

# Lagrangian fibrations on hyperkähler manifolds

## — On a question of Beauville

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Journées Complexes Lorraines – May 4th, 2011

- ① Hyperkähler manifolds: Beauville's question
- ② Fibrations on non-projective hyperkähler manifolds
- ③ Transporting fibrations along deformations

Joint work with **Christian Lehn** and **Sönke Rollenske**.

## Definition

A compact Kähler manifold  $X$  is called **hyperkähler** if

- 1  $\pi_1(X) = \{e\}$ ,
- 2  $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \sigma$ , where  $\sigma$  is holomorphic symplectic.

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## Remarks:

- also called **irreducible holomorphic symplectic**,
- differential–geometric characterisation: Holonomy =  $Sp(n)$ ,
- $\sigma$  induces a trivialisation  $\omega_X \cong \mathcal{O}_X$ ,
- together with tori and CY-manifolds: basic building blocks of compact Kähler manifolds with  $c_1(X) = 0$ .

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## ① dimension 2: K3 – surfaces

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- 2 **Douady spaces of points:** If  $X$  is a K3 – surface, consider

$$\begin{array}{ccc} X \times X & \longleftarrow & \mathrm{Bl}_{\Delta}(X \times X) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^2(X) & \longleftarrow & X^{[2]} \end{array}$$

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- 3 **generalised Kummer manifolds:** similar construction
- 4 **O'Grady:** special moduli spaces of sheaves on K3 – surfaces (dimension 6 and 10)

Up to deformation: **All known examples !**

Study maps defined on hyperkähler manifolds:

## Theorem (Matsushita) and Definition

Let  $X$  be a hyperkähler manifold and  $f: X \rightarrow B$  a fibration. Then,

- 1  $f$  is purely equidimensional,  $\dim X = 2 \dim B$ ,
- 2 every fibre of  $f$  is Lagrangian: the restriction of  $\sigma$  to  $f^{-1}(b)_{\text{red,reg}}$  is zero,
- 3 every smooth fibre of  $f$  is an abelian variety.

We call such an  $f$  a **Lagrangian fibration**.



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## Hyperkähler SYZ-conjecture:

Every hyperkähler manifold has a deformation that admits a Lagrangian fibration.

# Examples of Lagrangian fibrations

## On K3 – surfaces:

$X$  K3 – surface,  $C \subset X$  smooth elliptic curve. Then,  $C^2 = 0$ , and  $\varphi := \varphi|_C: X \rightarrow \mathbb{P}_1$  is a Lagrangian fibration; call such  $X$  **elliptic**.

**Note:** the elliptic curve  $C$  is one of the fibres of  $\varphi$ .

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## On $X^{[2]}$ 's:

Let  $\varphi: X \rightarrow \mathbb{P}_1$  be an elliptic K3 – surface. Then,  $\varphi$  induces a map

$$\Phi: X^{[2]} \rightarrow \mathrm{Sym}^2(\mathbb{P}_1) = \mathbb{P}_2$$

which is a Lagrangian fibration.

**Note:** general fibre  $= \varphi^{-1}(p) \times \varphi^{-1}(q)$  is projective.

## Question (Beauville 2010)

Let  $X$  be a hyperkähler manifold which contains a Lagrangian submanifold  $L$  that is isomorphic to a complex torus. Is  $L$  a fibre of a (meromorphic) Lagrangian fibration on  $X$  ?

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We call such an  $L$  a **Lagrangian subtorus** of  $X$ .

## Theorem (GLR 2011)

Let  $X$  be a hyperkähler manifold,  $L \subset X$  a Lagrangian subtorus. **Assume the pair  $(X, L)$  admits a small deformation  $(X', L')$  such that  $X'$  is non-projective.**

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**Assume the pair  $(X, L)$  admits a small deformation  $(X', L')$  such that  $X'$  is non-projective.**

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## Remarks:

- if  $X$  itself is **non-projective**, then fibration is holomorphic,
- if  $X$  is **projective** and  $(X, L)$  **deforms** as required above, then the fibration has a "smooth hyperkähler model".
- deformability of  $(X, L)$  is a **topological condition**
- in **dim. 4**: every pair  $(X, L)$  deforms



# Main ingredients of the proof

- 1 reformulate Beauville's question in terms of Barlet spaces
- 2 study algebraic reductions of non-projective hyperkähler manifolds: [Campana – Oguiso – Peternell 2010]
- 3 analyse the following "short exact sequence":

$$\mathrm{Def}(L \subset X) \hookrightarrow \mathrm{Def}(X, L) \twoheadrightarrow \mathrm{Def}(X)$$

# 1) Deforming $L$ in $X$

## Question

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Consider

$$H^0(L, N_{L/X}) \cong H^0(L, \Omega_L^1) \cong H^0(L, \mathcal{O}_L^{\oplus n}) \cong \mathbb{C}^n.$$

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# Reformulation in terms of Barlet spaces

Let  $\mathfrak{B}$  be the component of the Barlet space of  $X$  that contains  $[L]$ . We obtain the following diagram:

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\varepsilon} & X \\ \downarrow \pi & & \\ \mathfrak{B} & & \end{array} \quad (1)$$



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This leads to

## Lemma

The evaluation map  $\varepsilon$  is generically finite. Furthermore,  $X$  admits an almost holomorphic Lagrangian fibration with fibre  $L$  if and only if  $\varepsilon$  is bimeromorphic.

(2)

## 2) Non-projective hyperkähler manifolds

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### Analysis of the algebraic reduction: [COP]

- ① If  $a(X) = 0$ , then  $X$  is **isotypically semisimple**:  
 $\exists$  simple Kähler manifold  $S$  such that

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- ② If  $a(X) \geq 1$ , then **either**
- a)  $a(X) = n$ , and the algebraic reduction  $f: X \rightarrow B$  is a Lagrangian fibration, **or**
  - b)  $1 \leq a(X) < n$ , and the very general fibre  $F$  of the algebraic reduction  $f: X \dashrightarrow B$  is isotypically semisimple with  $a(F) = 0$ .

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**Claim:**  $L$  is a fibre of  $f$ .

**Proof:**

- $L$  is projective, hence "curve-connected",
- $f$  contracts every curve in  $X$  ([COP]).

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Hence, we have shown:

### Proposition

Let  $X$  be a non-projective hyperkähler manifold with a Lagrangian subtorus  $L \subset X$ . Then, the algebraic reduction  $f: X \rightarrow B$  is a Lagrangian fibration with fibre  $L$ .



### 3) The universal deformation space

The deformation theory of  $X$  is well-understood:

- the universal deformation space  $\text{Def}(X)$  is smooth,
- there is a locally biholomorphic period mapping  $\text{Def}(X) \rightarrow \mathcal{D}$  into an open subset  $\mathcal{D}$  of a quadric in  $\mathbb{P}(H^2(X, \mathbb{C}))$ ,
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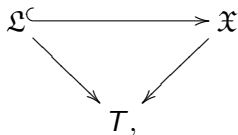
In particular, we have the following

Fact

Non-projective deformations of  $X$  are dense in  $\text{Def}(X)$ .

# Transporting fibrations along deformations

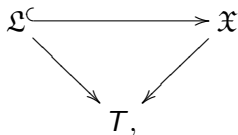
**General case:**  $X$  is projective, and there exists a smooth deformation of the pair  $(X, L) = (\mathfrak{X}_0, \mathfrak{L}_0)$  over a small disc  $T$ ,



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After shrinking  $T$  we have:

- $\mathfrak{X}_t$  hyperkähler  $\forall t \in T$ ,
- $\exists$  a dense subset  $T_{np} \subset T$  such that  $\mathfrak{X}_t$  is not projective  $\forall t \in T_{np}$ ,
- $\mathfrak{L}_t \subset \mathfrak{X}_t$  Lagrangian  $\forall t \in T$ .

Let  $\mathfrak{B}(\mathfrak{X}/T)$  be the component of the relative Barlet space of  $\mathfrak{X}$  over  $T$  containing all the  $[\mathfrak{L}_t]$ .

$$\begin{array}{ccccc}
 & & \mathfrak{U}_T & & \\
 & \swarrow \bar{\pi} & \downarrow & \searrow \bar{\epsilon} & \\
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 & \searrow & & \swarrow & \\
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We know:

- $\bar{\epsilon}_t: (\mathfrak{U}_T)_t \rightarrow \mathfrak{X}_t$  is an isomorphism  $\forall t \in T_{np}$  (non-proj. result)
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$\Rightarrow \varepsilon_0$  is bimeromorphic.

$\Rightarrow X$  has an almost hol. Lagrangian fibration with fibre  $L$ .



**Thank you for your attention !**