Exercise Sheet 3
Algebraic Topology II

Please write your name on your solution sheet. The submission deadline is thursday, 10th of May, 14h (post box "Jonas Schnitzer", 3rd floor, Ernst-Zermelo-Straße)

Exercise 1 (10 points) Prove or disprove:

(i) An abelian group A is called torsion-free, if for all $a \in A \setminus \{0\}$ and all $n \in \mathbb{Z}$ we have $na \neq 0$. Every torsion-free abelian group is free.

(ii) Let A be an abelian group. The functor $\mathrm{Tor}_{\mathbb{Z}}(\cdot, A)\colon \mathcal{A}b \to \mathcal{A}b$ is cocontinuous.

(iii) Let X, Y, E be well-pointed, $f\colon X \to Y$, then $Z(f \wedge \mathrm{id}_E) \cong Zf \wedge E$.

(iv) Let X, Y, E be well-pointed, $f\colon X \to Y$, then $C(f \wedge \mathrm{id}_E) \cong Cf \wedge E$.

(v) Let X, Y, E be well-pointed, $f\colon X \to Y$, then $C(f \vee \mathrm{id}_E) \cong Cf \vee E$.

Exercise 2 (10 points = 4+4+2 points) Let R be a principal ideal domain, $r,\ s \in R^{\times}$, and let B be an R-module. prove the following statements:

(i) $\mathrm{Tor}(R/r, B) \cong \hom(R/r, B) \cong \{\, b \in B \mid br = 0 \,\} \subset B$

(ii) $\mathrm{Ext}(R/r, B) \cong (R/r) \otimes B \cong B/rB$

(iii) $\mathrm{Tor}(R/r, R/s) \cong \mathrm{Ext}(R/r, R/s) \cong (R/r) \otimes (R/s) \cong \hom(R/r, R/s) \cong R/(r, s)$

where we denote by (r, s) the ideal in R generated by r und s.

Exercise 3 (10 points = 4+3+3 points) An extension of $\mathbb{Z}$-modules of A by B is a short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

up to isomorpisms, where two sequences of this form are called isomorphic, if there is a map of sequences, which is given by the identity on A and B. Prove the following statements:

(i) A free resolution of A induces a unique map of sequences up to chain homotopy

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{a} & A_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

(ii) The map $f \in \hom(A_1, B)$ is well-defined up to $h \circ a$ with $h \in \hom(A_0, B)$, thus we get a class $[f] \in \operatorname{coker}(\hom(A_0, B) \to \hom(A_1, B)) = \mathrm{Ext}_R(A, B)$.

(iii) The assignement from *ii.*) from the set of extensions to the set $\mathrm{Ext}_R(A, B)$ is a bijection.

Exercise 4 (10 points = 4+2+4 points) We consider $A = \mathbb{Z}/n$ and construct the Moore space MA_k for $k \geq 2$ by glueing a $(k+1)$-cell with a map $\varphi\colon S^k \to S^k$ of degree n to S^k. By mapping the k-skeleton S^k to a point, we get the collapsing map $f\colon MA_k \to MA_k/S^k \cong S^{k+1}$. Additionally, let $g\colon MA_k \to S^{k+1}$ be the constant map. Prove the following statements:

(i) $f_* = g_* = 0\colon \tilde{H}_\bullet(MA_k;\mathbb{Z}) \longrightarrow \tilde{H}_\bullet(S^{k+1};\mathbb{Z})$,

(ii) $g_* = 0\colon \tilde{H}_\bullet(MA_k;A) \longrightarrow \tilde{H}_\bullet(S^{k+1};A)$,

(iii) $f_{k+1}\colon \tilde{H}_{k+1}(MA_k;A) \xrightarrow{\cong} \tilde{H}_{k+1}(S^{k+1};A) \cong A$.