Lagrangian fibrations on hyperkähler manifolds — On a question of Beauville

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Overview

1. Hyperkähler manifolds: Beauville's question
2. Fibrations on non–projective hyperkähler manifolds
3. Transporting fibrations along deformations

Joint work with **Christian Lehn** and **Sönke Rollenske**.

Hyperkähler manifolds

Definition

A compact Kähler manifold X is called hyperkähler if

1. $\pi_1(X) = \{e\}$,
2. $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \sigma$, where σ is holomorphic symplectic.

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Remarks:

- also called irreducible holomorphic symplectic,
- differential–geometric characterisation: Holonomy $= Sp(n)$,
- σ induces a trivialisation $\omega_X \cong \mathscr{O}_X$,
- together with tori and CY-manifolds: basic building blocks of compact Kähler manifolds with $c_1(X) = 0$.

Examples of hyperkähler manifolds

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2. **Douady spaces of points:** If X is a K3 – surface, consider

$$\begin{tikzcd} X \times X \arrow[d] & \mathrm{Bl}_{\Delta}(X \times X) \arrow[l] \arrow[d] \\ \mathrm{Sym}^2(X) & X^{[2]} \arrow[l] \end{tikzcd}$$

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3. **generalised Kummer manifolds:** similar construction
4. **O'Grady:** special moduli spaces of sheaves on K3 – surfaces (dimension 6 and 10)

Up to deformation: **All known examples !**

Lagrangian fibrations

Study maps defined on hyperkähler manifolds:

Theorem (Matsushita) and Definition

Let X be a hyperkähler manifold and $f\colon X \to B$ a fibration. Then,

1. f is purely equidimensional, $\dim X = 2\dim B$,
2. every fibre of f is Lagrangian: the restriction of σ to $f^{-1}(b)_{\mathrm{red,reg}}$ is zero,
3. every smooth fibre of f is an abelian variety.

We call such an f a Lagrangian fibration.

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Hyperkähler SYZ–conjecture:

Every hyperkähler manifold has a deformation that admits a Lagrangian fibration.

On K3 – surfaces:

X K3 – surface, $C \subset X$ smooth elliptic curve. Then, $C^2 = 0$, and $\varphi := \varphi_{|C|} \colon X \to \mathbb{P}_1$ is a Lagrangian fibration; call such X elliptic.

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On $X^{[2]}$'s:
Let $\varphi\colon X \to \mathbb{P}_1$ be an elliptic K3 – surface. Then, φ induces a map

$$\Phi\colon X^{[2]} \to \mathrm{Sym}^2(\mathbb{P}_1) = \mathbb{P}_2$$

which is a Lagrangian fibration.

Note: general fibre $= \varphi^{-1}(p) \times \varphi^{-1}(q)$ is projective.

Beauville's Question

Question (Beauville 2010)

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We call such an L a Lagrangian subtorus of X.

Main Results

Theorem (GLR 2011)

Let X be a hyperkähler manifold, $L \subset X$ a Lagrangian subtorus. **Assume the pair** (X, L) **admits a small deformation** (X', L') **such that** X' **is non–projective.**

Theorem (GLR 2011)

Let X be a hyperkähler manifold, $L \subset X$ a Lagrangian subtorus. **Assume the pair (X, L) admits a small deformation (X', L') such that X' is non–projective.**
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Remarks:

- if X itself is **non–projective**, then fibration is holomorphic,
- if X is **projective and** (X, L) **deforms** as required above, then the fibration has a "smooth hyperkähler model".
- deformability of (X, L) is a **topological condition**
- in **dim. 4**: every pair (X, L) deforms

Main ingredients of the proof

1. reformulate Beauville's question in terms of Barlet spaces
2. study algebraic reductions of non–projective hyperkähler manifolds: [**C**ampana – **O**guiso – **P**eternell 2010]
3. analyse the following "short exact sequence":

$$\mathrm{Def}(L \subset X) \hookrightarrow \mathrm{Def}(X, L) \twoheadrightarrow \mathrm{Def}(X)$$

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Question

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Any infinitesimal deformation of L in X is effective.

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Consider

$$H^0(L, N_{L/X}) \cong H^0(L, \Omega_L^1) \cong H^0(L, \mathcal{O}_L^{\oplus n}) \cong \mathbb{C}^n.$$

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Reformulation in terms of Barlet spaces

Let $\mathfrak{B}$ be the component of the Barlet space of X that contains $[L]$. We obtain the following diagram:

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & X \\ \mathfrak{B} & \end{tikzcd} \tag{1}$$

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This leads to

Lemma

The evaluation map ε is generically finite. Furthermore, X admits an almost holomorphic Lagrangian fibration with fibre L if and only if ε is bimeromorphic.

(2)

2) Non–projective hyperkähler manifolds

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Analysis of the algebraic reduction: [COP]

1. If $a(X) = 0$, then X is isotypically semisimple:
   $\exists$ simple Kähler manifold S such that

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$$\begin{array}{ccc} Y & \xrightarrow{\text{gen. finite}} & X \\ \Big\downarrow{\scriptstyle \text{gen. finite}} & & \\ S \times \cdots \times S. & & \end{array}$$

2. If $a(X) \geq 1$, then **either**
   a) $a(X) = n$, and the algebraic reduction $f : X \to B$ is a Lagrangian fibration, **or**
   b) $1 \leq a(X) < n$, and the very general fibre F of the algebraic reduction $f : X \dashrightarrow B$ is isotypically semisimple with $a(F) = 0$.

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Claim: L is a fibre of f.

Proof:

- L is projective, hence "curve–connected",
- f contracts every curve in X ([COP]).

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- L is projective, hence "curve–connected",
- f contracts every curve in X ([COP]).

Hence, we have shown:

Proposition

Let X be a non–projective hyperkähler manifold with a Lagrangian subtorus $L \subset X$. Then, the algebraic reduction $f : X \to B$ is a Lagrangian fibration with fibre L.

3) The universal deformation space

The deformation theory of X is well–understood:

- the universal deformation space $\mathrm{Def(X)}$ is smooth,
- there is a locally biholomorphic period mapping $\mathrm{Def(X)} \to \mathscr{D}$ into an open subset $\mathscr{D}$ of a quadric in $\mathbb{P}\big(H^2(X, \mathbb{C})\big)$,
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In particular, we have the following

Fact

Non–projective deformations of X are dense in $\mathrm{Def(X)}$.

General case: X is projective, and there exists a smooth deformation of the pair $(X, L) = (\mathfrak{X}_0, \mathfrak{L}_0)$ over a small disc T,

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\begin{tikzcd}
\mathfrak{L} \arrow[rr, hook] \arrow[dr] & & \mathfrak{X} \arrow[dl] \\
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such that at least one fibre $\mathfrak{X}_{t_0}$ is not projective.

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After shrinking T we have:

- $\mathfrak{X}_t$ hyperkähler $\forall t \in T$,
- $\exists$ a dense subset $T_{np} \subset T$ such that $\mathfrak{X}_t$ is not projective $\forall t \in T_{np}$,
- $\mathfrak{L}_t \subset \mathfrak{X}_t$ Lagrangian $\forall t \in T$.

Let $\mathfrak{B}(\mathfrak{X}/T)$ be the component of the relative Barlet space of $\mathfrak{X}$ over T containing all the $[\mathfrak{L}_t]$.

$$\begin{tikzcd} & \mathfrak{U}_T \arrow[dl, "\bar{\pi}"'] \arrow[dd] \arrow[dr, "\bar{\varepsilon}"] & \\ \mathfrak{B}(\mathfrak{X}/T) \arrow[dr] & & \mathfrak{X} \arrow[dl] \\ & T & \end{tikzcd} \tag{2}$$

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We know:

- $\bar{\varepsilon}_t \colon (\mathfrak{U}_T)_t \to \mathfrak{X}_t$ is an isomorphism $\forall t \in T_{np}$ (non–proj. result)
- $\bar{\varepsilon}_0 \colon (\mathfrak{U}_T)_0 \to \mathfrak{X}_0 = X$ is generically finite (Lemma; small lie)

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$\Rightarrow$ ε_0 is bimeromorphic.

$\Rightarrow$ X has an almost hol. Lagrangian fibration with fibre L. $\square$

Thank you for your attention !