JProf. Abhishek Oswal
Dr. Andreas Demleitner WS 2024/25

Exercises "Algebraic Number Theory"

Sheet 6

Problem 1: (5 points) [Continuation of Sheet 5, Problem 3]
Let p be an odd prime number and $\zeta = \zeta_p$ a primitive pth root of unity. Show that $\mathbb{Q}(\zeta)$ contains a unique quadratic number field $\mathbb{Q}(\sqrt{d})$ ($d \in \mathbb{Z} \setminus \{0, 1\}$ squarefree) and determine d in terms of p.

Problem 2: (3+2 points)

(1) Show that the ideals

$$\left(2,\ 1+\sqrt{-5}\right),\ \left(3,\ 1+\sqrt{-5}\right) \text{ and } \left(3,\ -1+\sqrt{-5}\right)$$

are prime in $\mathbb{Z}[\sqrt{-5}]$.

(2) Show that the ideal $(5) \subseteq \mathbb{Z}[\sqrt{-5}]$ not prime. Furthermore, find a prime ideal $\mathfrak{p} \subseteq \mathbb{Z}[\sqrt{-5}]$ such that $(5) = \mathfrak{p}^2$.

Problem 3: (5 points)
Let A be a Dedekind domain. Show that A is a UFD if and only if A is a PID.

Problem 4: (5 points)
Let A be a Dedekind domain and $(0) \neq \mathfrak{a}, \mathfrak{b} \subset A$ be ideals with prime factorization

$$\mathfrak{a} = \mathfrak{p}_1^{\mu_1} \cdot \ldots \cdot \mathfrak{p}_k^{\mu_k} \text{ and } \mathfrak{b} = \mathfrak{p}_1^{\nu_1} \cdot \ldots \cdot \mathfrak{p}_k^{\nu_k}.$$

Here, standard terms and conditions apply: $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are pairwise distinct prime ideals and $\mu_i, \nu_j \geq 0$. Show that

$$\mathfrak{a} + \mathfrak{b} = \prod_{i=1}^{k} \mathfrak{p}_i^{\min\{\mu_i, \nu_i\}}, \qquad \mathfrak{a} \cap \mathfrak{b} = \prod_{i=1}^{k} \mathfrak{p}_i^{\max\{\mu_i, \nu_i\}}.$$

Abgabedetails:

Wann? Bis spätestens Donnerstag, 28. November 2024, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.