Localisation in equivariant cohomology and the Duistermaat-Heckman Theorem

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Abstract

These are lecture notes for two talks given at a seminar of the SFB-TR 12 "Symmetries and Universality in Mesoscopic Systems" in April 2006.

We study equivariant differential forms on (symplectic) manifolds endowed with an action of a compact Lie group K. We prove localisation results under certain assumptions on the fixed point set of the action, give references for the proofs in the general cases, and indicate some applications.

1 Introduction

In physical disciplines such as geometric optics, statistical mechanics and the study of classically chaotic systems via random matrix theory or via σ-models, it is often necessary to evaluate oscillatory integrals of the form

$$\int_M e^{itH(x)}\beta(x),$$

where (M^{2l},ω) is a symplectic manifold, H is a Hamiltonian function on M and β is the Liouville form $\frac{\omega^l}{l!(2\pi)^l}$ (see for example [GS90], [Zir99], [AM04]).

In this note, we investigate certain situations in which the exact evaluation of these integrals is possible as a consequence of general results in equivariant cohomology.

As an application, the last chapter discusses the measure $H_*(d\beta)$, for which we give explicit formulas in terms of geometric data associated to M, ω and H.

2 Equivariant differential forms and their cohomology

The exposition follows [BGV92]. Let M be an n-dimensional real manifold, G a connected real Lie group with Lie algebra $\mathfrak{g}$ and $G\times M\to M$ an action of G on M.

The action of G on M induces actions on $\mathcal{C}^\infty(M)$, on $\Gamma(M,TM) =: \mathcal{X}(M)$ and on $\Gamma(M,\bigwedge^k(T^*M)) := \mathcal{A}^k(M)$ for all $k\in\mathbb{N}$. Let $\mathcal{A}(M) = \oplus_{k=0}^n\mathcal{A}^k(M)\otimes\mathbb{C}$ be the algebra of complex-valued differential forms on M.

Furthermore, every $\xi \in \mathfrak{g}$ induces a vector field on M via

$$\xi_M(f)(x) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-t\xi) \bullet x).$$

The mapping from $\mathfrak{g}$ to $\mathcal{X}(M)$ given by $\xi \mapsto \xi_M$ is a Lie algebra homomorphism.

Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of complex-valued polynomial functions on $\mathfrak{g}$. We may view the tensor product $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ as the algebra of polynomial maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. Note that $\mathbb{C}[\mathfrak{g}] \cong S(\mathfrak{g}^*) \otimes \mathbb{C}$ and therefore $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M) \cong S(\mathfrak{g}^*) \otimes \mathcal{A}(M)$ is an object similar to a Clifford-Weyl algebra.

Example 2.1. *Let ω be any k-form ($k \geq 1$) on M. Then the assignment $\xi \mapsto i_{\xi_M}\omega$ defines an element of $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, in fact, in $\mathfrak{g}^* \otimes \mathcal{A}(M)$.*

There is a natural G-action on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ given by

$$(g \bullet \alpha)(\xi) = g \bullet (\alpha(\mathrm{Ad}(g)\xi)) \quad \forall g \in G, \forall \xi \in \mathfrak{g}.$$

Let $\mathcal{A}_G(M) := (\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))^G$. This is a subalgebra of $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$. An element $\alpha \in \mathcal{A}_G(M)$ satisfies

$$\alpha(\mathrm{Ad}(g)\xi) = g \bullet (\alpha(\xi))$$

and will be called an *equivariant differential form*. The algebra $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ has a $\mathbb{Z}$-grading that is given by

$$\deg(P \otimes \beta) = 2\deg(P) + \deg(\alpha).$$

We will see the reason for the choice of this particular grading later. This $\mathbb{Z}$-grading induces a $\mathbb{Z}/2\mathbb{Z}$-grading, which makes $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ into an infinite-dimensional super-space. This in turn implies that $End(\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))$ is a Lie-superalgebra.

The element of $End(\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))$ that is given by

$$(d_{\mathfrak{g}}\alpha)(\xi) = d(\alpha(\xi)) - i_{\xi_M}(\alpha(\xi))$$

is called the *equivariant exterior differential*. It increases the degree by one: consider $\alpha \in S^k(\mathfrak{g}^*) \otimes \mathcal{A}^l(M)$. Then $\xi \mapsto i_{\xi_M}(\alpha(\xi))$ is an element of $S^{k+1} \otimes \mathcal{A}^{l-1}(M)$. This gives the reason for the choice of the grading and shows that $d_{\mathfrak{g}}$ is an on odd element of $End(\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))$.

Furthermore, the subalgebra of equivariant differential forms is invariant under $d_{\mathfrak{g}}$, i.e.

$$d_{\mathfrak{g}}(\mathcal{A}_G(M)) \subset \mathcal{A}_G(M).$$

For the proof write out the definitions and use the relation $g_*(\xi_M)_x = (\mathrm{Ad}(g)\xi)_{(g \bullet x)}$. Recalling Cartan's „magic formula", $\mathcal{L}_X = i_X \circ d + d \circ i_X$ for any $X \in \mathcal{X}(M)$, we see that for all $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, we have

$$(d_{\mathfrak{g}}^2(\alpha))(\xi) = -\mathcal{L}_{\xi_M}(\alpha(\xi)).$$

Since $\alpha(\xi) = \alpha(\mathrm{Ad}(\exp(t\xi))\xi) = \exp(t\xi) \bullet \alpha(\xi)$ holds for all $t \in \mathbb{R}$, the restriction of $d_{\mathfrak{g}}$ to $\mathcal{A}_G(M)$ satisfies $d_{\mathfrak{g}}^2 = 0$. This implies that $(\mathcal{A}_G(M), d_{\mathfrak{g}})$ is a complex, i.e. we have a sequence

$$0 \xrightarrow{d_{\mathfrak{g}}} \mathcal{A}_G(M)^{(0)} \xrightarrow{d_{\mathfrak{g}}} \mathcal{A}_G(M)^{(1)} \xrightarrow{d_{\mathfrak{g}}} \cdots$$

and $d_{\mathfrak{g}} \circ d_{\mathfrak{g}} = 0$. Here, $\mathcal{A}_G^{(k)}(M)$ denotes the set of equivariant differential forms of degree k.

Elements $\alpha \in \mathcal{A}_G(M)$ with $d_{\mathfrak{g}}(\alpha) = 0$ are called *equivariantly closed* and elements α such that there exists a $\beta \in \mathcal{A}_G(M)$ with $d_{\mathfrak{g}}(\beta) = \alpha$ are called *equivariantly exact.* We make the following

Definition 2.2. *The* equivariant cohomology $H_G^*(M)$ *of* M *is the cohomology of the complex* $(\mathcal{A}_G(M), d_{\mathfrak{g}})$.

Next, we discuss some properties of equivariant cohomology:

First, we notice that a homomorphism of Lie groups induces a pullback map on equivariant forms $\mathcal{A}_G(M) \to \mathcal{A}_H(M)$ that is defined using the restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$. This in turn induces a map $H_G^*(M) \to H_H^*(M)$.

Second, if we apply the results of the previous paragraph to the inclusion $H = \{e\} \hookrightarrow G$, we obtain a map $\mathcal{A}_G(M) \to \mathcal{A}_{\{e\}}(M)$. However, $\mathcal{A}_{\{e\}}(M)$ is just the algebra of ordinary differential forms on M. This map is explicitly given by

$$\begin{aligned} \mathcal{A}_G(M) &\to \mathcal{A}(M) \\ \alpha &\mapsto \alpha(0). \end{aligned}$$

If $\beta = \alpha(0) \in \mathcal{A}(M)$, we call α an *equivariant extension* of β.

Next, if M is a compact and oriented manifold, then we can define the integration map:

$$\int_M : \mathcal{A}_G(M) \to \mathbb{C}[\mathfrak{g}]^G$$

which is given by $\left(\int_M \alpha\right)(\xi) = \int_M \alpha(\xi)_{top}$. Here, $\alpha(\xi)_{top}$ denotes the top degree part of $\alpha(\xi)$. If α is equivariantly exact, i.e. $\alpha = d_{\mathfrak{g}}\beta$ for some $\beta \in \mathcal{A}_G(M)$ and $\dim M = n$, then $\alpha(\xi)_{[n]} = d\left(\beta(\xi)_{[n-1]}\right)$. Thus, Stokes' theorem implies that $\int_M \alpha \in \mathbb{C}[\mathfrak{g}]^G$ just depends on the equivariant cohomology class of α.

Example 2.3. *Let p a point and G any Lie group. Consider the trivial G-action on p. Then, we have*

$$H_G^*(p) \cong \mathbb{C}[\mathfrak{g}]^G.$$

The following example discusses the opposite extreme case:

Example 2.4. *Let $M = \mathbb{C}^*$ and let $G = S^1$ act on $\mathbb{C}^*$ by multiplication. Then, we have*

$$\begin{aligned} &H^0_{S^1}(\mathbb{C}^*) = \mathbb{C} \\ &H^q_{S^1}(\mathbb{C}^*) = 0 \quad \forall q \geq 1. \end{aligned}$$

In fact, this second example is a special case of the following

Proposition 2.5. *Let $G \times M \to M$ be a free and proper action. In other words, M/G exists as a manifold and $M \to M/G$ is a principal fibre bundle. Then, we have*

$$H_G^*(M) \cong H^*(M/G).$$

A proof of the proposition can be found in [GS99].

Let us now consider an intermediate situation:

Example 2.6. *Let $G = S^1$ act on $M = \mathbb{C}$ by multiplication. Then, we have*

$$H^*_{S^1}(\mathbb{C}) \cong \mathbb{C}[x],$$

where x is a formal variable. Note that as a special case of example 2.3, we have $H^*_{S^1}(\{0\}) \cong \mathbb{C}[x]$. Hence, $H^*_{S^1}(\mathbb{C}) \cong H^*_{S^1}(\{0\})$.

Let us now come to the main example for equivariant differential forms considered in this note and thus make the connection to the integrals we finally want to evaluate.

Example 2.7. *Let (M, ω) be a symplectic manifold. Recall that for a function $H \in \mathcal{C}^\infty(M)$, we define the Hamiltonian vector field generated by H by the equation $dH = i_{X_H}\omega$. Let G be a Lie group that has a Hamiltonian action $G \times M \to M$ on M, i.e. there exists an equivariant momentum map $\mu : M \to \mathfrak{g}^*$ that fulfills the defining equation $d\mu^\xi = i_{\xi_M}\omega$. Let us define an element of $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ by*

$$\omega_{\mathfrak{g}}(\xi) = \mu^\xi + \omega =< \mu(\cdot), \xi > +\omega.$$

We calculate:

$$\begin{aligned} \omega_g(\mathrm{Ad}(g)\xi)_x &= \mu(x)(\mathrm{Ad}(g)\xi) + \omega \\ &= \mu(g^{-1} \bullet x)(\xi) + \omega \\ &= \mu^\xi(g^{-1} \bullet x) + \omega \\ &= (g \bullet (\omega_{\mathfrak{g}}(\xi)))_x. \end{aligned}$$

Hence, $\omega_{\mathfrak{g}}$ is equivariant. Obviously, $\omega_{\mathfrak{g}}(0) = \omega$. Hence, $\omega_{\mathfrak{g}}$ is an equivariant extension of ω. The fact that ω is closed and the defining equation for the momentum map immediatly imply that $\omega_{\mathfrak{g}}$ is equivariantly closed:

$$d_{\mathfrak{g}}(\omega_{\mathfrak{g}})(\xi) = d(\mu^\xi + \omega) - i_{\xi_M}(\mu^\xi + \omega) = d\mu^\xi - i_{\xi_M}\omega = 0.$$

A closely related setup is the following: as above, let (M, ω) be a symplectic manifold, let $H \in \mathcal{C}^\infty(M)$ and let X_H be its Hamiltonian vector field. It generates a one-parameter subgroup ϕ_t of the symplectic diffeomorphism group of M. This gives a symplectic $\mathbb{R}$-action on M by $t \bullet x = \phi_t(x)$. In fact, this action is Hamiltonian with momentum map mu : $M \to \mathbb{R}^*$ that is defined by its 1-component ($1 \in \mathbb{R} = Lie(\mathbb{R})$):

$$\mu^1(x) = H(x).$$

Hence, the construction introduced above works and we have

$$\omega_{\mathbb{R}}(\xi) = \xi \cdot H + \omega \in \mathcal{A}(M) \quad \forall \xi \in \mathbb{R}.$$

Hence, we see that it is good idea to discuss integration of equivariantly closed forms. This will be done in the next section.

3 Localisation formula

Let M be an n-dimensional manifold that is acted upon by a compact group K. Then we can find a Riemannian metric $(\cdot,\cdot)$ on M that is K-invariant. In this section, we will see how such a metric may be used to study equivariant differential forms.

The crucial result which will later on imply the localisation formula is the following

Proposition 3.1. *Let K be a compact Lie group and $\alpha \in \mathcal{A}_K(M)$ be an equivariantly closed form on M. For $\xi \in \mathfrak{k} = Lie(K)$, let M^ξ be the set of zeroes of ξ_M. Then, for each $\xi \in \mathfrak{k}$, the differential form $\alpha(\xi)_{[n]}$ is d-exact outside M^ξ.*

Proof. Let $\xi \in \mathfrak{k}$ and define $d_\xi := d - i_\xi \in End(\mathcal{A}(M))$ (a „hidden supersymmetry“). It fulfills $d_\xi(\alpha(\xi)) = 0$, since α is equivariantly closed. We construct a differential form θ on M with the following two properties:

- $\mathcal{L}_{\xi_M}\theta = 0$,
- $d_\xi\theta$ is invertible as an element of $\mathcal{A}(M \setminus M^\xi)$.

Let $(\cdot,\cdot)$ denote a K-invariant Riemannian metric on M. Define

$$\theta(X) = (\xi_M, X) \qquad \forall X \in \mathcal{X}(M).$$

Then θ is invariant under the action of the compact torus $T_\xi := \overline{<\exp(\mathbb{R}\xi)>} \subset K$ thus fulfills the first condition. For the second condition notice that $d_\xi(\theta) = (\xi_M, \xi_M) + d\theta$, which is invertible (by a geometric series) in $\mathcal{A}(M \setminus M^\xi)$.

We claim that for every $\alpha \in \mathcal{A}_K(M)$ that is equivariantly closed, we have

$$\alpha(\xi) = d_\xi\left((d_\xi\theta)^{-1} \wedge \theta \wedge \alpha(\xi)\right). \tag{1}$$

Indeed, we have the following three equalities, which together with the fact that d_ξ is a derivation imply the claim.

1. $d_\xi^2\theta = 0$
2. $d_\xi(\alpha(\xi)) = 0$, since $\alpha \in \mathcal{A}_G(M)$.
3. $d_\xi(d_\xi\theta^{-1}) = 0$.

By taking the highest degree part on each side of (1), we obtain the result. □

Remark 3.2. *Inspecting the proof we gave above, we see that in fact the following slightly more general result is true:*

Proposition 3.3. *Let G be a (not necessarily compact) Lie group acting on a manifold M. Let $\xi \in \mathfrak{g}$ be an elliptic element, i.e. an element such that $\overline{<\exp(\mathbb{R}\xi)>} =: T_\xi$ is a compact torus in G. Let α be an element of $\mathcal{A}_T(M)$ which is T-equivariantly closed. Let M^ξ be the set of zeroes of the vector field ξ_M (this is equal to the fixed point set of the T_ξ-action on M). Then, for each $\eta \in \mathfrak{t} = Lie(T) \subset \mathfrak{g}$, the differential form $\alpha(\eta)_{[n]}$ is d-exact on $M \setminus M^\xi$.*

The set of elliptic elements of a real Lie group can be explicitly described by methods developed in [HS05]. It has the structure of a homogeneous fibre bundle over the non-compact symmetric space G/K, where K is a maximally compact subgroup of G.

This shows that the ambient group K or G does not play a role in our considerations and hence, the theory is actually concerned with actions of compact abelian groups, the tori T_ξ.

Another point view which is often adopted is the following: let $X \in \mathcal{X}(M)$ be a vector field on a compact manifold M. Assume that it preserves some Riemannian metric g on M, i.e. $\mathcal{L}_X g = 0$. Since the isometry group of the compact Riemannian manifold (M, g) is a compact (real) Lie group, the closure of the one-parameter subgroup in $Iso(M, g)$ that is generated by X is a compact torus. This shows that the „localisation" results obtained above are valid.

Proposition 3.1 shows that contributions to an integral $\int_M \alpha(\xi)$ of an equivariantly closed form α are concentrated on arbitrary small neighbourhoods of the set M^ξ. So, as a next step, we investigate the action of the compect tori T_ξ on neighbourhoods of M^ξ. To keep the discussion elementary, we make the following:

Assumption: the zeroes of ξ_M are isolated.

Let $p \in M^\xi$ be an isolated zero of ξ_M. The torus T_ξ fixes p and hence, we consider the isotropy representation of T_ξ on T_pM. This action gives rise to an action of the Lie algebra $Lie(T_\xi)$ of T_ξ by differentiation. Hence, $\xi \in Lie(T_\xi)$ acts on T_pM. Let $\mathcal{L}_p(\xi)$ be the endomorphism of T_pM that is induced by ξ.

Since p is an isolated fixed point, $\mathcal{L}_p(\xi)$ is invertible. Indeed, suppose that $v_p \in T_pM$ is annihilated by $\mathcal{L}_p(\xi)$. Then all points $\exp_p(sv_p)$, $s \in (-\varepsilon, +\varepsilon)$ are fixed by T_ξ. Here, $\exp_p : T_pM \to M$ denotes the Riemannian exponential map. This implies $v_p = 0$.

Since $T_\xi \subset K$ preserves the Riemannian metric $(\cdot, \cdot)$, the isotropy representation preserves the inner product $(\cdot, \cdot)_p$ and therefore, its infinitesimal action $\mathcal{L}_p(\xi)$ is contained in $\mathfrak{o}(T_pM, (\cdot, \cdot)_p)$. Recalling that p is an isolated fixed point, we see that all eigenvalues of $\mathcal{L}_p(\xi)$ are purely imaginary. Hence, $\dim M$ is even and there exists a basis $e_1, \ldots, e_n$ of T_pM and real numbers $\lambda_1, \ldots, \lambda_j$ such that for $1 \leq j \leq l := \frac{n}{2}$, we have

$$\begin{aligned} \mathcal{L}_p(\xi)(e_{2j-1}) &= \lambda_j e_{2j} \\ \mathcal{L}_p(\xi)(e_{2j}) &= -\lambda_j e_{2j-1}. \end{aligned}$$

In other words, the matrix of $\mathcal{L}_p(\xi)$ with respect to the ordered basis $\{e_1, \ldots, e_n\}$ is given by

$$\begin{pmatrix} \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & -\lambda_l \\ \lambda_l & 0 \end{pmatrix} \end{pmatrix}.$$

The λ_j are uniquely determined and $\det(\mathcal{L}_p(\xi)) = \prod_{j=1}^{l} \lambda_j^2$. We set

$$\det{}^{\frac{1}{2}}(\mathcal{L}_p(\xi)) = \prod_{j=1}^{l} \lambda_j.$$

With these notations we can state the first localisation result as follows:

Theorem 3.4. *Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ acting on a compact oriented manifold. Let $\alpha \in \mathcal{A}_K(M)$ be $d_{\mathfrak{k}}$-closed. Let $\xi \in \mathfrak{k}$ be such that ξ_M has only isolated zeroes. Then*

$$\int_M \alpha(\xi) = (-2\pi)^l \sum_{p \in M^\xi} \frac{\alpha(\xi)(p)}{\det^{\frac{1}{2}}(\mathcal{L}_p(\xi))}.$$

Here, $\alpha(\xi)(p) = \alpha(\xi)_{[0]}(p)$ is the zero degree part of $\alpha(\xi)$.

Proof. Let $\xi \in M^\xi$. The action of T_ξ can be linearised in a neighbourhood $U(p)$ of p, i.e. it is equivalent to the linear action of T_ξ on T_pM. This implies that there are coordinates $\{x_1, \ldots, x_n\}$ near p, such that ξ_M is given in this new coordinates as

$$\xi_M = \lambda_1 \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) + \ldots \lambda_l \left(x_n \frac{\partial}{\partial x_{n-1}} - x_{n-1} \frac{\partial}{\partial x_n} \right)$$

As in the proof of Proposition 3.1, we are going to use a special 1-form on M:

Let θ^p be the 1-form in $U(p)$ that is given by

$$\theta^p = \lambda_1^{-1} (x_2 dx_1 - x_1 dx_2) + \cdots + \lambda_l^{-1} (x_n dx_{n-1} - x_{n-1} dx_n)$$

Then, θ^p is invariant under the action of T_ξ on $U(p)$ and $\theta^p(\xi_M)_x = \|\xi_M\|^2$.

We now use a T_ξ-invariant partition of unity subordinate to the covering of M given by the T_ξ-invariant sets $U(p)$ and $M \setminus M^\xi$ to construct a one form θ on M with the following properties:

1. $\mathcal{L}_{\xi_M} \theta = 0$
2. $d_\xi \theta$ is invertible in $\mathcal{A}(M \setminus M^\xi)$
3. $\theta = \theta^p$ in a neighbourhood of p

As we have seen in the proof of Proposition 3.1, this implies that $\alpha(\xi)_{[n]} = d \left(\frac{\theta \wedge \alpha(\xi)}{d_\xi \theta} \right)_{[n-1]}$. Consider the neighbourhood $B_\varepsilon(p)$ of p in M that is given by

$$B_\varepsilon(p) = \{x \in U(p); \|x\|^2 < \varepsilon\}.$$

Furthermore, let $S_\varepsilon(p) = \partial(B_\varepsilon(p)) = \{x \in U(p); \|x\|^2 = \varepsilon\}$. Then, we calculate:

$$\begin{aligned} \int_M \alpha(\xi) &= \lim_{\varepsilon \to 0} \int_{M \setminus \bigcup_p B_\varepsilon(p)} \alpha(\xi) \\ &= \lim_{\varepsilon \to 0} \int_{M \setminus \bigcup_p B_\varepsilon(p)} d \left(\frac{\theta \wedge \alpha(\xi)}{d_\xi \theta} \right) \\ &= - \sum_{p \in M^\xi} \lim_{\varepsilon \to 0} \int_{S_\varepsilon(p)} \frac{\theta \wedge \alpha(\xi)}{d_\xi \theta} \end{aligned}$$

The sign exchange comes from the change of orientation of $S_\varepsilon(p)$. Let p be one of the points in M^ξ. Near p, we have $\theta = \theta^p$. If we rescale the coordinates by a factor of $\varepsilon^{\frac{1}{2}}$, the sphere $S_\varepsilon(p)$ becomes the unit sphere $S_1(p)$, while $\theta(d_\xi\theta)^{-1}$ is invariant under this change of coordinates. Hence, we get

$$\int_{S_\varepsilon(p)} \frac{\theta \wedge \alpha(\xi)}{d_\xi \theta} = \int_{S_1(p)} \frac{\theta \wedge \alpha_\varepsilon(\xi)}{d_\xi \theta}.$$

Here, $\alpha_\varepsilon(\xi)$ denotes the pull-back of $\alpha(\xi)$ under the coordinate change, which is given by rescaling. As $\varepsilon \to 0$, $\alpha_\varepsilon(\xi)$ tends to the constant $\alpha(\xi)(p)$.

It remains to compute

$$-\int_{S_1(p)} \theta(d_\xi\theta)^{-1} = \int_{S_1(p)} \theta(1-d\theta)^{-1} = \int_{S_1(p)} \theta(d\theta)^{l-1} = \int_{B_1(p)} (d\theta)^l$$

But

$$(d\theta)^l = (-2)^l l! (\prod_j \lambda_j)^{-1} dx_1 \wedge \cdots \wedge dx_n.$$

Since the volume of the $2l$-dimensional unit ball is $\frac{\pi^l}{l!}$, the proof is completed. □

If the zero set of ξ_M contains non-isolated points, essentially the same result is true. Due to positive-dimensional connected components of M^ξ one has to take into account curvature properties of the normal bundle $\mathcal{N}$ of M^ξ in M:

Theorem 3.5. *Let K be a compact Lie group acting on a compact manifold M. Let α be an equivariantly closed form on M. For $\xi \in \mathfrak{k}$, let M^ξ be the zero set of ξ_M. Let $\mathcal{N}$ be the normal bundle of M^ξ in M. Choose an orientation on $\mathcal{N}$ and impose the corresponding orientation on M^ξ. Then,*

$$\int_M \alpha(\xi) = \int_{M^\xi} (-2\pi)^{rk(\mathcal{N})/2} \frac{\alpha(\xi)}{\det^{\frac{1}{2}}(\mathcal{L}^{\mathcal{N}}(\xi) + R^{\mathcal{N}})},$$

where $R^{\mathcal{N}}$ is the curvature of any metric connection in $\mathcal{N}$.

Proof. See [BGV92]. □

4 The Duistermaat-Heckman Theorem and its relation to stationary-phase approximation

As we have seen earlier, a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) gives rise to an equivariant differential form $\omega_{\mathfrak{g}}(\xi) =< \mu, \xi > + \omega$. Applying the results of the previous section gives

Theorem 4.1 (Duistermaat-Heckman, [DH82]). *Let (M^{2l}, ω) be a compact symplectic manifold. Let the compact Lie group K act on M in a Hamiltonian fashion. Let $\beta = \frac{\omega^l}{l!(2\pi)^l}$ be the Liouville form. If $\xi \in \mathfrak{k}$ is such that M^ξ consists of isolated points, then we have*

$$\int_M e^{it\mu^\xi(x)} \beta(x) = \left(\frac{i}{t}\right)^l \sum_{p \in M^\xi} \frac{e^{it\mu^\xi(p)}}{\det^{\frac{1}{2}} \mathcal{L}_p(\xi)}.$$

Proof. The form $\omega_{\mathfrak{g}} \in \mathcal{A}_G(M)$ is equivarantly closed. This implies that

$$e^{it\omega_{\mathfrak{g}}} = e^{it<\mu,\cdot>}e^{it\omega}$$

is equivariantly closed as well. The theorem now follows by applying the Localisation Theorem 3.4 to the integral

$$\int_M e^{it\mu^{\xi}(x)}\beta(x) = (2\pi it)^{-l}\int_M e^{it\omega_{\mathfrak{g}}(\xi)}.$$

□

Remark 4.2. *As we have a version of the localisation theorem in the case of non-isolated zeroes, there is also a version of the Duistermaat-Heckman Theorem in this case (see [DH83]).*

We now consider relations to the „method of stationary phase “. If we look at integrals of the form

$$\int_M e^{itf(x)}dvol \qquad (2)$$

for some function $f \in \mathcal{C}^{\infty}(M)$, intuitively, the rapid oscillation of $e^{itf(x)}$ should lead to cancellations in the integral (2) for large t. This is indeed the case and leads to asymptotic expansions in terms of the Taylor series of f at the critical points. One precise formulation of this phenomenon is

Theorem 4.3. *Let M be an n-dimensional compact manifold with volume form dvol, $f : M \to \mathbb{R}$ a Morse function, i.e. a function such that all critical points are isolated and non-degenerate. Then, for large t, we have*

$$\int_M e^{itf(x)}dvol(x) = \sum_{p\in Crit(f)} \left(\frac{2\pi}{t}\right)^l \frac{e^{\frac{i\pi}{4}\,\mathrm{sign}(\mathrm{Hess}_f(p))}}{\sqrt{|\det \mathrm{Hess}_f(p)|}}\, e^{itf(p)} + O(t^{-l-1}).$$

Proof. See for example [GS77]. □

As we have seen, we are able to rather explicitly evaluate oscillatory integrals involving components μ^{ξ} of the momentum map or (as a special case) Hamiltonian functions H on a symplectic manifold (M,ω). Furthermore, recalling that

$$d\mu^{\xi} = i_{\xi_M}\omega$$

and that ω is non-degenerate, we see that

$$Crit(\mu^{\xi}) = M^{\xi} = M^{T_{\xi}}.$$

In addition, the Hessian of the function μ^{ξ} at the point $p \in M^{\xi}$ is given by the formula

$$\mathrm{Hess}_{\mu^{\xi}}(p)(v_p, w_p) = -\omega(\mathcal{L}_p(\xi)v_p, w_p) \qquad \forall v_p, w_p \in T_pM.$$

This relates the Hessian of μ^{ξ} to the action of ξ on T_pM. Putting all this together, one can show the following

Corollary 4.4. *Let (M, ω) be a compact symplectic manifold, $H \in \mathcal{C}^\infty(M)$ a Hamiltonian such that the associated Hamiltonian vector field X_H is periodic and has discrete zeroes. Then, the error term in the stationary phase approximation of the integral $\int_M e^{itH}\beta$ vanishes:*

$$\int_M e^{itf}\beta = \sum_{p \in Crit(H)} \left(\frac{2\pi}{t}\right)^l \frac{e^{\frac{i\pi}{4}\operatorname{sign}(\mathrm{Hess}_H(p))}}{\sqrt{|\det \mathrm{Hess}_H(p)|}} e^{itH(p)}.$$

Note that the formula holds for all $t \in \mathbb{R}$, and the proof does not depend on the lemma of stationary phase, so the formula should not be regarded as an exact approximation, but as a result of localisation in equivariant cohomology.

5 The Duistermaat-Heckman measure

We will now describe an important application of the localisation formulae that we obtained in the previous sections.

Let (M, ω) be a $2l$-dimensional compact symplectic manifold. Then the Liouville form $\beta := \frac{\omega^l}{l!(2\pi)^l}$ defines a measure ν on M: for all closed subsets $A \subset M$, we set

$$\nu(A) = \int_A \beta.$$

Let T be a compact torus acting in a Hamiltonian fashion on M. Let $\mu : M \to \mathfrak{t}^*$ be a momentum map for this action. Since M is assumed to be compact, we can consider the push-forward measure $\mu_*(\nu)$ that is defined by

$$\mu_*(\nu)(A) = \nu(\mu^{-1}(A))$$

for all closed subsets A of $\mathfrak{t}^*$.

Definition 5.1. *The measure $\mu_*(\nu)$ is called the* Duistermaat-Heckman measure *on $\mathfrak{t}^*$ induced by the action of T on M.*

In their original paper [DH82], Duistermaat and Heckman prove the following: the measure $\mu_*(\nu)$ has got a density $f = \frac{d(\mu_*(\nu))}{d\lambda_{\mathfrak{t}^*}}$ with respect to the Lebesgue measure $\lambda_{\mathfrak{t}^*}$ on $\mathfrak{t}^*$. Let $R \subset P$ a connected component of the set of regular values of μ. Then the restriction of f to R is a polynomial.

Here, we are going to give an explicit global describtion of the Duistermaat-Heckman measure in the cases that our localisation formulae apply to.

So assume that the action of T on M has only isolated fixed points. For $p \in M^T$, let $\alpha_1^p, \ldots, \alpha_l^p \in \mathfrak{t}^*$ be the weights of the torus action on T_pM. Recall that for a postitive Borel measure η of finite total volume on a vector space V, the *Fourier transform* is defined to be

$$\hat{\eta}(\gamma) = \int_V e^{i\gamma(v)} d\eta(v) \qquad \forall \gamma \in V^*.$$

Hence, we see that Theorem 4.1 implies that for each *regular* $\xi \in \mathfrak{t}$, i.e. for each $\xi \in \mathfrak{t}$ such that $\alpha_j^p(\xi) \neq 0 \quad \forall p \in M^T$ and $\forall j = 1, \ldots, l$, the following formula holds:

$$\widehat{\mu_*(\nu)}(\xi) = \int_M e^{<\mu(x),\xi>} \beta(x) = i^l \sum_{p \in M^\xi} \frac{e^{it\mu^\xi(p)}}{\det^{\frac{1}{2}} \mathcal{L}_p(\xi)} = i^l \sum_{p \in M^\xi} \frac{e^{it\mu^\xi(p)}}{\prod_{j=1}^l \alpha_j^p(\xi)}.$$

Next, we present a special class of measures on a vector space V. Later on, we will show that the Duistermaat-Heckman measure on $\mathfrak{t}$ is built from measures of this simple type.

Let V be vector space and let $v = \{v_1, \ldots, v_m\}$ ($m \geq \dim V$) be a spanning set of vectors in V that generates a proper polyhedral cone $\mathcal{C}_v$ in V, i.e.

$$\mathcal{C}_v = \mathbb{R}^+ \cdot v_1 + \cdots + \mathbb{R}^+ \cdot v_m.$$

Let L_v be the map from $\left(\mathbb{R}^{\geq 0}\right)^m \subset \mathbb{R}^m$ to V defined by

$$L_v(s_1, \ldots, s_m) = \sum_{j=1}^m s_j v_j, \quad \text{where } s_j \geq 0.$$

Since the cone $\mathcal{C}_v$ is proper, this map is proper and we can define

$$H_v := (L_v)_*(\lambda_{\mathbb{R}^m}),$$

the push-forward of the Lebesgue measure on $\mathbb{R}^m$ (restricted to the positive quadrant) to V. Note that the support of this measure is $\mathcal{C}_v$.

Using Fourier analysis, one can show that the Duistermaat-Heckman measure is a sum of measures of this type.

First, we have to introduce some renormalisation: consider

$$\mathcal{N} := \{\xi \in \mathfrak{t} \,|\, \alpha_j^p(\xi) = 0 \text{ for some } p \in M^T \text{ and some } j = 1, \ldots, l\}.$$

This is a union of hyperplanes. Each connected component of the complement $\mathfrak{t} \setminus \mathcal{N}$ is called a *Weyl chamber*. Each regular element $\xi \in \mathfrak{t}$ is contained in the interior of such a chamber. Fix a ξ_0 and call the cooresponding chamber $\mathfrak{t}_+$, the *positive Weyl chamber*. Define for all $p \in M^T$ and for all $j = 1, \ldots, l$:

$$\beta_j^p := \operatorname{sign}(\alpha_j^p(\xi_0)) \cdot \alpha_j^p.$$

The set $\{\beta_j^p\}_{p \in M^T, j=1,\ldots,l}$ is called a *renormalisation* of the set of weights. It does not depend on the choice of ξ_0 inside a fixed chamber $\mathfrak{t}_+$. For $p \in M^T$, let $\epsilon(p) = \prod_{j=1}^l \operatorname{sign}(\alpha_j^p(\xi_0))$. Furthermore, let δ_η be the delta distribution supported at $\eta \in \mathfrak{t}$.

With these notations, the precise result is the following (see [GP90], [PW94]):

Theorem 5.2. *Let T be a compact torus acting in a Hamiltonian fashion on a compact symplectic manifold M. Let $\mu : M \to \mathfrak{t}^*$ be a momentum map for this action. Assume that the action of T has only isolated fixed points. Let, ν be the Liouville measure. Then, we have*

$$\mu_*(\nu) = \sum_{p \in M^T} \epsilon(p) \cdot \delta_{\mu(p)} * H_{(\beta_1^p, \ldots, \beta_l^p)}.$$

Here, $*$ denotes the convolution of measures.

Remark 5.3. *Note that* $\delta_{\mu(p)} * H_{(\beta_1^p,\dots,\beta_l^p)}$ *is just the push-forward of the Lebesgue measure on* $\left(\mathbb{R}^{\geq 0}\right)^m \subset \mathbb{R}^m$ *to* $\mathfrak{t}^*$ *via the map* $\mu(p) + L_{(\beta_1^p,\dots,\beta_l^p)}$*. That means it is supported on the cone with vertex* $\mu(p)$ *and generators* $\beta_1^p, \dots, \beta_l^p$*.*

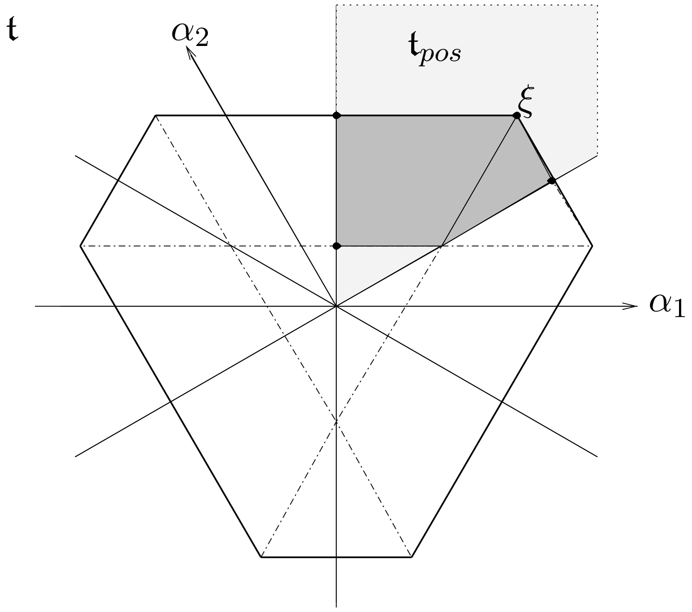
Let us now look at two examples:

Example 5.4. *Let* $M = S^2$ *be the unit sphere in* $\mathbb{R}^3$ *equipped with the symplectic form that is defined by the canonical volume element* β*. The height function* $H : (x_1, x_2, x_3) \mapsto x_3$ *generates an* S^1*-action with momentum map* $H : M \to \mathbb{R}$*. Applying Theorem 5.2, we get*

$$H_*(\beta) = H_1 * \delta_{-1} \; - \; H_1 * \delta_1 = \mathbb{I}_{[-1,1]} \cdot \lambda_{\mathbb{R}}.$$

Here, $\mathbb{I}_A$ *denotes the characteristic function of a set* $A \subset \mathbb{R}$*.*

Example 5.5. *Let* $M = SU(3) \bullet \xi$ *be an adjoint orbit of* $SU(3)$ *through an element* ξ *in a chosen positive Weyl-Chamber* $\mathfrak{t}_{pos}$ *in* $\mathfrak{su}(3)$*. The momentum map for the action of the maximal torus* T *of* $SU(3)$ *on* M *is just the orthogonal projection* $M \to \mathfrak{t}$*. In general, the image* $\mu(M) \subset \mathfrak{t}$ *is the convex hull of* $\mu(M^T)$*. However, in our particular example,* M^T *is equal to the Weyl group orbit through* ξ*. Hence, the picture*[1] *looks as follows:*



It is now easy to read off the measure from this picture. Consider for example the contribution of $\xi \in \mu(M^T)$*: the weights of the action on* $T_\xi M$ *can in our case be read of from the momentum polytope: we have* $\alpha_1^\xi = -\alpha_1$*,* $\alpha_2^\xi = -\alpha_2$ *and* $\alpha_3^\xi = -(\alpha_1 + \alpha_2)$*. Similarly, one can analyse the situation at the other vertices. The measure is then obtained by renormalising and adding up the contributions of the different vertices.*

[1]picture by Patrick Schützdeller; see [Sch06] for more details

References

[AM04] Alexander Altland and Jan Müller, *Field theory approach to quantum interference in chaotic systems*, http://arxiv.org/abs/nlin.CD/0411055, 2004.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften, vol. 298, Springer-Verlag, Berlin, 1992.

[DH82] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), no. 2, 259–268.

[DH83] ______, *Addendum to: "On the variation in the cohomology of the symplectic form of the reduced phase space"*, Invent. Math. **72** (1983), no. 1, 153–158.

[GP90] Victor Guillemin and Elisa Prato, *Heckman, Kostant, and Steinberg formulas for symplectic manifolds*, Adv. Math. **82** (1990), no. 2, 160–179.

[GS77] Victor Guillemin and Shlomo Sternberg, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977, Mathematical Surveys, No. 14.

[GS90] ______, *Symplectic techniques in physics*, second ed., Cambridge University Press, Cambridge, 1990.

[GS99] Victor W. Guillemin and Shlomo Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999.

[HS05] Peter Heinzner and Gerald Schwarz, *Cartan decomposition of the moment map*, http://arxiv.org/abs/math.CV/0502515, 2005.

[PW94] Elisa Prato and Siye Wu, *Duistermaat-Heckman measures in a non-compact setting*, Compositio Math. **94** (1994), no. 2, 113–128.

[Sch06] Patrick Schützdeller, *Convexity properties of moment maps of real forms acting on Kählerian manifolds*, Dissertation, Ruhr-Universität Bochum, 2006.

[Zir99] M.R. Zirnbauer, *Another critique of the replica trick*, http://arxiv.org/abs/cond-mat/9903338, 1999.

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