Manifolds — SS 2021

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Inhaltsverzeichnis

Introduction

In this rather long introduction, we will give a broad overview over some typical topics in manifold theory. Some details will be treated later in the course.

Basic Definitions

0.1. Definition. A *topological manifold* is a paracompact Hausdorff space that is locall homeomorphic to $\mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1}$.

0.2. Remark. "Paracompact" means that every open covering $\mathcal{U}$ of a space X has a locally finite subcovering $\mathcal{U}_0 \subset \mathcal{U}$, that is, each $x \in X$ has a neoghbourhood V such that $V \cap U = \emptyset$ for all but finitely many $U \in \mathcal{U}_0$.

An alternative definition of manifolds replaces "paracompact" by "second countable".

(1) Second countable implies paracompact.
(2) Paracompact and Hausdorff imply normal (T1 and T4 hold) and metrisable.
(3) Paracompact and Hausdorff also imply the existence of partitions of unity subordinate to any open covering.
(4) A manifold embeds into R^N for some large N if and only if it is second countable.
(5) A manifold is second countable if and only if it consists of countably many connected components.

In particular, the so-called "long line" is not considered to be a manifold.

0.3. Remark. Manifolds are always allowed to have a nonempty boundary in this course. A point $x \in M$ is a boundary point, $x \in \partial M$, if and only if there exists a neighbourhood $U \subset M$ of x and a local homeomorphism $\varphi\colon U \to V \subset \mathbb{R}^n_+$ with $\varphi(x) \in \{0\} \times \mathbb{R}^{n-1}$.

By invariance of domain, "there exists U and φ" can equivalently be replaced by "for all U and φ." Moreover, the dimension n of M is uniquely determined by the topology.

0.a. Structures on Manifolds

A class of homeomorphisms $F\colon U \to V$ with $U, V \subset \mathbb{R}^n_+$ open will be called a "pseuodgroup" if and only if it is suitable to define a notion of an atlas on a topological manifolds. This is sufficient to deduce a more formal definition of the term pseudogroup (see exercise class).

We will mainly consider the pseudogroups

- $\mathcal{T}op$ of all homeos as above,
- $\mathcal{PL}$ of all piecewise linear homeomorphisms, and
- $\mathcal{D}iff$ of all $\mathcal{C}^\infty$ diffeomorphisms.

In the following, let *CAT* denote one of the three pseudogroups above. Other examples one could conisder include $\mathcal{H}ol$, $\mathcal{A}ff$, and also orientable variants of the above, that is, $\mathcal{ST}op$, $\mathcal{SPL}$ and $\mathcal{SD}iff$.

0.4. Definition. A *CAT*-structure on a topological manifold M is an atlas Γ on M such that all coordinate changes belong to *CAT*.

A *CAT*-manifold is a topological manifold with a *CAT*-structure.

0.5. Remark. Here some ways to construct new CAT-manifolds from old.

(1) In case of the pseudogroups $CAT = \mathcal{T}op$, $\mathcal{PL}$ or $\mathcal{D}iff$ (and some others, but not $\mathcal{H}ol$), each CAT homeomorphism
$$\varphi\colon U \to V$$
induces a CAT homeomorphism
$$\partial\varphi\colon \partial U \to \partial V\ .$$
Hence, the boundary of a CAT-manifold (M, Γ) is again equipped with a CAT-structure $\partial\Gamma$.

(2) If M_1, M_2 are CAT-manifolds, there is a natural CAT-structure on $M_1 \times M_2$.

(3) There is a standard CAT-structure on the unit interval $I = [0, 1]$.

If we include orientations, we would need some conventions to pick a preferred orientation for the first two constructions above.

This would be the right place to turn CAT-manifolds into a category. However, the type of morphisms we want to allow will vary. Moreover, unless we are interested in local CAT-homeomorphisms, we would have to give an extra definition of a CAT-map (which is natural in most cases). By the remark above, our category would necessarily be a bordism category, because there is a well defined functor ∂ with the property that $\partial \circ \partial = \emptyset$.

To get an overview over our three types of structures, we introduces a fourth one called $\mathcal{PD}$ ("piecewise differentiable"). Then we have forgetful functors

$$\begin{tikzcd}
 & \mathcal{PL} \arrow[d] & \\
\mathcal{D}iff \arrow[r] & \mathcal{PD} \arrow[r] & \mathcal{T}op\ .
\end{tikzcd}$$

If we choose our definition of piecewise differentiablity cleverly, for example by demanding that a $\mathcal{PD}$-homeo $f\colon U \to V$ be smooth between closed simplices of some triangulations of U and V, then the functor $\mathcal{PL} \to \mathcal{PD}$ has a left inverse, and our diagram above becomes a chain
$$\mathcal{D}iff \longrightarrow \mathcal{PL} \longrightarrow \mathcal{T}op\ .$$
The functor $\mathcal{D}iff \to \mathcal{PL}$ is sometimes called "Whitehead triangulation."

We would like to classify CAT-manifolds up to CAT-isomorphisms, but that turns out to be too coarse. Instead, we will introduce two different notions, which require us to consider manifolds with corners as well. This is unproblematic in $CAT = \mathcal{PL}$ and $\mathcal{T}op$ because there are natural CAT-homeomorphisms
$$[0, \infty)^k \xrightarrow{\cong} [0, \infty) \times \mathbb{R}^{k-1}\ .$$
In particular, for two CAT-manifolds with boundary M_1 and M_2, one has
$$\partial(M_1 \times M_2) = \partial M_1 \times M_2 \cup_{\partial M_1 \times \partial M_2} M_1 \times \partial M_2\ .$$
In $\mathcal{D}iff$, we will simply ignore this problem for the moment, and work with smooth manifolds with corners, which are outside our category.

0.6. Definition. Let $CAT = \mathcal{PL}$ or $\mathcal{D}iff$. Two CAT-structures Γ_0, Γ_1 on a topological manifold M are called

(1) (CAT-) *concordant* if $\partial\Gamma_0 = \partial\Gamma_1$ and there exists a CAT-structure Γ on $M \times I$ such that for $i = 0$, 1,
$$\partial\Gamma|_{M \times \{i\}} = \Gamma_i \text{ for } i = 0, 1 \qquad \text{and} \qquad \partial\Gamma|_{\partial M \times I} = \partial\Gamma_0 \times \Gamma_I = \partial\Gamma_1 \times \Gamma_I\ .$$
We call $(M \times I, \Gamma)$ a *concordance.*

(2) (*CAT*-) *isotopic* if there exists a *CAT*-concordance $(M \times I, \Gamma)$ as above and a (*CAT*-) *isotopy* $h\colon M_{\Gamma_0} \times I \to M \times I$, that is, a *CAT*-isomorphism where $h(x,t) \in M \times \{t\}$ for all $x \in M$ and all $t \in I$, such that $h_0 = \mathrm{id}_M$ and $h_1\colon M_{\Gamma_0} \to M_{\Gamma_1}$ are *CAT*-isomorphisms.

Note that an isotopy is entirely determined by the maps $h_t\colon M \to M$ with $h(x,t) = (h_t(x), t)$. We write the full map h to have a meaningful notion of *CAT*-isomorphism.

0.7. Remark. Obviously, we have implications

$$\text{concordant} \quad \Longleftarrow \quad \text{isotopic} \quad \Longrightarrow \quad \text{isomorphic.}$$

- If $\dim M \geq 5$ concordance implies isotopy for $CAT = \mathcal{PL}$ or $\mathcal{D}\mathit{iff}$.
- Any two smooth structures on $\mathbb{R}^4$ are concordant, but not isotopic.
- Let Σ^7 be an exotic smooth 7-sphere. The smooth structure pulls back by an orientation reversing map to a smooth structure that we denote by $-\Sigma^7$, in fact, the connecetd sum of Σ^7 and $-\Sigma^7$ is diffeomorphic to the standard sphere. However, among the 28 smooth structures on S^7 (up to isotopy), there are only two of order 2 or less. For the remaining 26, we see that Σ^7 and $-\Sigma^7$ are isomorphic, but not isotopic.

0.b. Microbundles

To *straighten* a topological manifold (put a $\mathcal{PL}$-structure on it), or to *smoothen* a $\mathcal{T}\!op$- or $\mathcal{PL}$-manifold (put a $\mathcal{D}\mathit{iff}$-structure on it), it turns out to be helpful to consider tangent bundles, even for non-smooth manifolds.

0.8. Definition. A ($\mathcal{T}\!op$-) *microbundle* $\mathfrak{e} = (E, \iota, p)$ of *rank* k on a topological space X consists of a total space E, a zero section $\iota\colon X \to E$ and a projection $p\colon E \to X$, such that $p \circ \iota = \mathrm{id}_X$, and such that for each $x \in X$ there exist neighbourhood $U \subset X$ of x and $V \subset E$ of $\iota(x)$ together with a homeomorphism $V \cong U \times \mathbb{R}^k$, such that the following diagram commutes.

$$\begin{tikzcd}
& V \arrow[dr, "p"] \arrow[dd, "\cong"] & \\
U \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\times\{0\}"'] & & U \\
& U \times \mathbb{R}^k \arrow[ur, "\mathrm{pr}_1"'] &
\end{tikzcd}$$

Two microbundles are called *equivalent* if there exists a homeomorphism between neighbourhoods of their zero sections that are compatible with the structure maps ι and p.

0.9. Example. Here are some easy examples.

(1) The trivial microbundle $\underline{\mathbb{R}}^k = (X \times \mathbb{R}^k, \,\cdot \times \{0\}, \mathrm{pr}_1)$ of rank k is given by

$$X \xrightarrow{\times\{0\}} X \times \mathbb{R}^k \xrightarrow{\mathrm{pr}_1} X\ .$$

(2) Any locally trivial fibrebundle with fibre $\mathbb{R}^k$ and structure group $\mathrm{Homeo}(\mathbb{R}^n, 0)$ becomes a microbundle. This includes of course topological vector bundles.

(3) Each topological manifold M has a tangent microbundle $\mathfrak{t}_M = (M \times M, \Delta, \mathrm{pr}_1)$,

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_1} M\ .$$

where $\Delta\colon M \to M \times M$ denotes the diagonal map.

0.10. Remark. (1) A microbundle is something like the germ of a fibre bundle with fibre $\mathbb{R}^k$ and a distinguished section.

(2) There are analogous definitions of CAT-microbundles over CAT-manifolds, where $CAT = \mathcal{PL}$ or $\mathcal{D}\!\mathit{iff}$. From the commutative diagram in Definition 0.8, we can deduce some desirable properties of the maps ι and p.

0.11. Remark. Many well-known constructions with vector bundles work just as well with microbundles.

(1) Pullback of CAT-microbundles along (suitably defined) CAT-maps is possible, and if f, $g\colon M \to N$ are CAT-homotopic and $\mathfrak{e}$ is a microbundle on N, then $f^*\mathfrak{e}$ and $g^*\mathfrak{e}$ are equivalent.
(2) There is a notion of a Whitney sum $\mathfrak{e} \oplus \mathfrak{f}$ of microbundles.
(3) Any microbundle $\mathfrak{e}$ over a compact CW complex has a complement $\mathfrak{f}$ in the sense that $\mathfrak{e} \oplus \mathfrak{f}$ is equivalent to a trivial microbundle.

The properties above allow one to set up a "microbundle K-theory" called k_{CAT}, see [**Mi**].

0.12. Theorem (Kister-Mazur, see [**Ki**]). *Every CAT-microbundle of rank k contains a neighbourhood of the zero section that is isomorphic to a CAT-fibre bundle with fibre $\mathbb{R}^k$ and structure group consisting of the CAT-isomorphisms of $\mathbb{R}^k$ fixing 0. This fibre bundle is unique up to isomorphism.*

0.13. Remark. If $\mathfrak{e}$ is a $\mathcal{D}\!\mathit{iff}$-microbundle, we can consider the "vertical tangent bundle" $V = \iota^* \ker(dp)$. It is isomorphic to the fibre bundle of Kister's theorem.

If M is a smooth manifold, then the tangent vector bundle TM is equivalent to the microbundle $\mathfrak{t}_M$ of example 0.9 (3).

We will see in the course of the lecture that fibre-bundles, and hence also microbundles, can be classified in a way similar to vector bundles. There are classifying spaces $BCAT$ for $CAT = \mathcal{T}\!\mathit{op}$, $\mathcal{PL}$ and $\mathcal{D}\!\mathit{iff}$ such that K-theory classes of CAT-microbundles over a sufficiently nice space X are in one-to-one correspondence with homotopy classes of maps $X \to BCAT$. The spaces $BCAT$ are infinite loop spaces by a result of Boardman and Vogt [**BV**], so microbundle K-theory k_{CAT} can be upgraded to a cohomology theory. The spaces $B\mathcal{D}\!\mathit{iff}$ and the classifying space BO for real vector bundles are homotopy equivalent, so $k_{\mathcal{D}\!\mathit{iff}}$ is a "connective version" of the classical real K-theory KO that satisfies Bott periodicity. Because we are in a category of spaces with base points, we write X_+ for the disjoint union of X with an additional base point.

0.14. Theorem (Hirsch-Cairns, Kirby-Siebenmann [**KS**, IV, Thm 4.1]). *Let (CAT_0, CAT_1) be either $(\mathcal{T}\!\mathit{op}, \mathcal{PL})$ or $(\mathcal{T}\!\mathit{op}, \mathcal{D}\!\mathit{iff})$ or $(\mathcal{PL}, \mathcal{D}\!\mathit{iff})$, and let $\pi\colon BCAT_1 \to BCAT_0$ the natural map. Let M be a CAT_0-manifold of dimension $n \geq 5$ with a fixed compatible CAT_1-structure on the boundary ∂M. Then the set of CAT_1-structures on M that reduce to the given CAT_0-structure are in one-to-one correspondence with the vertical homotopy classes of lifts $\bar\tau$ in the diagram*

$$\begin{tikzcd} \partial M_+ \arrow[r, "\sigma"] \arrow[d, hook] & BCAT_1 \arrow[d, "\pi"] \\ M_+ \arrow[r, "\tau"] \arrow[ur, dashed, "\bar\tau"] & BCAT_0 \end{tikzcd}$$

up to vertical homotopy, where σ and τ classify the tangent microbundles of ∂M and M.

0.15. Remark. The statement of this theorem is surprisingly simple. The map π is compatible with the infinite loop space structures on $BCAT_0$ and $BCAT_1$. In particular, π is a map in an infinite fibre sequence

$$\cdots \longrightarrow CAT_0/CAT_1 \longrightarrow BCAT_1 \xrightarrow{\pi} BCAT_0 \longrightarrow B(CAT_0/CAT_1) \longrightarrow \cdots$$

This turns the existence and classification of compatible CAT_1-structures on a CAT_0-manifold into a simple homotopy theoretic problem, see [**ACKKNPRR**, Thm 16.16].

(1) A compatible CAT_1-structure exists if and only if the map $M/\partial M \to B(CAT_0/CAT_1)$ induced by σ and τ is homotopic to the trivial map.
(2) If one such structure exists, then the abelian group $[M/\partial M, CAT_0/CAT_1]$ acts simply transitively on the set of all compatible CAT_1-structures.

So in a way, the existence and classification problem in Theorem 0.14 is similar to the existence and classification of orientations on manifolds, or of spin structures on oriented manifolds.

Kirby and Siebenmann have shown that $\mathcal{T}op/\mathcal{PL}$ has the homotopy type of a $K(3, \mathbb{Z}/2)$. Hence, the only obstruction for the existence of a $\mathcal{PL}$-structure extending σ is a class $\mathrm{ks}(M, \sigma) \in H^4(M, \partial M; \mathbb{Z}/2)$, the so-called *Kirby-Siebenmann class,* and $H^3(M, \partial M; \mathbb{Z}/2)$ acts simply transitively on all such $\mathcal{PL}$-structures.

From the above, it is not hard to see that $\pi_k(\mathcal{PL}/\mathcal{D}\mathit{iff}) \cong \pi_k(\mathcal{PL}/O)$ is exactly the group of exotic k-spheres. These groups are nonzero for many $k \geq 7$, which makes it more difficult to find obstructions against smoothing and to classify different smooth structures on a given $\mathcal{T}op$- or $\mathcal{PL}$-manifold.

Finally note that not all similar problems have such easy answers. For example, it is not that easy to put a holomorphic structure on a smooth manifolds. And of course, the theorem fails drastically in dimension 4.

0.16. Remark. Note that the Hirsch-Cairns theorem, which concerns the special case $(\mathcal{PL}, \mathcal{D}\mathit{iff})$, actually holds in all dimensions. Because $\mathcal{PL}/\mathcal{D}\mathit{iff}$ is 6-connected, this means that every $\mathcal{PL}$-manifold M of dimension $n \leq 7$ carries a smooth structure, which is unique if $n \leq 6$.

Here are two important intermediate steps in the proof of Theorem 0.14. The first one explains why it suffices to consider CAT_1-structures on the stable tangent microbundle. Note that two CAT-microbundles on M define the same class in the reduced K-group $\tilde{k}_{CAT}(M)$ if and only if they become CAT-equivalent after taking Whitney sums with trivial microbundles.

0.17. Theorem (Stable Smoothing Theorem [**Mi**, Thm 5.13]). *Let ξ be a CAT_1-bundle with fibre $\mathbb{R}^k$ over a connected CAT_0-manifold M. Then $M \times \mathbb{R}^q$ can be given a CAT_1-structure for some $q \geq 0$ with a tangent microbundle that is stably isomorphic to ξ if and only if $[\xi] \in k_{CAT_1}$ maps to the class of the tangent microbundle of M in $\tilde{k}_{CAT_0}$.*

0.18. Theorem (Product Structure Theorem [**HM**, I, Thm 4.1], [**KS**, I, Thm 5.1]). *Let M be a CAT_0-manifold and let Σ be a CAT_1-structure on $M \times \mathbb{R}^k$ for some $k \geq 1$. Let ρ be a CAT_1-structure on an open subset $U \subset M$ such that $\Sigma|_{U \times \mathbb{R}^k}$ equals the product CAT_1-structure $\rho \times \mathbb{R}^k$.*

Assume that $m = \dim M \geq 6$ or $m = 5$ and $\partial M \subset U$. Then there exists a CAT_1-structure σ on M extending ρ and a concordance relative $U \times \mathbb{R}^k$ from Σ to $\sigma \times \mathbb{R}^k$.

It is an easy consequence that σ is unique up to concordance relative to U. Theorem 0.14 follows by first considering CAT_1-structures on $M \times \mathbb{R}^k$ for $k \geq 1$ sufficiently large, then reduce them to CAT_1-structures on M.

0.c. Fourdimensional Manifolds

Smooting theory works best in dimensions ≥ 5. There are two explanations for this. First, the so-called "Whitney-trick" for smooth manifolds is only available in dimension 5 or larger (astonishingly enough, some theorems based on the Whitney trick hold in the topological category already in dimension 4 and larger). Second, there are strikingly different results for smooth 4-manifolds. Here, we will mention a few of them. Note that by Remark 0.16, there is no difference between smooth and $\mathcal{PL}$-manifolds, so we will only talk about topological and smooth manifolds for simplicity.

Recall that on an oriented $2k$-manifold, there is a non-degenerate pairing

$$H^k(M) \times H^k(M, \partial M) \to \mathbb{Z} \qquad \text{with} \qquad (\alpha, \beta) \mapsto \alpha \cdot \beta = (\alpha \smile \beta)[M, \partial M] ,$$

where $[M, \partial M]$ denotes the fundamental class of M specified by the orientation. Here, a pairing of two free $\mathbb{Z}$-modules A and B is called *non-degenerate* if it induces isomorphisms $A^* \cong B$ and $B^* \cong A$. If M is closed, this induces a non-degenerate bilinear form on $H^k(M)$ called the *intersection form* of M. It can equivalently be described by considering intersections of k-cycles in M, hence the name.

0.19. Definition. A symmetric bilinear form on a $\mathbb{Z}$-module A is called *even* if $a \cdot a \in 2\mathbb{Z}$ for all $a \in A$. It is called *unimodular* if it induces an isomorphism $A \cong A^*$. It has *signature* $n_+ - n_-$ if n_+ (n_-) denotes the maximal dimension of a subspace of $A \otimes \mathbb{R}$ on which the form is positive (negative) definite.

The signature of the intersection form of a $4k$-dimensional manifold M is also called the signature $\operatorname{sign}(M)$ of M.

0.20. Remark. On a smooth closed oriented 4-manifold, the intersection form is even if and only if the manifold is spin. This is because for $\alpha \in H^2(M; \mathbb{Z}_2) = H^2(M, \partial M; \mathbb{Z}_2)$, we have

$$\alpha \smile \alpha = \operatorname{Sq}^2 \alpha = \alpha \smile v_2(M) = \alpha \smile w_2(TM) .$$

The first equation is the definition of the Steenrod square. The second is the definition of the Wu class $v_2(M)$. The last holds because $w(TM) = \operatorname{Sq}(v(M))$. More precisely, $w_2(TM) = v_2(M) + v_1(M) \smile v_1(M)$ and $v_1(M) = w_1(TM) = 0$ because M is orientable. Because the intersection form is nondegenerate, $w_2(TM) \neq 0$ implies that we find α such that $\alpha \smile w_2(TM) \neq 0$, hence the claimed equivalence.

0.21. Theorem (Rokhlin, Ochanine). *Let M be a smooth closed oriented spin $8\ell + 4$-manifold. Then $16 \mid \operatorname{sign}(M)$.*

0.22. Example. There are two most basic unimodular even pairing on $\mathbb{Z}$-modules called the *hyperbolic form* and the *E_8-form,* given in a suitable basis by the matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{pmatrix}$$

Note that the first one is indefinite, whereas the second one is negative definite. Indeed, there is no negative definite unimodular even bilinear form on a free $\mathbb{Z}$-module of smaller dimension. The intersection form of the (two-dimensional complex, hence really fourdimensional) $K3$-surface is given by $E_8^{\oplus 2} \oplus H^{\oplus 3}$. It has signature -16 and hence shows that the divisibility in Rokhlin's theorem is sharp.

0.23. Example. One can construct a smooth oriented spin $4k$-manifold M with nonempty boundary and with intersection E_8 by a procedure called *plumbing.* The building block is the disk bundle of an oriented vector bundle over S^{2k} of rank $2k$ and Euler class -2. The zero-section of this bundles generates H_{2k} and has self-intersection number -2 by construction.

Next, one glues eight such blocks in a way described by the matrix E_8, such that the blocks correspond to the standard basis elements of $\mathbb{Z}^8$. Over each $D^2 \subset S^2$, the bundle above is trivial and hence homeomorphic to $D^2 \times D^2$. For each "1" in the matrix, one glues two blocks along copies of $D^2 \times D^2$ as above, but swapping the factors, so that the fibre in one of the blocks is identified with the base of the other and vice versa. Then the corresponding generators of H_2 intersect once with intersection number 1 if we have chosen the orientations correctly.

After smoothing the corners, we obtain a $(2k-1)$-connected smooth $4k$-manifold P_{4k} with boundary. If $k = 1$, one can show that $\partial P_4 \cong S^3/\Gamma$, where $\Gamma \subset SU(2) \cong S^3$ is the so-called *binary icosahedral group.* Because it is perfect, ∂P_4 is a $\mathbb{Z}$-homology sphere. If $k > 1$, one can show that ∂P_{4k} is homeomorphic to S^{4k-1}. If $k > 1$ is odd, ∂P_{4k} cannot be diffeomorphic to S^{4k-1} because then one could glue D^{4k} smoothly to P_{4k} along the boundary, obtaining a closed manifold with intersection form E_8 and hence a counterexample to Ochanine's theorem.

Also let $Q_{4k} = P_{4k} \natural P_{4k}$ denote the boundary connected sum of two copies of P_{4k}, then $\partial Q_{4k} = \partial P_{4k} \# \partial P_{4k}$ has fundamental group $\Gamma * \Gamma$, which is still perfect.

0.24. Theorem (Freedman [**F**, Theorem 1.4′]). *The* 3*-manifolds* ∂P_4 *and* ∂Q_4 *bound contractible* 4*-manifolds.*

Gluing P_4 or Q_4 to such a contractible manifold along the boundary produces closed topological manifolds with intersection forms E_8 and $E_8 \oplus E_8$, respectively. These manifolds are usually denoted $|E_8|$ and $|E_8 \oplus E_8|$.

0.25. Definition. A topological manifold M is called *almost smoothable* if there exists a smooth structure on $M \setminus \{\mathrm{pt}\}$.

Recall the Kirby-Siebenmann invariant $\mathrm{ks}(M) \in H^4(M; \mathbb{Z}/2)$ from Remark 0.15. For a closed 4-manifold M, we have $H^4(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$, hence $\mathrm{ks}(M)$ takes values in $\mathbb{Z}/2$.

0.26. Theorem (Freedman [**F**, Thm 1.5]). *For every unimodular symmetric bilinear form* ω *on a free* $\mathbb{Z}$*-module* A *there exists an oriented closed almost smoothable* 4*-manifold* M *with intersection form isomorphic to* ω*.*

If ω *is even, it is unique up to oriented homeomorphism. Otherwise, there exist exactly two oriented homeomorphism classes* $[M]$ *and* $[M']$ *of almost smoothable* 4*-manifolds with* $\mathrm{ks}(M) = 0$ *and* $\mathrm{ks}(M') \neq 0$*.*

Freedman proves that $|E_8|$ and $|E_8 \oplus E_8|$ are almost smoothable, hence they constitute examples of the theorem above. However, they are not smoothable.

0.27. Theorem (Donaldon). *If* M *is a closed orientable* 4*-manifold with definite intersection form, then the intersection form can be diagonalised over* $\mathbb{Z}$*.*

Note that $K3$-surfaces have a non-diagonalisable intersection form. But are no counterexample because their intersection form is indefinite.

0.28. Remark. If the intersection form is even, then one can prove that $8 \mid \mathrm{sign}(M)$. Freedman proves that $|E_8|$ and $|E_8 \oplus E_8|$ are almost smoothable, hence they constitute examples of the theorem above. By a result of Siebenmann,

$$\mathrm{ks}(M) = \mathrm{sign}(M)/8 \mod 2 ,$$

which of course fits with Rokhlin's theorem 0.21.

Hence $\mathrm{ks}(|E_8|) \neq 0$ and $\mathrm{ks}(|E_8 \oplus E_8|) = 0$. Since neither $|E_8|$ nor $|E_8 \oplus E_8|$ can be smoothed and the Hirsch-Cairns theorem (Theorem 0.14 for $(\mathcal{PL}, \mathcal{D}\mathit{iff})$) holds in all dimensions, neither $|E_8|$ nor $|E_8 \oplus E_8|$ can be straightened. So the Kirby-Siebenmann theorem (Theorem 0.14 for $(\mathcal{T}\mathit{op}, \mathcal{PL})$) fails in dimension 4.

But it holds in dimensions ≥ 5. In particular, $|E_8 \oplus E_8| \times \mathbb{R}$ can be straightened and smoothed (but $|E_8| \times \mathbb{R}$). This shows that the Product Structure Theorem 0.18 also fails for $(\mathcal{T}\mathit{op}, \mathcal{PL})$ in dimension 4.

0.29. Example. If the intersection form is odd, the manifolds M and M' from Theorem 0.26 are homotopy equivalent. The manifold $M\times\mathbb{R}$ can be straightened, but $M'\times\mathbb{R}$ cannot. For example $\mathbb{C}P^2$ has intersection form (1) Hence, there exists a "fake" $\mathbb{C}P^2$ called "Chern manifold" Ch that is homotopy equivalent to $\mathbb{C}P^2$, but $\mathrm{Ch}\times\mathbb{R}^k$ cannot be straightened (or smoothed) for any $k\geq 0$.

To construct exotic smooth structures on $\mathbb{R}^4$, we need one more ingredient from 4-dimensional topology. Freedman also proves that there exists a smooth structure $\mathcal{A}$ on $\mathbb{R}^4$, compact subsets $K\subset V$, $K'\subset R'=(\mathbb{R}^4,\mathcal{A})$, and a diffeomorphism

$$\varphi\colon V\setminus K\to R'\setminus K' \ .$$

For a suitable $r_0>0$, we have $B_{r_0}(0)\supset K'$, where we take the standard metric on $\mathbb{R}^4$ for reference (which is probably badly behaved with respect to $\mathcal{A}'$). For all $s\geq r_0$, we consider

$$R_s=\bigl(B_s(0),\mathcal{A}|_{B_s(0)}\bigr)\subset R' \ ,$$

then R_s is homeomorphic to $\mathbb{R}^4$, so we get an uncountable family of smooth structures $\mathcal{A}_s$ on $\mathbb{R}^4$.

0.30. Theorem (Taubes [**T**, Theorem 1.1]). *The smooth manifolds $(\mathbb{R}^4,\mathcal{A}_s)$ are pairwise not diffeomorphic.*

Taubes argues by contradiction. Assume there exists a diffeomorphism $(\mathbb{R}^4,\mathcal{A}_r)\to(\mathbb{R}^4,\mathcal{A}_s)$ for $s>r>r_0$. It corresponds to a diffeomorphism

$$\psi\colon\bigl(B_r(0),\mathcal{A}|_{B_r(0)}\bigr)\xrightarrow{\cong}\bigl(B_s(0),\mathcal{A}|_{B_s(0)}\bigr) \ .$$

Then there exists $s'\in(r,s)$ such that

$$K'\subset\psi^{-1}\bigl(B_{s'}(0)\bigr) \ ,$$

and we obtain a diffeomorphism

$$\psi_0=\psi|_{B_r(0)\setminus\overline{\psi^{-1}(B_{s'}(0))}}\colon B_r(0)\setminus\overline{\psi^{-1}(B_{s'}(0))}\xrightarrow{\cong}B_s(0)\setminus\overline{B_{s'}(0)}$$

with respect to the smooth structure $\mathcal{A}$.

We can use ψ_0 and a suitable restriction φ_0 of Freedman's diffeomorphism φ to construct a smooth, non-compact manifold

$$M=V'\cup_{\varphi_0}W\cup_{\psi_0}W\cup_{\psi_0}\cdots$$

where $K\subset V'\subset V$ and $W=B_s(0)\setminus\overline{\psi^{-1}(B_{s'}(0))}$.

Taubes' argument works for more general smooth manifolds with periodic ends as above, where W and the domain $N\subset W$ on which ψ_0 is defined satisfy

(1) $\pi_1(W)$ has no nontrivial representation in $SU(2)$.
(2) $H^1(N;\mathbb{R})=H^2(N;\mathbb{R})=0$.
(3) If one glues W to itself along N, the resulting closed manifold has definite intersection form of the same sign as the intersection form of V'.

Here, these conditions are are met because $\pi_1(W)=0$, $H^1(N)=H^2(N)=0$, and because the closed manifold in (3) is homotopy equivalent to $S^1\times S^3$, which has trivial intersection form.

0.31. Theorem (Taubes [**T**, Theorem 1.4]). *Assume that M is end-periodic and satisfies the conditions above. Then the intersection pairing on M is diagonalisable in the sense that there exists abelian groups*

$$\Lambda^{-1}\subset\Lambda^0\subset\cdots\subset H^2(M)\qquad\textit{with}\qquad H^2(M)=\bigcup_i\Lambda^i$$

such that the intersection form is unimodular and diagonalisable on each Λ^i for $i\geq 1$ and $\Lambda^{-1}\otimes\mathbb{R}=H^2(V';\mathbb{R})$.

In our example, $H^2(M) = H^2(V') = E_8 \oplus E_8$, which is of finite rank, and the intersection is not diagonalisable. Hence, the existence of a diffeomorphism ψ contradicts Theorem 0.31. This finishes the proof of Theorem 0.30.

Exotic Spheres

Spheres are not only the most accessible examples of smooth manifolds with more than one smooth structure up to concordance. By the Smoothing Theorem 0.14 and Remark 0.15, the group of exotic spheres in dimension k is isomorphic to the group $\pi_k(\mathcal{PL}/O)$, where $\mathcal{PL}/O$ is the homotopy fibre of the map of classifying spaces $B\mathcal{D}\mathit{iff} \cong BO \to B\mathcal{PL}$ that is "responsible" for the existence and classification of smooth structures. So, exotic spheres can help to understand the structure of this space. We write $\mathcal{PL}$ rather than $\mathcal{T}\mathit{op}$ simply to avoid $\pi_3(\mathcal{T}\mathit{op}/O) \cong \pi_3(\mathcal{T}\mathit{op}/\mathcal{PL}) \cong \mathbb{Z}/2$, but as we are interested in $k \geq 7$, we may as well work in $\mathcal{T}\mathit{op}/O$.

Let us begin with the generalised Poincaré conjecture, as far as it is known today. By a *CAT-homotopy sphere* we denote a *CAT*-manifold M (necessarily without boundary) that is homotopy equivalent to $S^{\dim M}$.

0.32. Theorem (Poincaré-conjecture). *Assume that M is an n-dimensional CAT-homotopy sphere.*

(1) *Then M is homeomorphic to S^n.*
(2) *If $CAT = \mathcal{PL}$ and $n \neq 4$ then M is $\mathcal{PL}$-isomorphic to S^n.*
(3) *If $CAT = \mathcal{D}\mathit{iff}$ and $n \in \{1, 2, 3, 5, 6, 12, 61\}$, then M is diffeomorphic to S^n.*

The cases $n \leq 2$ follow from long known classification results for n-dimensional manifolds. Of course $n = 3$ follows in all three categories from Perelman's work on the Ricci flow. Freedman proved in 1982 the four-dimensional $\mathcal{T}\mathit{op}$-version. The $\mathcal{PL}$- and $\mathcal{D}\mathit{iff}$-version for $n = 4$ remain open. Smale proved in 1960 that any $\mathcal{PL}$- or $\mathcal{D}\mathit{iff}$-homotopy sphere is homeomorphic to S^n, so there is a "loss in category." We will prove this later in this course using Smale's h-cobordism theorem. In 1962, Smale solved the $\mathcal{PL}$-version for $n \geq 5$. In 1966, Newman finally solved the $\mathcal{T}\mathit{op}$-version for $n \geq 5$.

On the other hand, the first counterexamples to the $\mathcal{D}\mathit{iff}$-Poincaré conjecture for $n = 7$ were already constructed by Milnor in 1956. In [**KM**] and the "inofficial sequel" [**L**], the classification of smooth structures is done to a very large extend, see below. In particular, the dimensions listed in (3) come from their classification together with some later work by other authors.

0.33. Definition. Let Θ_n denote the set of smooth structures on S^n up to isotopy.

Note that [**KM**] use h-cobordism instead of concordance. By the h-cobordism theorem, this amounts to the same as isotopy.

0.34. Lemma. *The set Θ_n is a group under connected sum. The standard sphere is the neutral element, and orientation reversal is the inverse.*

One can check that this group structure is compatible with the one on $\pi_n(\mathcal{PL}/O)$.

In order to understand exotic spheres, Kervaire and Milnor choose the following path. First, they show that the tangent bundle (and hence the normal bundle) of any exotic sphere is stably trivial. Any trivialisation ("framing") defines a class in framed bordism. So if $\tilde{\Theta}_n$ denote the group of exotic spheres with stable normal framing, we have homomorphisms

$$\Theta_n \longleftarrow \tilde{\Theta}_n \longrightarrow \Omega^n_{\mathrm{fr}}(\mathrm{pt}) \xrightarrow{\cong} \pi^s_n .$$

Hence, one can try to study Θ_n using techniques from framed bordism.

The difference of two framings on Σ^n is described by a map $g\colon \Sigma^n \to O$. The image of g in π_n^s is given by the so-called J-homomorphism, which can be described as follows. Let $S^n * S^{q-1} \cong S^{n+q}$ denote the (reduced) join of S^n and S^{q-1}. A pointed map $\alpha\colon S^n \to O(q)$ defines a pointed map

$$S^n * S^{q-1} \longrightarrow SS^{q-1} \qquad \text{with} \qquad (x, y, t) \longmapsto \bigl(\underbrace{\alpha(x)}_{\in O(q)}(y), t\bigr) ,$$

where S on the right denotes reduced suspension. To see that this is well-defined, it suffices to check that when $x_0 \in S^n$ and $y_0 \in S^{q-1}$ denote the base points, then (x_0, y_0, t) is mapped to the base point of S^{q-1} for all t. But this holds because $\alpha(x_0) = \mathrm{id}$ as α was assumed to be pointed.

0.35. Definition. The map $J_n\colon \pi_n(O) \to \pi_n^s$ induced as $q \to \infty$ is called the *J-homomorphism.*

From the above, one obtains a homomorphism

$$\Theta_n \longrightarrow \operatorname{coker}(J_n) .$$

In general, it is neither injective nor surjective.

0.36. Definition. Let $bP_{n+1} \subset \Theta_n$ denote the subgroup of exotic spheres that bound parallelisable manifolds.

This group is exactly the kernel of the map above. Geometrically, one can visualise the group structure on bP_{n+1} using boundary connected sums of the bounding parallelisable manifolds. Note that the cokernel of the J-homomorphism is the "hard" part of the stable stem, and similarly, the quotient Θ_n/bP_{n+1} is the "hard" part of Θ_n. On the other hand, the group bP_{n+1} vanishes for even n, and it is cyclic for odd n of an order that one can compute. For small n, here is a table of the various groups.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
π_n^s	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_6$	$\mathbb{Z}_{540}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{480} \times \mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\operatorname{coker}(J_n)$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
bP_{n+1}	0	0	0	0	0	0	$\mathbb{Z}_{28}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_{992}$	0	0	0	$\mathbb{Z}_{8128}$	0

Let us now start with the programme laid out above.

0.37. Lemma. *Let Σ be a smooth n-sphere. Then Σ is* stably parallelisable, *that is, its tangent bundle is stably trivial.*

Note that the only parallelisable spheres are S^n for $n \in \{0, 1, 3, 7\}$. Note also that the lifts $\bar\tau$ in Theorem 0.14 are homotopic to the constant map by the Lemma above for all smooth structures on S^n. This does not prevent the maps $\bar\tau$ for different smooth structures to be different as lifts of τ.

The tangent bundle of a n-manifold M can be trivialised if and only if its (stable) normal bundle ν can be trivialised. A trivialisation of the normal bundle is also known as a *framing* of M. A framing on M defines an element of the n-th stable homotopy group π_n^s of spheres by the Pontryagin construction. Concretely, there exists a smooth map $F\colon S^{q+n} \to S^q$ for q sufficiently large ($q > n$ suffices) such that the south pole $s \in S^q$ is a regular value, $F^{-1}(s) \cong M$, and the trivialisation of the normal bundle is induced by $dF|_{F^{-1}(s)}\colon \nu \to T_sS^q \cong \mathbb{R}^q$. Moreover, Pontryagin construction gives a group homomorphism, where the group structure on framed manifold is disjoint union, which is framed cobordant to the connected sum.

0.38. Lemma. *Let Σ be a smooth n-sphere. Then two stable trivialisations of $T\Sigma$ differ by an element $\alpha \in \pi_n(O)$, and the corresponding elements in π_n^s differ by $J_n(\alpha)$.*

It is well known that π_n^s is finite for $n > 0$, hence the same holds for $\operatorname{coker}(J_n)$ and so also for Θ_n/bP_{n+1}. The map $\Theta_n/bP_{n+1} \to \operatorname{coker}(J_n)$ is either an isomorphism or injective withe index 2. The latter happens if and only if there exists a n-dimensional closed framed manifold M with *Kervaire invariant* 1, see Definition 0.45 and Remark 0.47 below.

Let B_i denote the Bernoulli numbers, which satisfy

$$\frac{t}{2}\coth\frac{t}{2} = \sum_{i=0}^{m} \frac{B_{2i}}{(2i)!} t^{2i} \ .$$

0.39. Theorem. *Assume $n \geq 5$. Then*

1. *If n is even, then $bP_{n+1} = 0$.*
2. *If $n \equiv 1 \bmod 4$, then $\# bP_{n+1} \leq 2$.*
3. *If $n = 4\ell - 1$ then*

$$\# bP_{4\ell} = 2^{2\ell-2}\,(2^{2\ell-1} - 1)\ \operatorname{num}\frac{4B_{2\ell}}{\ell} \ .$$

The proof uses surgery. In the first step, one removes all lower homotopy groups of the bounding manifold W.

0.40. Proposition. *Let $\Sigma \in bP_{n+1}$ and $n \geq 4$. Then there exists a $(i-1)$-connected compact stably parallelisable manifold W with $\partial W \cong \Sigma$ if $2i \leq n+1$.*

If $n = 2j$ is even, with some care one can also eliminate $H_j(W)$. By Poincaré-Lefschetz duality, it follows that W is contractible, and we have proved (1) of the theorem.

0.41. Proposition. *Let W be a $2k$-dimensional smooth compact stably parallelisable manifold with $\partial W = \emptyset$ or ∂W a homotopy sphere.*

1. *If k is odd, ...*
2. *If k is even, then W is framed cobordant modulo ∂W to a manifold with even unimodular and definite intersection form.*

Note that the intersection form on W behaves as on a closed manifold because ∂W has no (co-)homology in the middle dimensions. In particular, it is still unimodular. It is also even because W is stably parallelisable, and hence all Stiefel Whitney classes and all Wu classes vanish. Again, this proposition can be proved using surgery techniques.

If $n \equiv 0 \bmod 4$, we know that $8 \mid \operatorname{sign}(M)$. Examples of such manifolds W can be constructed using plumbing exactly as in Example 0.23. Next, one shows that framed surgery (relative to ∂W) can transform W into a contractible manifold if and only if $\operatorname{sign}(M) = 0$. Hence, we obtain a surjective map

$$\mathbb{Z} \longrightarrow bP_{4\ell} \ .$$

In particular, $bP_{4\ell}$ is cyclic.

0.42. Definition. An *almost framed* manifold is a manifold M with a stable trivialisation of $TM|_{M\setminus\{*\}}$.

We take the non-framed point as the basepoint of M. There is also a natural notion of almost framed cobordism. However, if $\dim M \neq 0 \bmod 4$, then an almost framing can always be extended to a framing.

0.43. Proposition. *We have $a \in \ker(\mathbb{Z} \to bP_{4\ell})$ if and only if there exists an almost framable 4ℓ-manifold N with $\operatorname{sign}(N) = 8a$.*

One can construct a degree 1 map $\xi\colon N \to S^{4\ell}$ which collapses all but a neighbourhood of the basepoint. Then there exists a stable real oriented vector bundle $E \to S^{4\ell}$ such that $\nu \cong \xi^* E$. By the Hirzebruch signature theorem, $\operatorname{sign}(N)$ can be computed using Pontryagin numbers. The only nonzero Pontryagin number is $p_\ell(TN)[N] = -p_\ell(\nu)[N] = -p_\ell(E)[S^{4\ell}]$, so Hirzebruch's theorem gives

$$\frac{\operatorname{sign}(N)}{8} = \pm \frac{2^{2\ell-3}(2^{2\ell-1}-1)\,B_{2\ell}}{(2\ell)!}\, p_\ell(E)[S^{4\ell}]\ .$$

On the other hand, stable vector bundles on $S^{4\ell}$ correspond to $\pi_{4\ell-1}(SO) \cong \mathbb{Z}$, and there is a generator E_1 with

$$p_\ell(E_1)[S^{4\ell}] = \pm a_\ell (2\ell-1)!\ , \qquad \text{where } a_\ell = \begin{cases} 1 & \text{if } \ell \text{ is even, and} \\ 2 & \text{if } \ell \text{ is odd.} \end{cases}$$

Finally, an element $\alpha \in \pi_{4\ell-1}(SO)$ can only appear as an obstruction against framing N is the induced framing of $S^{4\ell-1}$ is zero-bordant, that is, if $\alpha \in \ker(J_{4\ell-1})$. By a result of Adams, the kernel of $J_{4\ell-1}$ is generated by an element of order

$$j_{4\ell-1} = \operatorname{denom}\left(\frac{B_{2\ell}}{4\ell}\right)\ .$$

Combing the three equations above, we can compute

$$\# bP_{4\ell} = a_\ell\, 2^{2\ell-2}(2^{2\ell-1}-1)\ \operatorname{num}\frac{B_{2\ell}}{4\ell}\ .$$

Note that the constant a_ℓ can be subsumed into the numerator expression, giving the expression in Theorem 0.39 (3).

Let us now consider bP_{2k} for odd k. Let $\Sigma = \partial W$ be a homotopy sphere and let W be framed. As before, we may assume that W is $(k-1)$-connected. Then the intersection form on $H_k(W)$ is symplectic, with respect to a basis represented by embedded k-spheres. There are exactly two possible framings on S^k if k is odd. For each basis element $\alpha \in H_k(W)$, the framing of W induces a framing $\varphi(\alpha) \in \mathbb{Z}_2$, independent of the embedding.

0.44. Proposition. *If $k \neq 3$, 7, then $\varphi(\alpha) = 0$ if and only if the embedded sphere has trivial normal bundle in W. For $k = 3$, 7, the normal bundle is trivial and $\varphi(\alpha) = 0$ if any only if surgey at the respective sphere can be framed.*

The map φ factors over a quadratic refinement

$$\varphi_2\colon H_k(W) \to \mathbb{Z}_2$$

of the intersection form modulo 2, that is,

$$\varphi_2(\alpha+\beta) - \varphi_2(\alpha) - \varphi_2(\beta) = \langle \alpha, \beta \rangle\ ,$$

which is symplectic.

0.45. Definition. A quadratic refinement ψ of a symplectic form modulo 2 has *Arf invariant*

$$A(\psi) = \sum_i \psi(\alpha_i)\psi(\beta_i) \in \mathbb{Z}_2\ ,$$

where $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ is a symplectic basis of the underlying symplectic form.

If W is a $2k$-dimensional $(k-1)$-connected framed manifold with k odd and $\partial W = \emptyset$ or ∂W a homotopy sphere, then the *Kervaire invariant* $c(W) \in \mathbb{Z}_2$ of W with the given framing is the Arf invariant of the quadratic function φ_2 constructed above.

Note that the Kervaire invariant does not depend on the framing of W except if $k = 3$ or $k = 7$.

0.46. Proposition. *A $2k$-dimensional $(k-1)$-connected framed manifold W with k odd and either $\partial W = \emptyset$ or ∂W a homotopy sphere is framed cobordant relative to its boundary to a homotopy sphere or a contractible manifold respectively if and only if its Kervaire invariant vanishes.*

One can construct a W with Kervaire invariant $c(W) = 1$ by plumbing. Hence, we have a surjective map $\mathbb{Z}_2 \to bP_{2k}$ very similar to the map $\mathbb{Z} \to bP_{4\ell}$ considered above. The map $\mathbb{Z}_2 \to bP_{2k}$ is trivial for $k \neq 3, 7$ if and only if the standard sphere S^{2k-1} with the nontrivial framing is the framed boundary of a manifold W. If this happens, attaching a disk to ∂W produces an almost framed compact manifold N. In dimensions not congruent 0 mod 4, the framing can be extended. Thus we see that $bP_{2k} = 0$ if and only if there exists a compact framed manifold N with Kervaire invariant 1.

0.47. Remark. A closed framed manifold with Kervaire invariant 1 has dimension $n = 2^j - 2$ for $1 < j < 8$. Examples are known for $j = 1, \ldots, 6$, that is, for $n = 2, 6, 14, 30$ and 62. The case $j = 7$ with $n = 126$ is still open.

Finally, let us summarise the considerations above in the "Kervaire-Milnor braid" of four interwoven long exact sequences. At most places, exactness of these sequences follows from the arguments above.

$$
\begin{tikzcd}[column sep=small]
\pi^s_{n+1} \arrow[rr, bend left, "p"] \arrow[dr, "\pi_{n+1}"] & & P_{n+1} \arrow[rr, bend left, "b"] \arrow[dr, "b"] & & \Theta_n \arrow[rr, bend left, "0"] \arrow[dr, "\iota"] & & \pi_{n-1}(O) \arrow[rr, bend left, "J_{n-1}"] \arrow[dr, "f"] & & \pi^s_{n-1} \\
& A_{n+1} \arrow[ur, "p"] \arrow[dr, "o"] & & \tilde{\Theta}_n \arrow[ur] \arrow[dr, "\iota"] & & A_n \arrow[ur, "o"] \arrow[dr, "p"] & & \tilde{\Theta}_{n-1} \arrow[ur, "\iota"] \arrow[dr] & \\
\Theta_{n+1} \arrow[ur, "\iota"] \arrow[rr, bend right, "0"'] & & \pi_n(O) \arrow[ur, "f"] \arrow[rr, bend right, "J_n"'] & & \pi^s_n \arrow[ur, "\pi_n"] \arrow[rr, bend right, "p"'] & & P_n \arrow[ur, "b"] \arrow[rr, bend right, "b"'] & & \Theta_{n-1} \, .
\end{tikzcd}
$$

The fat black exact sequence gives us the decomposition of exotic spheres into bP-spheres and non-bP-spheres. It consists of the forgetful map ι from exotic spheres to almost framed bordism classes $A_\bullet$. This is well-defined because there is essentially only one possible framing on $\Sigma \setminus \{*\}$. By "punching out a disk", one gets the map p from $A_\bullet$ to framed bordism classes $P_\bullet$ of framed manifolds with boundary a homotopy sphere. We have

$$
P_n \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod 4, \\ \mathbb{Z}_2 & \text{if } n \equiv 2 \bmod 4, \text{ and} \\ 0 & \text{if } n \text{ is odd,} \end{cases}
$$

where the isomorphisms are given by sign $/8$ and the Kervaire invariant, respectively. The map b denotes taking the boundary (and forgetting the framing).

The thin black sequence decomposes into many short exact sequences. The map f changes the framing on the standard sphere. It is followed by a surjective map that just forgets the framing. The fat gray sequence describes the obstruction o to frame an almost framed manifold. The map $J_\bullet$ is the J-homomorphism. Here π denotes the inverse Pontryagin construction, followed by forgetting the frame at one point. Note that π_n is surjective and o vanishes except if $n \equiv 0$ mod 4. Finally, the thin gray sequence consists of a boundary map ∂ similar to b above, but which remembers the framing. The map ι includes framed exotic spheres into the framed bordism group $\Omega_\bullet^{\mathrm{fr}} \cong \pi_\bullet^s$, and p again means “punching out a disk”.

Literatur

[ACKKNPRR] G. Ananthakrishna, A. Conway, D. Kosanović, C. Kremer, I. Nonino, M. Powell, A. Ray, and B. Ruppik, Topological Manifolds, Lecture Notes 2021, `https://maths.dur.ac.uk/users/mark.a.powell/topological-manifolds-lecture-notes.pdf`

[BV] J. M. Boardman and R. M. Vogt, Homotopy-everything H-spaces, Bull. Amer. Math. Soc. 74 (1968), 1117–1122

[F] M. H. Freedman, The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982), 357–453

[HM] M. W. Hirsch and B.Mazur, Smoothings of Piecewise Linear Manifolds, Princeton Univ. Press and Univ. of Tokyo Press, Princeton NJ, 1974

[KM] M. A.Kervaire and J. W. Milnor, Groups of Homotopy Spheres I, Annals of Math. (2) 77 (1963), 504–537

[KS] R. C. Kirby and L. C. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Princeton Univ. Press, Princeton NJ, 1977

[Ki] J. M. Kister, Microbundles are fibre bundles Ann. of Math. (2) 80, 190–199 (1964)

[L] J. P. Levine, Lectures on groups of homotopy spheres, in A. Ranicki, N. Levitt, F. Quinn (Eds.), Algebraic and Geometric Topology, Lect. Notes in Math. 1126, Springer, New Brunswick NJ, 1983

[Mi] J. W. Milnor, Microbundles I, Topology 3, 53–80 (1964)

[T] C. H. Taubes, Gauge Theory on Asymptotically Period 4-Manifolds, J. Diff. Geo. 25 (1987) 363–430