element of $\mathbb{Z}[i]$ which has norm p is irreducible (and hence prime).

Problem 3: (3 points) Prove or disprove: is $\mathbb{Z}[\sqrt{3}]$ the ring of integers of $\mathbb{Q}(\sqrt{3})$?

Problem 4: (3 points) Prove or disprove: is $\mathbb{Z}[\sqrt{5}]$ the ring of integers of $\mathbb{Q}(\sqrt{5})$?

Bonus Problem 5: (5 points) Let $K = \mathbb{Q}(\alpha)$, where α is a root of $X^3 + X^2 - 2X + 8 \in \mathbb{Z}[X]$. Find the minimal polynomial of

$$\beta := \frac{\alpha + \alpha^2}{2}$$

over $\mathbb{Q}$. Is $\beta \in \mathcal{O}_K$?

Abgabedetails:

Wann? Bis spätestens Donnerstag, 24. Oktober 2024, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.