JProf. Abhishek Oswal
Dr. Andreas Demleitner

Exercises "Algebraic Number Theory"

Sheet 11

Problem 1: (5 points)
Compute the fundamental units of $\mathbb{Q}(\sqrt{d})$ for each $d \in \{2, 3, 5, 6, 7\}$.

Problem 2: (4 points)
It is also interesting to investigate Pell's equation $x^2 - Dy^2 = 1$ for non-squarefree D. Let p be a prime and $d \in \mathbb{Z}_{>0}$ squarefree. Show that $x^2 - p^2dy^2 = 1$ has infinitely many integer solutions.

Problem 3: (2+3 points)

(1) Show that if a number field K has a real embedding, then ± 1 are the only roots of unity contained in K.

(2) Let $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}_{<0}$ squarefree. Show that the group of roots of unity contained in K are given as follows:

$$\begin{cases} \{\pm 1,\ \pm\sqrt{-1}\}, & \text{if } d = -1 \\ \{\pm 1,\ \pm\omega,\ \pm\omega^2\}, & \text{if } d = -3 \\ \{\pm 1\}, & \text{otherwise.} \end{cases}$$

Here, $\omega := \frac{-1+\sqrt{-3}}{2}$.

Problem 4: (1+2+3 points)
Let K be a number field with exactly r real and s pairs of complex conjugate embeddings. According to Dirichlet's Unit Theorem, the unit group $\mathcal{O}_K^*$ is a finitely generated abelian group, whose free part is isomorph to $\mathbb{Z}^{r+s-1}$. The easiest non-trivial cases are thus the ones where $r + s - 1 = 1$, i.e., the following ones:

(i) $(r, s) = (2, 0)$, that is, K is a totally real quadratic number field: here, we know that finding a generator of the free part of $\mathcal{O}_K^*$ is related to solving Pell's equation.

(ii) $(r, s) = (1, 1)$, that is, K is a cubic number field with precisely one real embedding.

(iii) $(r, s) = (0, 2)$, that is, K is a totally imaginary quartic number field.

One may therefore wonder how one is able to find generators of the free part of $\mathcal{O}_K^*$ in cases (ii) and (iii). Here, we shall discuss (ii), and you may use *Artin's Inequality* without proof:

Artin's Inequality: *Let K be a cubic number field with exactly one real embedding. We view K as a subfield of $\mathbb{R}$ using that embedding. Let $u > 1$ be a unit in $\mathcal{O}_K$. Then*

$$|d_K| < 4u^3 + 24.$$

In the following, let K be a cubic number field with precisely one real embedding, and we view K as a subfield of $\mathbb{R}$ using that embedding.

(1) Show that there exists exactly one generator of the free part of $\mathcal{O}_K^*$ which is > 1. We call it the *fundamental unit* of $\mathcal{O}_K$.

(2) Let ε be the fundamental unit of $\mathcal{O}_K$. Show that: If $u > 1$ is a unit of $\mathcal{O}_K$ satisfying

$$4u^{3/m} + 24 \leq |d_K|$$

for an integer $m \geq 2$, then $u = \varepsilon^k$ for some $1 \leq k < m$. Conclude that if u satisfies the above inequality for $m = 2$, then $u = \varepsilon$.

(3) Show that $-\alpha^{-1}$ is the fundamental unit of $\mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{R}$ is a root of the polynomial $X^3 + 2X + 1 \in \mathbb{Q}[X]$. (You may use a computer to compute expressions involving α numerically.)

Abgabedetails:

Wann? Bis spätestens Donnerstag, 16. Januar 2025, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.