Exercises "Algebraic Number Theory"

Sheet 7

Problem 1: (5 points)
Let K be a number field and $\alpha \in \mathcal{O}_K$ a primitive element for K, i.e., $K = \mathbb{Q}(\alpha)$. Show that

$$\mathfrak{C} := \{\gamma \in \mathcal{O}_K \mid \gamma\mathcal{O}_K \subseteq \mathbb{Z}[\alpha]\}$$

is a non-zero ideal of $\mathcal{O}_K$.

For the following exercise(s), you may use the following result without proof:

Theorem (Dedekind). *Let K be a number field, $\alpha \in \mathcal{O}_K$ a primitive element for K and p a prime number such that $p\mathcal{O}_K$ is coprime to $\mathfrak{C}$ (as defined in Problem 1). Denote by $f \in \mathbb{Z}[X]$ the minimal polynomial of α and let*

$$\overline{f} = \overline{q}_1^{e_1} \cdot \ldots \cdot \overline{q}_r^{e_r}$$

the decomposition of the reduction $\overline{f} \in \mathbb{F}_p[X]$ of f into irreducibles (i.e., the $\overline{q}_i$ are pairwise distinct, irreducible, and $e_i \geq 1$). Then the prime decomposition of $p\mathcal{O}_K$ is given by

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_r^{e_r}, \qquad \mathfrak{p}_i := (p, q_i(\alpha)),$$

where q_i is any preimage of $\overline{q}_i$ under the quotient map $\mathbb{Z}[X] \to \mathbb{F}_p[X]$.

Problem 2: (5 points)
Let $K = \mathbb{Q}(\sqrt{-19})$. Find the prime decomposition of $p\mathcal{O}_K$ for each $p \in \{2, 3, 5, 7\}$.

Problem 3: (4 points)
Show that the class group of $\mathbb{Q}(\sqrt{-5})$ contains an element of order 2.

Problem 4: (6 points)
Let A be a Dedekind domain. Let $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$ be two non-zero ideals. Show that there is an ideal $(0) \neq \mathfrak{b} \subseteq A$ which satisfies the following two properties simultaneously:

(i) $\mathfrak{b}$ is coprime to $\mathfrak{a}_2$ (i.e., no prime ideal occurring in the prime decomposition of $\mathfrak{a}_2$ occurs in the prime decomposition of $\mathfrak{b}$), and

(ii) $\mathfrak{a}_1\mathfrak{b}$ is a principal ideal.

Hint: Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime ideals dividing $\mathfrak{a}_2$. Then the sets

$$M_i := \mathfrak{a}_1 \cdot \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_{i-1} \cdot \mathfrak{p}_{i+1} \cdot \ldots \cdot \mathfrak{p}_r \setminus \mathfrak{p}_i\mathfrak{a}_1$$

are non-empty for all i. (Why?)

Abgabedetails:

Wann? Bis spätestens Donnerstag, 05. Dezember 2024, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.