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Exercises “Algebraic Number Theory”

Sheet 4

Problem 1: (4 points)
Let K be a number field and $\alpha \in \mathcal{O}_K$. Show that $N_{K/\mathbb{Q}}(\alpha)$ and $\mathrm{tr}_{K/\mathbb{Q}}(\alpha)$ are integers. Conclude that α is a unit in $\mathcal{O}_K$ if and only if $N_{K/\mathbb{Q}}(\alpha) \in \{\pm 1\}$.

Problem 2: (1+1+2+1+1 points)
Let K be any field and

$$f = a_n X^n + \ldots + a_1 X + a_0 \in K[X], \qquad a_n \neq 0.$$

Let $\beta_1, \ldots, \beta_n$ be the roots of f in a splitting field of f (or an algebraically closed field containing K). The *discriminant* of f is defined by

$$\Delta(f) := a_n^{2n-2} \prod_{i<j} (\beta_i - \beta_j)^2.$$

Show the following basic properties of the discriminant:

(1) $\Delta(f)$ does not depend on the numbering of $\beta_1, \ldots, \beta_n$.

(2) $\Delta(f) \neq 0$ if and only if f has no multiple roots.

(3) $\Delta(f) \in K$.

(4) If L/K is a finite separable field extension and $\alpha \in L$, then $\Delta(m_\alpha) \neq 0$, where $m_\alpha \in K[X]$ is the minimal polynomial of α over K.

Furthermore:

(5) Compute the discriminant of a real quadratic polynomial $f = aX^2 + bX + c \in \mathbb{R}[X]$, $a \neq 0$.

Problem 3: (5 points)
Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square free. Compute the discriminant of the quadratic number field $\mathbb{Q}(\sqrt{d})$.

Problem 4: (5 points)
Let L/K be a finite separable field extension. By the primitive element theorem, there exists $\alpha \in L$ such that $L = K[\alpha]$. Show that the discriminant of L/K equals $\Delta(m_\alpha)$, the discriminant of the minimal polynomial of α over K.
Hint: You might want to use the formula for the determinant of a Vandermonde matrix.

Abgabedetails:

Wann? Bis spätestens Donnerstag, 14. November 2024, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.