JProf. Abhishek Oswal WS 2024/25
Dr. Andreas Demleitner

Exercises "Algebraic Number Theory"

Sheet 12

Problem 1: (5 points)
The Battle of Hastings (October 14, 1066; Normans vs. Saxons) "The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon war hatched would break his lance and cut his coat of mail... When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle-cries 'Ut', 'Olicrosee!', Godemite!'." [Fictious historical text, taken from Neukirch.]

Question: How many troops did Harold II have at the battle of Hastings (counting Harold himself, too)?

Remarks:

- Although the above text may be appreciated for its color, here is a clearer restatement: before Harold joins his men, they are in thirteen squares, each square consisting of an equal number of men. Once Harold joins them, they all together rearrange themselves to form a single large square.
- You may of course assume that the size of Harold's army (including himself) is ≥ 2.
- According to Wikipedia, the world population in 1066 was around 250–350 million.
- You may want to use a computer.

Definition: A number field L is called a *CM-field*, if it is totally imaginary and contains a totally real subfield K such that $[L : K] = 2$. (One can show that K is uniquely determined by L.)

Problem 2: (2+3 points)

(1) Show that cyclotomic fields are CM. Explicitly give the totally real subfield K in the definition of a CM-field by finding an α such that $K = \mathbb{Q}(\alpha)$.

(2) Let L/K be an extension of number fields, and suppose that $L \neq K$. Show that $\mathcal{O}_K^*$ has finite index in $\mathcal{O}_L^*$ if and only if L is CM and K is the totally real subfield of L such that $[L : K] = 2$.

Problem 3: (5 points) Let $K \subseteq \mathbb{R}$ be a number field. Show that $\mathcal{O}_K^*$ is dense in $\mathbb{R}$ if and only if one of the following two conditions holds:

(i) $[K : \mathbb{Q}] \geq 4$, or

(ii) K is a totally real cubic number field.

Problem 4: (5 points) Let α be a root of $X^3 - X - 2 \in \mathbb{Z}[X]$, and $K = \mathbb{Q}(\alpha)$. It was shown in Sheet 5, Problem 1, that $\mathcal{O}_K = \mathbb{Z}[\alpha]$ and that $d_K = -104$. Let p be a prime number. Briefly explain why one of the following five possibilies occurs:

(i) p is totally split in K,

(ii) p is totally ramified in K,

(iii) $p\mathcal{O}_K$ is prime,

(iv) $p\mathcal{O}_K = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2$ for maximal ideals $\mathfrak{p}_1 \neq \mathfrak{p}_2$ of $\mathcal{O}_K$,

(v) $p\mathcal{O}_K = \mathfrak{p}_1 \cdot \mathfrak{p}_2$ for maximal ideals $\mathfrak{p}_1 \neq \mathfrak{p}_2$ of $\mathcal{O}_K$.

Does each of the five possibilies occur? If yes, give a corresponding prime p. If not, prove its non-existence.

Hint: The largest prime number you should consider is 31.

If you want to be a good noodle, you can also provide the prime ideal factorization of $p\mathcal{O}_K$ in each case. However, this is not necessary for solving the problem.

Abgabedetails:

Wann? Bis spätestens Donnerstag, 23. Januar 2025, 12:00.

Wo? Sie haben zwei Optionen:

- Ins Postfach von Demleitner im 3. Stock des mathematischen Instituts
- Geben Sie unserem Tutor Jannek Link das Blatt in der Vorlesung

Wie? Abgabe in Gruppen bis zu zwei Personen erlaubt, sogar erwünscht. Alle Namen und Matrikelnummern auf das Blatt schreiben. Abgabe in deutscher Sprache ist in Ordnung.