Kählerian Reduction in Steps

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I) Symplectic Reduction

One of the fundamental principles in symplectic geometry: symplectic reduction

- allows to reduce the analysis of a mechanical system with symmetries to the analysis of a smaller one,

- is a method of producing new symplectic manifolds from known ones.

First instances of this principle already appear in the works of Euler (*Theoria motus corporum solidorum seu rigidorum*, 1765) and Lie (*Theorie der Transformationsgruppen*, 1876).

One of the first formalised versions is due to Marsden and Weinstein (1974):

Proposition 1 (MW74). *Let* (X, ω) *be a symplectic manifold,* L *a Lie group, let* $L \times X \to X$ *be a Hamiltonian action with momentum map* $\mu : X \to \mathfrak{l}^*$*. Assume that*

- $\mathcal{M}_L = \mu^{-1}(0) \subset X$ *is smooth,*

- *the action of* L *on* $\mathcal{M}_L$ *is free.*

Then, $\mathcal{M}_L/L$ *is smooth and carries a symplectic form* ω_{red} *such that*

$$\pi_L^*(\omega_{red}) = \omega|_{\mathcal{M}_L}$$

holds, where $\pi_L : \mathcal{M}_L \to \mathcal{M}_L/L$ *denotes the quotient map.*

If $H \in \mathcal{C}^\infty(M)^L$*, then there exits a* ***reduced Hamiltonian*** H_{red} *on* $\mathcal{M}_L/L$ *such that*

$$\pi_L^*(H_{red}) = H|_{\mathcal{M}_L}.$$

Removing these very restrictive regularity assumptions leads to the following question:

What kind of space is $\mathcal{M}_L/L$ in general and which structures of our starting object X descend to the quotient?

In particular,

- Does $\mathcal{M}_L/L$ have a smooth structure? If yes, what are the smooth functions?

- If X is a complex manifold, does $\mathcal{M}_L/L$ carry a complex structure? If yes, what are the holomorphic functions?

- If X is Kähler, does the quotient admit a Kähler structure?

- If X is algebraic, what can be said about $\mathcal{M}_L/L$?

- Do line bundles descend to the quotient? Does quantisation commute with reduction?

II) Kählerian reduction

Theorem 2 (HHL94). *Let X be a Kähler manifold on which the complex-reductive group $H = L^{\mathbb{C}}$ acts holomorphically. Assume that the action of L on X is Hamiltonian with momentum map $\mu_L : X \to \mathfrak{l}^*$. Let $\mathcal{M}_L := \mu_L^{-1}(0)$.*
We define the set of ***semistable points*** *as*

$$X(\mathcal{M}_L) := \{x \in X | \overline{H \cdot x} \cap \mathcal{M}_L \neq \emptyset\}.$$

Then, the quotient $X(\mathcal{M}_L)/\!\!/H =: Q_H$ exists as a complex space and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_L & \hookrightarrow & X(\mathcal{M}_L) \\ \pi_L \downarrow & & \downarrow \pi_H \\ \mathcal{M}_L/L & \xrightarrow{\simeq} & Q_H. \end{array}$$

Furthermore, the complex space Q_H carries a Kähler structure induced by the homeomorphism $\mathcal{M}_L/L \simeq Q_H$ which is smooth along a natural stratification of Q_H.

Complex spaces

The notion of complex space generalises the notion of complex manifold. In particular, complex spaces are allowed to have singularities.

In our setup, even when starting with a complex manifold, in almost every case, the quotient Q_H will be singular.

Example 3. *Consider* $L = H = \mathbb{Z}_2 = \{\pm 1\}$.

$$\mathbb{Z}_2 \times \mathbb{C}^2 \to \mathbb{C}^2,$$
$$(t,(z,w)) \mapsto (tz, tw)$$

is Hamiltonian w.r.t. the std. form on $\mathbb{C}^2$*. We have* $\mathcal{M}_L = \mathbb{C}^2$*. The quotient* $\mathbb{C}^2/\mathbb{Z}_2 = Q_H$ *is biholomorphic to*

$$\{(x,y,z) \in \mathbb{C}^3 | xz = y^2\} \subset \mathbb{C}^3.$$

This is a quadratic cone.

However, these complex analytic singularities are controllable:

Let A denote the set of singular points of Q_H. Then,

- the set $A \subset Q_H$ is analytic; in particular, $A \subset Q_H$ has measure zero in Q_H

- the smooth part $Q_H \setminus A$ is open, dense, and connected.

- by iteration, we get a canonical stratification of Q_H into complex manifolds

Moreover, by a theorem of Boutot, the quotient Q_H has only **rational singularities**. This to some extend gives us control over the analytic cohomology groups of Q_H.

Singularities of Kähler structures

Let $(S^1)^{\mathbb{C}} = \mathbb{C}^*$ act on $\mathbb{C}^2$ by

$$t \bullet (z, w) = (tz, t^{-1}w).$$

The action of S^1 is Hamiltonian. We have

$$\mathcal{M}_L = \{(z,w) | |z|^2 - |w|^2 = 0\},$$
$$\mathbb{C}^2(\mathcal{M}_L) = \mathbb{C}^2.$$

The quotient Q_H is biholomorphic to $\mathbb{C}$ (smooth !), the quotient map is given by

$$(z, w) \mapsto zw.$$

The Kähler form $\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$ has a global potential (a **strictly plurisubharmonic function**):

$$\omega = i\partial\bar{\partial}\rho,$$

where $\rho(z, w) = |z|^2 + |w|^2$. The restriction $\rho|_{\mathcal{M}_L}$ is S^1-invariant and hence induces a continuous function

$$\rho_{red} \in \mathcal{C}^0(\mathcal{M}_L/L) = \mathcal{C}^0(Q_H) = \mathcal{C}^0(\mathbb{C}).$$

This is a continuous strictly pluri-subharmonic function on Q_H and in this sense defines a Kähler structure. We compute

$$\rho_{red}(v) = |v|.$$

Regularity of ρ_{red}:

- $\rho_{red} : Q_H \to \mathbb{R}$ is **not** smooth.

- ρ_{red} **is** smooth on the open, dense, connected set $\mathcal{S} = \mathbb{C}^*$ of Q_H, where it defines a Kähler form; $Q_H \setminus \mathcal{S}$ is analytic.

Stratifications / Finding $\mathcal{S}$

Look at the orbit geometry of $X = \mathbb{C}^2$. We see

$$\left\{ \begin{array}{c} \mathbb{C}^* - \text{orbits} \\ \text{through} \\ \mathcal{M}_L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{closed} \\ \mathbb{C}^* - \text{orbits} \\ \text{in } X(\mathcal{M}_L) \end{array} \right\}.$$

Isotropy groups of closed orbits:

- $\mathbb{C}^*$ fixes $(0,0)$,
- every other point in $\mathcal{M}_L$ has trivial $\mathbb{C}^*$-isotropy.

Hence, $\mathcal{S}$ is the image (under $\pi_{\mathbb{C}^*}$) of the set of closed orbits with the „smallest“ isotropy groups among all isotropy groups of closed orbits.

In the general case, one can show that the quotient has a stratification $\{\mathcal{S}_\gamma\}$ (the **quotient stratification**) into smooth manifolds along which the Kähler structure is smooth.

It is given by the conjugacy classes of isotropy groups of closed orbits in $X(\mathcal{M}_L)$.

III) Kählerian reduction in steps

Let (X, ω) be a Kähler manifold. Let L and $\tilde{L}$ be Lie groups. Let $K = L \times \tilde{L}$ act in a Hamiltonian fashion on X with momentum map $\mu : X \to \mathfrak{k}^* = \mathfrak{l}^* \oplus \tilde{\mathfrak{l}}^*$.

Then, the action of L on X is Hamiltonian with momentum map $\mu_L = pr_{\mathfrak{l}} \circ \mu$ and we can form $\mu_L^{-1}(0)/L =: Q_H$.

The following questions arise:

- Does the reduced system on Q_H have $\tilde{L}$-symmetry ? Is there a Kählerian reduction of Q_H by $\tilde{L}$?

- If yes, what is the relation between $\mu^{-1}(0)/K$ and the Kählerian reduction of Q_H by $\tilde{L}$?

Consider the action of $\tilde{L}^{\mathbb{C}} = \mathbb{C}^*$ on $X = \mathbb{C}^2$ given by

$$t \bullet (z,w) = (tz, tw).$$

This action commutes with the action of $L^{\mathbb{C}}$ and hence $L^{\mathbb{C}} \times \tilde{L}^{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^* =: K^{\mathbb{C}}$ acts on $\mathbb{C}^2$. The action of $S^1 \times S^1$ is Hamiltonian with momentum fibre $\mathcal{M} = \{(0,0)\}$ and $X(\mathcal{M}) = \mathbb{C}^2$. We have

- $\tilde{L}^{\mathbb{C}}$ acts on Q_H via $t \bullet v = t^2 v$,
- the quotient map $\pi_{L^{\mathbb{C}}} : X \to Q_H$ is $\tilde{L}^{\mathbb{C}}$-equivariant,

We are now considering the action of $\tilde{L}^{\mathbb{C}}$ on the Kähler space Q_H which is stratified as $Q_H = \mathcal{S} \cup \{0\}$.

Note that

- the quotient of Q_H by the action of $\tilde{L}$ exists,

- the quotient stratification is $\tilde{L}^{\mathbb{C}}$-invariant,

- the action of $\tilde{L}$ on Q_H is Hamiltonian with momentum fibre $\mathcal{M}_{Q_H} = \pi_{L^{\mathbb{C}}}(\mathcal{M})$,

- the Kähler structure ρ_{red} is smooth along the stratification of Q_H that is given by intersection of the stratification w.r.t. the $\tilde{L}^{\mathbb{C}}$-action and the quotient stratification

Hence, the quotient $Q_H /\!\!/ \tilde{L}^{\mathbb{C}}$ can be given the structure of a stratified Kählerian space.

Comparing the constructions of the two quotients, we get

Proposition 4. *Kählerian reduction of stratified Kähler spaces can be done in steps:*

Let $K = L \times \tilde{L}$ *(or more generally:* $L \lhd K$*). With the notations introduced above, we have the following commutative diagram of stratified Kählerian spaces:*

$$\begin{tikzcd} X(\mathcal{M}) \arrow[r] \arrow[d] & X(\mathcal{M})/\!\!/K^{\mathbb{C}} \\ X(\mathcal{M})/\!\!/L^{\mathbb{C}} \arrow[ur] & \end{tikzcd}$$

In particular, $\left(X(\mathcal{M})/\!\!/L^{\mathbb{C}}\right)/\!\!/\tilde{L}^{\mathbb{C}}$ *exists and has the structure of a stratified Kählerian space. It is biholomorphic to* $X(\mathcal{M})/\!\!/K^{\mathbb{C}}$*. Furthermore, both the induced Kählerian structures and the stratifications coincide.*

IV) Algebraicity of quotients

Theorem 5 (G '06). *Let X be a smooth algebraic variety on which the connected complex-reductive group $L^{\mathbb{C}}$ acts regularly. Assume that X is Kähler and that the action of L on X is Hamiltonian with momentum map $\mu : X \to \mathfrak{l}^*$. Assume that $\mathcal{M}_L$ is compact.*

Then, the quotient $Q_H = X(\mathcal{M}_L)/\!\!/L^{\mathbb{C}}$ is a projective algebraic variety.

References

MW74 Marsden, J.; Weinstein, A.; *Reduction of symplectic manifolds with symmetry*. Rep. Mathematical Phys. **5** (1974), no. 1, 121–130.

HHL94 Heinzner, P.; Huckleberry, A. T.; Loose, F.; *Kählerian extensions of the symplectic reduction*. J. Reine Angew. Math. **455** (1994), 123–140.