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## Exercises "Algebraic Number Theory"

### Sheet 9

Problem 1: (4 points)
Let $K$ be a number field. Show that there is a finite extension $L/K$ such that $\mathfrak{a}\mathcal{O}_L$ is a principal fractional ideal for every fractional ideal $\mathfrak{a}$ of $K$.

Problem 2: (2+3 points)

(1) Use Dedekind's Theorem (see Sheet 7) to calculate the prime factorization of $2\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-11})$.

(2) Show that the quadratic number fields whose discriminants are

$$5, \quad 8, \quad 12, \quad 13, \quad -3, \quad -4, \quad -7, \quad -8, \quad -11$$

have class number 1, respectively.

Problem 3: (2+2 points)
Let $K = \mathbb{Q}(\zeta_5)$, where $\zeta_5$ is a primitive fifth root of unity. You may use without proof that $\mathcal{O}_K = \mathbb{Z}[\zeta_5]$.

(1) Find the invariants $[K : \mathbb{Q}]$, $d_K$, and $s$ (the number of pairs of complex conjugate embeddings of $K$).

(2) Show that the class number of $K$ is 1.

Problem 4: (2+3+2 points)

(1) Use the AM-GM-inequality to show that the sequence $(a_n)$ defined by

$$a_n := \left(1 + \frac{1}{n}\right)^n$$

is increasing. Deduce that the sequence $(b_n)$ defined by

$$b_n := \frac{n^n}{n!} \cdot \left(\frac{\pi}{4}\right)^{n/2}.$$

is increasing, too.

(2) Let $K$ be a number field of degree $n = [K : \mathbb{Q}]$. Show that $\sqrt{|d_K|} \geq b_n$.

(3) Use (1) and (2) to show that that if $K \neq \mathbb{Q}$, then $|d_K| \geq 2$.

*Remark:* This can be strengthened using Stickelberger's theorem that $d_k \equiv 0$ or $1 \pmod 4$. Thus, if $|d_K| \geq 2$, then $|d_K| \geq 3$.

**Abgabedetails:**

Wann? Bis spätestens Donnerstag, 19. Dezember 2024, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.