Poisson Geometry and Deformation Quantization

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Jonas Schnitzer
Department of Mathematics
University of Freiburg
Ernst-Zermelo-Strasse, 1
D-79104 Freiburg

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Chapter 1

Preliminaries

These are the lecture notes for a course on Poisson geometry and deformation quantization which is taught during winter term 2022/2023. There is no original work inside these notes and the material is taken, if not stated differently, from the following literature list:

- M. Crainic, R.L. Fernandes, I. Mărcuţ: *Lectures on Poisson Geometry*, American Mathematical Society (2021).
- C. Esposito: *Formality theory: from Poisson structures to deformation quantization*, Springer-Verlag Heidelberg, Berlin, New York (2015)
- B. Fedosov: *Deformation Quantization and Index Theory*, Akademie Verlag (1996)
- C. Laurent-Gengoux, A. Pichereau, P. Vanhecke: *Poisson Structures*, Springer-Verlag Heidelberg, Berlin, New York (2013)
- S. Waldmann: *Poisson-Geometrie und Deformationsquantisierung*, Springer-Verlag Heidelberg, Berlin, New York (2007)

The additional literature is neither exhaustive for the field nor necessary for this course, for a better historical overview we refer to the book of Waldmann from the list above. The requirement for this course is, besides linear algebra and analysis, differential geometry, so we assume the reader is familiar with the notion of *smooth manifolds*. The manifolds we are considering are (if not stated differently) Hausdorff and second countable. Moreover, we assume that the reader knows the following basic facts about manifolds/vector bundles:

- The tangent bundle TM of a manifold M is a smooth vector bundle.
- The smooth sections of the tangent bundle are exactly derivations of the algebra of smooth functions.
- Given two vector bundles over the same manifold one can take their direct product, their tensor/exterior/symmetric powers, dualize, etc and the result is still a smooth vector bundle.

Nevertheless, we introduce some less basic facts in the next sections which will be needed throughout the lecture. Note that this does not replace a lecture in differential geometry and is more intended to fix notation.

1.1 Vector fields and all that

We denote be $\mathfrak{X}(M)$ the Lie algebra of vector fields. For $X \in \mathfrak{X}(M)$ we denote by Φ_t^X its flow, i.e. for every $p \in M$ the unique solution to

$$\begin{cases} \Phi_0 = \mathrm{id}_M \\ \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^X(p) = X(\Phi_t^X(p)) \end{cases}$$

which is a smooth map defined on a maximal open subset $U \supset M \times \{0\}$. Moreover, we have that $\Phi_{t+s}^X(p) = \Phi_t^X(\Phi_s^X(p))$, whenever both of the sides are defined. For a function $f \in \mathscr{C}^\infty(M)$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^X)^* f = (\Phi_t^X)^* X(f).$$

For a smooth map $\phi\colon M \to N$, we denote by $T\phi\colon TM \to TN$ the tangent map defined by

$$T_p\phi\colon T_pM \ni X_p \mapsto (f \mapsto X_p(\phi^* f)) \in T_{\phi(p)}N.$$

Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called ϕ-related, if for all $p \in M$, we have

$$T_p\phi X(p) = Y(\phi(p)).$$

We write $X \sim_\phi Y$. If ϕ is a diffeomorphism, then we can define a push-forward of vector fields

$$\phi_*\colon \mathfrak{X}(M) \ni X \mapsto [p \mapsto T_{\phi^{-1}(p)}\phi X(\phi^{-1}(p))] \in \mathfrak{X}(N),$$

and we denote by $\phi^* = \phi_*^{-1}$. Note that X and $\phi_* X$ are ϕ-related.

Theorem 1.1.1 *Let M be a smooth manifold.*

(a) Let $X, Y \in \mathfrak{X}(M)$ be vector fields, then $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\Phi_t^X)^* Y = [X, Y]$

(b) Let $\phi\colon M \to N$ be a smooth map and let $X_i \sim_\phi Y_i$ for $i = 1, 2$, then $[X_1, X_2] \sim_\phi [Y_1, Y_2]$.

We are not only interested in the Lie algebra of vector fields on M, but instead in their exterior algebra:

$$\mathfrak{X}^\bullet(M) := \Gamma^\infty(\Lambda^\bullet TM) = \Lambda^\bullet_{\mathscr{C}^\infty(M)}\Gamma^\infty(TM) = \bigoplus_{i=0}^{\dim(M)} \Lambda^k_{\mathscr{C}^\infty(M)}\Gamma^\infty(TM).$$

Theorem 1.1.2 *Let M be a manifold, then there is a unique bracket $[\![\cdot\,,\cdot]\!]_{\mathrm{S}}\colon \mathfrak{X}^\bullet(M) \times \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$, such that*

- *$[\![X, Y]\!]_{\mathrm{S}} = [X, Y]$ for all $X, Y \in \mathfrak{X}^1(M)$.*
- *$[\![X, f]\!]_{\mathrm{S}} = X(f)$ for all $X \in \mathfrak{X}^1(M)$ and $f \in \mathscr{C}^\infty(M) = \mathfrak{X}^0(M)$.*
- *$[\![X, Y \wedge Z]\!]_{\mathrm{S}} = [\![X, Y]\!]_{\mathrm{S}} \wedge Z + (-1)^{(k-1)\ell} Y \wedge [\![X, Z]\!]_{\mathrm{S}}$ for all $X \in \mathfrak{X}^k(M)$, $Y \in \mathfrak{X}^\ell(M)$ and $Z \in \mathfrak{X}^\bullet(M)$.*
- *$[\![X, Y]\!]_{\mathrm{S}} = -(-1)^{(k-1)(\ell-1)}[\![Y, X]\!]_{\mathrm{S}}$ for all $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^\ell(M)$.*
- *$[\![X, [\![Y, Z]\!]_{\mathrm{S}}]\!]_{\mathrm{S}} = [\![[\![X, Y]\!]_{\mathrm{S}}, Z]\!]_{\mathrm{S}} + (-1)^{(k-1)(\ell-1)}[\![Y, [\![X, Z]\!]_{\mathrm{S}}]\!]_{\mathrm{S}}$ for all $X \in \mathfrak{X}^k(M)$, $Y in \mathfrak{X}^\ell(M)$ and $Z \in \mathfrak{X}^\bullet(M)$.*

PROOF (SKETCH): The idea is the following: one uses properties one, two and three to extend the usual Lie bracket for vector fields to $\mathfrak{X}^\bullet(M)$ and proves the remaining properties. □

Remark 1.1.3 Theorem 1.1.2 show that the triple $(\mathfrak{X}^\bullet, \wedge, [\![\,\cdot\,,\,\cdot\,]\!]_{\mathrm{S}})$ carries the structure of a *Gerstenhaber algebra*.

Now one can see that on factorizing multivector fields $X_1 \wedge \cdots \wedge X_k$ and $Y_1 \wedge \cdots \wedge Y_\ell$ for $X_i, Y_j \in \mathfrak{X}(M)$ one has

$$[\![X_1 \wedge \ldots \wedge X_k, Y_1 \wedge \cdots \wedge Y_\ell]\!]_{\mathrm{S}} = \sum_{i=1}^{k}\sum_{j=1}^{\ell}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \ldots \overset{i}{\wedge} \cdots \wedge X_k \wedge Y_1 \wedge \ldots \overset{j}{\wedge} \cdots \wedge Y_\ell \tag{1.1.1}$$

Moreover, in local coordinates (U, x) every multivector field $X \in \mathfrak{X}^k(M)$ is of the form

$$X\big|_U = \frac{1}{k!} X^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}$$

for unique $X^{i_1\cdots i_k} \in \mathscr{C}^\infty(U)$ and hence we have for $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^\ell(M)$ the formula

$$\begin{aligned}[\![X, Y]\!]_{\mathrm{S}}\big|_U = &\frac{1}{k!\ell!}\Big(kX^{i_1\ldots i_k}\frac{\partial Y^{j_1\ldots j_\ell}}{\partial x^{i_k}}\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k-1}}} \wedge \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_\ell}} \\ &- \ell X^{j_1\ldots j_\ell}\frac{\partial X^{i_1\ldots i_k}}{\partial x^{j_1}}\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \wedge \frac{\partial}{\partial x^{j_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_\ell}}\Big)\end{aligned}$$

The compuation is an exercise.

Note that for a smooth map $\phi\colon M \to N$ we say that $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^k(N)$ are ϕ-related, if for all $p \in M$

$$T_p\phi^{\otimes k}X(p) = Y(\phi(p)),$$

and write $X \sim_\phi Y$.

Lemma 1.1.4 *Let $\phi\colon M \to N$ be a smooth map and let $X_i \in \mathfrak{X}^k(M)$ and $Y_i \in \mathfrak{X}^k(N)$ be multivector fields for $i = 1, 2$, such that $X_i \sim_\phi Y_i$. Then*

$$[\![X_1, X_2]\!]_{\mathrm{S}} \sim_\phi [\![Y_1, Y_2]\!]_{\mathrm{S}}.$$

PROOF: The proof is an easy consequence of Theorem 1.1.1 in combination with Formula (1.1.1). □

The property of the Schouten bracket from Lemma 1.1.4 is called *naturality* and it implies moreover, that the Schouten bracket is diffeomorphism invariant, i.e. for a diffeomorphism $\phi\colon M \to N$, we have

$$\phi^*[\![X, Y]\!]_{\mathrm{S}} = [\![\phi^*X, \phi^*Y]\!]_{\mathrm{S}}$$

for all $X, Y \in \mathfrak{X}^\bullet(N)$.

1.2 Cartan Calculus

The dual picture is now the de Rham complex. We denote by $T^*M \to M$ the cotangent bundle of a manifold M, i.e. the dual of the tangent bundle $TM \to M$, and consider its exterior algebra

$$\Omega^\bullet(M) := \Gamma^\infty(\Lambda^\bullet T^*M) = \Lambda^\bullet_{\mathscr{C}^\infty(M)}\Gamma^\infty(T^*M) = \bigoplus_{i=0}^{\dim(M)} \Lambda^k_{\mathscr{C}^\infty(M)}\Gamma^\infty(T^*M).$$

Definition 1.2.1 *Let M be a manifold, then we define the $\mathbb{R}$-linear map $\mathrm{d}^k\colon \Omega^k(M) \to \Omega^{k+1}(M)$ by*

$$\begin{aligned}\mathrm{d}\omega(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1,\dots,\overset{i}{\wedge},\dots,X_{k+1}))\\ &\quad+\sum_{i<j}\omega([X_i,X_j],X_1,\dots,\overset{i}{\wedge},\dots,\overset{j}{\wedge},\dots,X_{k+1})\end{aligned}$$

for all $\omega \in \Omega^k(M)$ and $X_1,\dots,X_{k+1} \in \Gamma^\infty(TM)$ and $k \geq 1$. For $k = 0$ we define

$$\Omega^0(M) = \mathscr{C}^\infty(M) \ni f \mapsto (X \mapsto X(f)) \in \Omega^1(M).$$

$\mathrm{d} = \sum_k \mathrm{d}^k\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ *is called de Rham differential.*

Lemma 1.2.2 *The de Rham differential* d *has the following properties*

(a) $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^k\alpha\wedge\mathrm{d}\beta$ *for all* $\alpha \in \Omega^k(M)$ *and* $\Omega^\ell(M)$.

(b) $\mathrm{d}^2 = 0$.

PROOF: Exercise. □

In a local coordinate chart (U, x) we have that a differential form $\alpha \in \Omega^k(M)$ is of the form

$$\alpha\big|_U = \frac{1}{k!}\alpha_{i_1\dots i_k}\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$$

and with Lemma 1.2.2, we see that

$$\mathrm{d}\alpha\big|_U = \frac{1}{k!}\frac{\partial\alpha_{i_1\dots i_k}}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$$

For a manifold M, we can now introduce the quotient

$$\mathrm{H}^k_{\mathrm{dR}}(M) = \frac{\ker\mathrm{d}^k}{\operatorname{im}\mathrm{d}^{k-1}},$$

which is called the kth de Rham cohomology of M. The remarkable fact about this quotient is that even though its definition involves is the solution space of a partial differential equation ($\mathrm{d}^k\alpha = 0$) modulo *trivial solutions*, the information it contains is purely topological. Nevertheless, for this course we are only interested in the the de Rham complex itself and also very superficial.

It is not surprising that, since the bundles TM and T^*M are dual to each other, there are operations wich include both $\mathfrak{X}^\bullet(M)$ and $\Omega^\bullet(M)$. We define for a factorizing $X = X_1\wedge\dots\wedge X_k \in \mathfrak{X}^k(M)$ and $\alpha \in \Omega^\ell(M)$:

$$\iota_X\alpha = \iota_{X_1}\dots\iota_{X_k}\alpha, \tag{1.2.1}$$

where $\iota_{X_i}\alpha$ is the usual contraction of a vectorfield and a differential form. Additionally we set $\iota_f\alpha = f\alpha$ for $f \in \mathscr{C}^\infty(M)$. Moreover, we define the Lie derivative along a multivector field $X \in \mathfrak{X}^k(M)$ by Cartan's magic formula

$$\mathscr{L}_X = [\iota_X, \mathrm{d}] = \iota_X \,\mathrm{d} - (-1)^k \,\mathrm{d}\iota_X \colon \Omega^\bullet(M) \to \Omega^{\bullet-(k-1)}(M).$$

Note that for a vector field X and a k-form $\alpha \in \Omega^k(M)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\Phi_t^X)^*\alpha = \mathscr{L}_X\alpha.$$

Proposition 1.2.3 (Cartan calculus) *Let M be a smooth manifold. Then*

(a) $\iota_{X\wedge Y} = \iota_X\iota_Y$

(b) $[\iota_X, \iota_Y] = \iota_X\iota_Y - (-1)^{k\ell}\iota_Y\iota_X = 0$

(c) $\mathscr{L}_{X\wedge Y} = \iota_X\mathscr{L}_Y + (-1)^\ell\mathscr{L}_X\iota_Y$

(d) $[\mathrm{d}, \mathscr{L}_X] = \mathrm{d}\mathscr{L}_X - (-1)^{(k-1)}\mathscr{L}_X\,\mathrm{d} = 0$

(e) $[\mathscr{L}_X, \iota_Y] = \mathscr{L}_X\iota_Y - (-1)^{(k-1)\ell}\iota_Y\mathscr{L}_X = \iota_{[\![X,Y]\!]_{\mathrm{S}}}$

(f) $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_X\mathscr{L}_Y - (-1)^{(k-1)(\ell-1)}\mathscr{L}_Y\mathscr{L}_X = \mathscr{L}_{[\![X,Y]\!]_{\mathrm{S}}}$

for $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^\ell(M)$.

PROOF: Note that the every point, except for (e) and (f), follow directly from skew-symmetry and $\mathrm{d}^2 = 0$. (e) follows from Cartan's magic formula for two vector fields: one can inductively show the identity on factorizing tensors. (f) follows from (e), since

$$\begin{aligned}[\mathscr{L}_X, \mathscr{L}_Y] &= [\mathscr{L}_X, [\iota_Y, \mathrm{d}]] = [[\mathscr{L}_X, \iota_Y], \mathrm{d}] + (-1)^{\ell(k-1)}[\iota_Y, [\mathscr{L}_X, \mathrm{d}]] \\ &= [[\mathscr{L}_X, \iota_Y], \mathrm{d}] \overset{(e)}{=} [\iota_{[\![X,Y]\!]_{\mathrm{S}}}, \mathrm{d}] \\ &= \mathscr{L}_{[\![X,Y]\!]_{\mathrm{S}}}.\end{aligned}$$

□

Chapter 2

Poisson Geometry

Poisson geometry is the study of Poisson brackets, which were developed by Siméon Denis Poisson in 1809 order to study integrals of motion in mechanics. After that it was Carl Gustav Jacob Jacobi and Sophus Lie who studied Poisson brackets from different angles, which lead for example to the discovery of Lie algebras and Lie groups. The modern formulation of Poisson brackets is due to André Lichnerowicz in the 1970s and his work is arguably the starting of the geometric point of view. As a last historical remark we want to mention the fundamental work of Alan Weinstein in [12], who discovered many aspects of Poisson geometry which are up to now subjects of research.

In order to understand roughly the physical background, we consider a particle moving in the configuration space $\mathbb{R}^3$ with coordinates $(q^1(t), q^2(t), q^3(t))$. In order to describe its motion, we need to fix a Hamiltonian $H \in \mathscr{C}^\infty(T^*\mathbb{R}^3) = \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ or as it is called in in physics: an Energy. Usually, H is of the form

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(q) \tag{$*$}$$

for the standard coordinates (q, p) of $\mathbb{R}^3 \times \mathbb{R}^3$, where the first summand is the *kinetic energy* and the second summand is the *potential energy*. In the Hamiltonian formalism of classical mechanics the motion $(q^1(t), q^2(t), q^3(t))$ of the particle is a solution to the ordinary differential equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) = -\frac{\partial H}{\partial q^i}(q(t), p(t)). \tag{$**$}$$

If we define the binary operation $\{-,-\}\colon \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \times \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \to \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\{f, g\} = \sum_{i=1}^{3} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i},$$

then we can write the Equations $(**)$ in the form

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) = \{q^i, H\}(q(t), p(t)) \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) = \{p_i, H\}(q(t), p(t)).$$

Note that $\{-,-\}$ is a Lie bracket which is a derivation in both slots, and this is basically the starting point of Poisson geometry. In fact, to do mechanics, we need to fix three things:

(a) a *phase space*, which has sufficiently nice properties, i.e. is a smooth manifold M,

(b) a *Poisson bracket*, i.e. a Lie bracket $\{-,-\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, which is a derivation in both slots,

(c) a *Poisson subalgebra* $\mathscr{A}_{\mathrm{cl}} \subseteq \mathscr{C}^\infty(M)$ of *classical Observables* and

(d) an energy function $H \in \mathscr{A}_{\mathrm{cl}}$ (the *Hamiltonian*).

From the geometric point, we forget the chosen energy function and call the pair $(M, \{-,-\})$ a Poisson manifold. The study of these brackets is what is called *Poisson geometry*.

2.1 Poisson Brackets, Poisson Tensors and Poisson maps

Before we define Poisson manifolds, we define algebraic structure behind the geometric version:

Definition 2.1.1 *Let $\mathscr{A}$ be a commutative algebra over $\mathbb{k}$. A Poisson bracket is a bilinear map $\{\,\cdot\,,\,\cdot\,\}\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, such that*

(a) $\{a,b\} = -\{b,a\}$ (skew symmetry)

(b) $\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$ (Jacobi identity)

(c) $\{a,bc\} = \{a,b\}c + b\{a,c\}$ (biderivation)

for all $a,b,c \in \mathscr{A}$.

Remark 2.1.2 The first two properties of the Poisson bracket in Definition 2.1.1 are exactly the axioms for $(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$ being a Lie algebra.

The next section is full of examples, but let us, before entering the realm of Poisson manifolds, get rid of a triviality:

Example 2.1.3 We can endow every commutative algebra $\mathscr{A}$ with the trivial Poisson bracket, i.e. the bracket sending every two elements to 0. This bracket fulills axioms (a)-(c) for trivial reasons.

Given a smooth manifold there is obviously a commutative algebra canonically attached to it: $\mathscr{C}^\infty(M)$. The definition of a Poisson manifold is now straight forward:

Definition 2.1.4 *A Poisson manifold is a smooth manifold M together with a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathscr{C}^\infty(M)$.*

Lemma 2.1.5 *For every open subset U of a Poisson manifold $(M, \{\,\cdot\,,\,\cdot\,\})$ there exists a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_U$ on U, such that.*

$$\{f,g\}\big|_U = \{f\big|_U, g\big|_U\}_U.$$

for all $f,g \in \mathscr{C}^\infty(M)$.

PROOF: Let $f,g \in \mathscr{C}^\infty(U)$ be given. We define

$$\{f,g\}_U(x_0) = \{\tilde{f},\tilde{g}\}(x_0)$$

where $\tilde{f},\tilde{g} \in \mathscr{C}^\infty(M)$ are chosen such that there is an open neighbourhood $x_0 \in U' \subseteq U$ and $f\big|_{U'} = \tilde{f}\big|_{U'}$ and $g\big|_{U'} = \tilde{g}\big|_{U'}$. To prove that this bracket is well defined, we just have to check that $\{f,g\}$ vanishes at a point $x_0 \in M$, if there is an open neighbourhood $x_0 \in O$ with $g\big|_O = 0$. We therefore choose a function $\rho \in \mathscr{C}^\infty(M)$, such that $\rho(x_0) = 0$ and $\rho(x) = 1$ for all $x \notin O$. We have therefore $g = \rho g$ and hence

$$\{f,g\}(x_0) = \{f,\rho g\}(x_0) = \{f,\rho\}(x_0)g(x_0) + \rho(x_0)\{f,g\}(x_0) = 0.$$

This bracket is indeed Poisson, since the other properties can be checked using small enough neighbourhoods in which all the involved functions can be extended to M. $\square$

Proposition 2.1.6 *Let* $(\mathscr{C}^\infty(M), \{\,\cdot\,,\,\cdot\,\})$ *be a Poisson manifold, then*

(a) $\{\,\cdot\,,\,\cdot\,\}$ *is local, i.e.* $\operatorname{supp}\{f,g\} \subseteq \operatorname{supp} f \cap \operatorname{supp} g$ *and in local coordinates* (U,x) *of* M *we have*

$$\{f,g\}\big|_U = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \tag{2.1.1}$$

for local functions $\pi^{ij} \in \mathscr{C}^\infty(U)$ *defined by*

$$\pi^{ij} = \{x^i, x^j\} = -\pi^{ji}.$$

(b) there exists a bivector field $\pi \in \mathfrak{X}^2(M)$ *with* $[\![\pi,\pi]\!]_{\mathrm{S}} = 0$ *such that*

$$\{f,g\} = \mathrm{d}f \otimes \mathrm{d}g(\pi) = -[\![[\![f,\pi]\!]_{\mathrm{S}}, g]\!]_{\mathrm{S}}$$

The local functions in (2.1.1) *are exactly the coefficient function of the bivector field from (b), i.e.*

$$\pi = \frac{1}{2}\pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Proof: We start proving (a). Let $f,g \in \mathscr{C}^\infty(M)$ and let $U := M \setminus \operatorname{supp} g$, by Lemma 2.1.5 we know that there is a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_U$, such that

$$\{f,g\}\big|_U = \{f\big|_U, g\big|_U\}_U = \{f\big|_U, 0\}_U = 0.$$

Hence $\operatorname{supp}\{f,g\} \subseteq \operatorname{supp} g$ and with $\{f,g\} = -\{g,f\}$ we also get $\{f,g\} \subseteq \operatorname{supp} f$. To proceed with the proof, we notice that for all functions $f \in \mathscr{C}^\infty(M)$, we have $\{f,1\} = \{1,f\} = 0$, since

$$\{f,1\} = \{f, 1\cdot 1\} = \{f,1\}1 + 1\{f,1\} = 2\{f,1\}$$

and $0 = \{f,1\} = -\{1,f\}$. By bilinearity this holds even for all constant functions. Let us now pick a coordinate chart (U,x) and let $f,g \in \mathscr{C}^\infty(U)$, then we find functions $E_i, E_{ij}, T_i, T_{ij} \in \mathscr{C}^\infty(U)$, such that $E_i(x_0) = \frac{\partial f}{\partial x^i}(x_0)$, $T_i(x_0) = \frac{\partial g}{\partial x^i}(x_0)$

$$f(x) = f(x_0) + \sum_{i=1}^n E_i(x)(x^i - x_0^i) + \sum_{i,j=1}^n E_{ij}(x)(x^i - x_0^i)(x^j - x_0^j)$$

and

$$g(x) = g(x_0) + \sum_{i=1}^n T_i(x)(x^i - x_0^i) + \sum \sum_{i,j=1}^n T_{ij}(x)(x^i - x_0^i)(x^j - x_0^j)$$

close to x_0. We work now with the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_U$ from Lemma 2.1.5. We have, using that $\{\,\cdot\,,\,\cdot\,\}_U$ vanishes on constants and its Leibniz rule, that

$$\{f,g\}_U(x_0) = \{x^i, x^j\}_U(x_0) \frac{\partial f}{\partial x^i}(x_0) \frac{\partial g}{\partial x^j}(x_0)$$

and the claim is proven.

To prove (b) it is now enough to prove that $\pi\big|_U := \frac{1}{2}\{x^i, x^j\} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ glues together to a global object. Let therefore be given two coordinate charts (U,x) and (V,y), such that $U \cap V \neq \emptyset$, then

$$\{y^k(x), y^\ell(x)\} = \{x^i, x^j\} \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}$$

and hence

$$\begin{aligned}\pi^y_{U\cap V} &= \frac{1}{2}\{y^k, y^\ell\}\frac{\partial}{\partial y^k}\wedge\frac{\partial}{\partial y^\ell}\\ &= \frac{1}{2}\{x^i, x^j\}\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^k}\wedge\frac{\partial}{\partial y^\ell}\\ &= \frac{1}{2}\{x^i, x^j\}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}\\ &= \pi^x_{U\cap V}.\end{aligned}$$

In every chart (U, x) we can see that $\{f, g\}\big|_U = -[\![[\![f, \pi]\!]_S, g]\!]_S\big|_U$ and thus it is valid globally. Let us denote $X_f = -[\![f, \pi]\!]_S$, then we see

$$\begin{aligned}\{f, \{g, h\}\} &= [\![X_f, \{g, h\}]\!]_S\\ &= [\![X_f, [\![X_g, h]\!]_S]\!]_S\\ &= [\![[\![X_f, X_g]\!]_S, h]\!]_S + [\![X_g, [\![X_f, h]\!]_S]\!]_S.\end{aligned}$$

Moreover, we have

$$\begin{aligned}[\![X_f, X_g]\!]_S = -[\![X_f, [\![g, \pi]\!]_S]\!]_S &= [\![[\![X_f, h]\!]_S, \pi]\!]_S + [\![g, [\![X_f, \pi]\!]_S]\!]_S\\ &= -X_{\{f,g\}} + \frac{1}{2}[\![g, [\![f, [\![\pi, \pi]\!]_S]\!]_S]\!]_S.\end{aligned}$$

This implies that

$$\{f, \{g, h\}\} - \{\{f, g\}, h\} - \{g, \{f, h\}\} = -\frac{1}{2}[\![h, [\![g, [\![f, [\![\pi, \pi]\!]_S]\!]_S]\!]_S]\!]_S$$

and since differentials span the cotangent space at every point we get $\{f, \{g, h\}\} - \{\{f, g\}, h\} - \{g, \{f, h\}\} = 0 \iff [\![\pi, \pi]\!]_S = 0$. □

Remark 2.1.7 Note that, if there is a bivector field $\pi \in \mathfrak{X}^2(M)$, such that $[\![\pi, \pi]\!]_S = 0$, we can induce a Poisson bracket $\{\,\cdot\,, \cdot\,\}$ via the formula in (b) of Proposition 2.1.6. We will in the following switch freely between the two equivalent descriptions. A bivector field $\pi \in \mathfrak{X}^2(M)$ with $[\![\pi, \pi]\!]_S = 0$ is from now on called *Poisson bivector field.*

The Schouten bracket is a local operator itself, so it is enough to check the condition $[\![\pi, \pi]\!]_S = 0$ locally. In a chart (U, x), we have

$$[\![\pi, \pi]\!]_S\big|_U = 0 \quad\iff\quad \pi^{ij}\frac{\partial \pi^{kl}}{\partial x^j} + \pi^{lj}\frac{\partial \pi^{ik}}{\partial x^j} + \pi^{kj}\frac{\partial \pi^{li}}{\partial x^j} = 0 \ \text{ for all } i, k, l.$$

The proof is an exercise. We note that π being Poisson is the same as its coefficients are a solution to a certain quadratic PDE and one can study Poisson geometry from this point of view. But this is not part of these lecture notes.

Let us now focus on the Poisson tensor itself: even though it might very *singular*, we can still define one direction of the musical isomorphism from Riemannian geometry, but instead we "just" get a homomorphism:

Definition 2.1.8 *Let (M, π) be a Poisson manifold.*

(a) The musical homomorphism $\pi^\sharp\colon T^*M \to TM$ is the vector bundle map defined by

$$\pi^\sharp\colon T^*M \ni \alpha_p \mapsto \pi(\,\cdot\,, \alpha_p) \in TM.$$

(b) The rank of π at $p \in M$ is defined by

$$\mathrm{rank}(\pi)_p = \mathrm{rank}(\pi^\sharp(p)).$$

Lemma 2.1.9 *The rank map $M \ni p \mapsto \mathrm{rank}(\pi)_p \in \mathbb{N}$ takes only values in $2\mathbb{N}$ and is lower semicontinuous, i.e. for all points $x \in M$ there exists an open neighbourhood $U \subseteq M$, such that*

$$\mathrm{rank}(\pi)_x \leq \mathrm{rank}(\pi)_y$$

for all $y \in U$.

PROOF: Exercise. □

As a next step, we want to compare Poisson manifolds and therefore we need to define *Poisson maps*.

Definition 2.1.10 *Let $(M, \{\,\cdot\,,\,\cdot\,\}_M)$ and $(N, \{\,\cdot\,,\,\cdot\,\}_N)$ be two Poisson manifolds. A smooth map $\phi\colon M \to N$ is called Poisson map, if*

$$\phi^*\{f,g\}_N = \{\phi^* f, \phi^* g\}_M$$

for all $f, g \in \mathscr{C}^\infty(M)$.

Note that it is obvious, that one can concatenate two Poisson maps and the result is still a Poisson map, which ensures that Poisson manifolds together with Poisson maps form a category, the category $\mathfrak{PoissMfld}$. Note that, however it is sometimes preferable to define the category of Poisson manifolds with different morphisms, but this goes beyond the scope of this lecture notes.

Let us now investigate the relation of a Poisson map with the Poisson bivector field.

Proposition 2.1.11 *Let $\phi\colon M \to N$ a smooth map between the Poisson manifolds (M, π_M) and (N, π_N). The following are equivalent:*

(a) ϕ is a Poisson map.

(b) π_M and π_N are ϕ-related.

(c) For all $p \in M$, we have

$$T_p\phi \circ \pi_M(p)^\sharp \circ (T_p\phi)^* = \pi_N^\sharp(\phi(p)).$$

In particular, if ϕ is a (local) diffeomorphism , then $\phi^*\pi_N = \pi_M$.

PROOF: Let us assume (a). Let $f, g \in \mathscr{C}^\infty(N)$ and $p \in M$, then we have

$$\begin{aligned}
\mathrm{d}f_{\phi(p)} \otimes \mathrm{d}g_{\phi(p)}(T_p\phi \otimes T_p\phi(\pi_M(p))) &= \mathrm{d}f_{\phi(p)} \circ T_p\phi \otimes \mathrm{d}g_{\phi(p)} \circ T_p\phi(\pi_M(p)) \\
&= \mathrm{d}\phi^* f_p \otimes \mathrm{d}\phi^* g_p(\pi_M(p)) = \{\phi^* f, \phi *\}_M(p) \\
&= \phi^*\{f,g\}_N(p) = \{f,g\}_N(\phi(p)) \\
&= \mathrm{d}f_{\phi(p)} \otimes \mathrm{d}g_{\phi(p)}(\pi_N(\phi(p)).
\end{aligned}$$

Since the differentials on functions span the whole cotangent space, we get $T_p\phi \otimes T_p\phi(\pi) = \pi_N(\phi(p)$ and hence $\pi_M \sim_\phi \pi_N$. Let us now assume (b)and let $\alpha_{\phi(p)}, \beta_{\phi_p} \in T^*_{\phi(p)}N$ be arbitrary, then

$$\alpha_{\phi(p)}(T_p\phi \circ \pi_M(p)^\sharp \circ (T_p\phi)^*(\beta_{\phi(p)}) = \alpha_{\phi(p)}(T_p\phi\pi_M(p)(\,\cdot\,, \beta_{\phi(p)} \circ T_p\phi)$$

$$
\begin{aligned}
&= \alpha_{\phi(p)} \otimes \beta_{\phi(p)}(T_p\phi \otimes T_p\phi(\pi_M(p))) \\
&= \alpha_{\phi(p)} \otimes \beta_{\phi_p}(\pi_N(\phi(p)) \\
&= \alpha_{\phi(p)}(\pi_N(\phi(p)^\sharp \beta_{\phi(p)}).
\end{aligned}
$$

And again, since $\alpha_{\phi(p)}, \beta_{\phi_p} \in T^*_{\phi(p)}N$ were chosen arbitrary, we get $T_p\phi \circ \pi_M(p)^\sharp \circ (T_p\phi)^* = \pi_N(\phi(p))^\sharp$. Let us finally assume (c) and let $f, g \in \mathscr{C}^\infty(N)$, then for all $p \in M$ we have

$$
\begin{aligned}
\phi^*\{f,g\}_N(p) = \{f,g\}(\phi(p)) &= \mathrm{d}f_{\phi(p)}(\phi_N(p)^\sharp \,\mathrm{d}g_{\phi(p)}) \\
&= \mathrm{d}f_{\phi(p)}(T_p\phi \circ \pi_M(p)^\sharp \circ (T_p\phi)^* \,\mathrm{d}g_{\phi(p)}) \\
&= \mathrm{d}\phi^* f_p \otimes \mathrm{d}\phi^* g_p(\pi_M(p)) = \{\phi^* f, \phi^* g\}_M(p).
\end{aligned}
$$
□

2.2 Hamiltonian and Poisson Vector fields

After the first properties and definitions of Poisson manifolds we want to study their symmetries. Recall that in Riemannian geometry the isometries are always a Lie group, and to hope for such a strong statement is beyond any reason in Poisson geometry. This is why we only deal with infinitesimal symmetries, i.e. vector fields which preserve the Poisson structure in a reasonable way:

Definition 2.2.1 *Let (M, π) be a Poisson manifold.*

(a) A vector field $X \in \mathfrak{X}^1(M)$ is called Poisson vector field, if

$$\mathscr{L}_X \pi = [\![X, \pi]\!]_\mathrm{S} = 0.$$

(b) The vector field

$$X_H = \pi^\sharp(\mathrm{d}H) = [\![H, \pi]\!]_\mathrm{S}$$

is called Hamiltonian vector field of the function $H \in \mathscr{C}^\infty(M)$.

We collect in the following theorem the first properties of Hamiltonian and Poisson vector fields.

Theorem 2.2.2 *Let (M, π) be a Poisson manifold.*

(a) Each Hamiltonian vector field is also a Poisson vector field.

(b) For all $f, g \in \mathscr{C}^\infty(M)$ the following identities hold:

$$\{f, g\} = X_g(f) \qquad \text{and} \qquad [X_f, X_g] = -X_{\{f,g\}}.$$

(c) The Poisson vector fields are a Lie subalgebra of $\mathfrak{X}(M)$, moreover for a function $f \in \mathscr{C}^\infty(M)$ and a Poisson vector field X, we have

$$[X, X_f] = X_{X(f)},$$

so the Hamiltonian vector fields are a Lie ideal in the Poisson vector fields.

(d) A vector field X is a Poisson vector field, iff for all functions $f, g \in \mathscr{C}^\infty(M)$ the equality

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}$$

holds.

(e) A vector field X is a Poisson vectorfield, iff its (local) flow is a Poisson map.

PROOF: In the first four points we will see the strength of the Schouten calculus:

(a) Let $f \in \mathscr{C}^\infty(M)$ be arbitrary, then

$$\begin{aligned}\llbracket X_f, \pi \rrbracket_S &= \llbracket \llbracket f, \pi \rrbracket_S, \pi \rrbracket_S = \llbracket f, \llbracket \pi, \pi \rrbracket_S \rrbracket_S - \llbracket \llbracket f, \pi \rrbracket_S, \pi \rrbracket_S \\ &= -\llbracket \llbracket f, \pi \rrbracket_S, \pi \rrbracket_S = -\llbracket X_f, \pi \rrbracket_S\end{aligned}$$

and hence $\llbracket X_f, \pi \rrbracket_S = 0$.

(b) Let $f, g \in \mathscr{C}^\infty(M)$, then by using Proposition 2.1.6

$$\{f, g\} = -\{g, f\} = \llbracket \llbracket g, \pi \rrbracket_S, f \rrbracket_S = \llbracket X_g, f \rrbracket_S = X_g(f).$$

Moreover, we have

$$\begin{aligned}[X_f, X_g] &= \llbracket \llbracket f, \pi \rrbracket_S, \llbracket g, \pi \rrbracket_S \rrbracket_S = \llbracket \llbracket \llbracket f, \pi \rrbracket_S, g \rrbracket_S, \pi \rrbracket_S + \llbracket \llbracket \llbracket f, \pi \rrbracket_S, \pi \rrbracket_S, g \rrbracket_S \\ &= -\llbracket \{f, g\}, \pi \rrbracket_S + \llbracket \mathscr{L}_{X_f} \pi, g \rrbracket_S \\ &= -X_{\{f,g\}}\end{aligned}$$

(c) Let X, Y be two Poisson vector fields, then

$$\mathscr{L}_{[X,Y]} \pi = \llbracket \llbracket X, Y \rrbracket_S, \pi \rrbracket_S = \llbracket X, \llbracket Y, \pi \rrbracket_S \rrbracket_S - \llbracket Y, \llbracket X, \pi \rrbracket_S \rrbracket_S = 0.$$

(d) Let $f, g \in \mathscr{C}^\infty(M)$ and let X be a vector field, then

$$\begin{aligned}X(\{f, g\}) &= \llbracket X, \{f, g\} \rrbracket_S = -\llbracket X, \llbracket \llbracket f, \pi \rrbracket_S, g \rrbracket_S \rrbracket_S \\ &= -\llbracket \llbracket X(f), \pi \rrbracket_S, g \rrbracket_S - \llbracket \llbracket f, \mathscr{L}_X \pi \rrbracket_S, g \rrbracket_S - \llbracket \llbracket f, \pi \rrbracket_S, X(g) \rrbracket_S \\ &= \{X(f), g\} + \{f, X(g)\} - \llbracket \llbracket f, \mathscr{L}_X \pi \rrbracket_S, g \rrbracket_S.\end{aligned}$$

Hence, we see that if X is Poisson the equality holds. If now the equality holds, then we know that

$$0 = \llbracket \llbracket f, \mathscr{L}_X \pi \rrbracket_S, g \rrbracket_S = \mathrm{d}f \otimes \mathrm{d}g(\mathscr{L}_X \pi).$$

Since the differentials of functions span the cotangent space at every point and we get the claim.

(e) Let X be a vector field, then we have for its flow

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^X)^* \pi = (\Phi_t^X)^* (\mathscr{L}_X \pi).$$

If X is a Poisson vector field, we have that $\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^X)^* \pi = 0$ and hence $\pi = (\Phi_0^X)^* \pi = (\Phi_t^X)^* \pi$ for all t. If Φ_t^X is a Poisson map, we have that $\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^X)^* \pi = \frac{\mathrm{d}}{\mathrm{d}t}\pi = 0$ and and the equivalence follows. □

The quotient

$$\mathrm{H}^1_\pi(M) = \frac{\{\text{Poisson vector fields}\}}{\{\text{Hamiltonian vector fields}\}}$$

measures how many Poisson vector fields are not Hamiltonian vector fields and is called the first Poisson cohomology. Moreover, as a direct consequence of Theorem 2.2.2 we have:

Corollary 2.2.3 *Let (M, π) be a Poisson manifold. The vector space $\mathrm{H}^1_\pi(M)$ carries the structure of a Lie algebra induced by the usual Lie bracket from $\mathfrak{X}(M)$.*

In general, this quotient can be rather wild and hard to compute. In some special cases one can compute it as for the trivial Poisson structure.

Example 2.2.4 Let M be a manifold and $\pi = 0$ the trivial Poisson bivector field, then we have $X_f = 0$ for all $f \in \mathscr{C}^\infty(M)$, but it is clear that for all vector fields X, we have $\mathscr{L}_X \pi = 0$ and hence every vector field is a Poisson vector field.

In Example 2.2.4, we could see that sometimes two functions have the same Hamiltonian vector field even though they do not coincide (even if there difference is not constant). This motivates the following definition

Definition 2.2.5 *Let (M, π) be a Poisson manifold. A function $f \in \mathscr{C}^\infty(M)$ is called Casimir function, if $X_f = 0$. The vector space of all Casimir functions is also called zeroth Poisson cohomology and is denoted by $\mathrm{H}^0_\pi(M)$.*

Remark 2.2.6 (*) The notation $\mathrm{H}^0_\pi(M)$ and $\mathrm{H}^1_\pi(M)$ suggests that both vector spaces are part of a cohomology theory for Poisson manifolds and this is in fact true, but the exact formulation goes beyond the scope of this lecture notes.

Let us close this section with a lemma connecting Hamiltonian vector fields and Poisson maps:

Lemma 2.2.7 *Let $\phi \colon (M, \pi_M) \to (N, \pi_N)$ be a Poisson map. For every function $f \in \mathscr{C}^\infty(N)$, we have*

$$X_{\phi^* f} \sim_\phi X_f$$

PROOF: We have for $g \in \mathscr{C}^\infty(N)$

$$(T_p\phi X_{\phi^* f}(p))(g) = X_{\phi^* f}(p)(\phi^* g) = -\{\phi^* f, \phi^* g\}_M(p) = -\{f, g\}_N(\phi(p)) = X_f(\phi(p))(g)$$

and the claim is proven. □

2.3 Examples

2.3.1 Constant and linear Poisson Structures

We consider $M = V$, where V is a finite dimensional real vector space, with a given basis $\{e_i\}_{1 \le i \le n}$ and its dual $\{e^i\}_{1 \le i \le n}$. For a skew symmetric matrix $\{\pi^{ij}\}_{1 \le i,j \le n}$ we define

$$\pi = \frac{1}{2} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

where the coordinate x^i is defined by $V \ni v \mapsto e^i(v) \in \mathbb{R}$. One can see by using the local formula for $[\![\pi, \pi]\!]_{\mathrm{S}}$, that π is indeed a Poisson bivector. In this case we call π constant. Note that the definition of being a constant Poisson structure is independent of the chosen basis, i.e. if a Poisson structure on V is constant for one basis, it is constant for all.

Proposition 2.3.1 (linear Weinstein Splitting Theorem) *For a constant Poisson bivector π of rank $2d$ on V there is a basis $(f_1, \dots, f_d, g_1 \dots, g_d, h_1, \dots, h_{n-2d})$, such that*

$$\pi = \sum_{i=0}^{d} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i},$$

where the q^i's (resp. the p^i's) correspond to the basis elements f_i (resp. g_i).

PROOF: Exercise. □

From Proposition 2.3.1, one can see that in this case the Casimir functions are exactly the functions which are constant in q and p directions.

Let us now define what a linear Poisson structure is. Let therefore V be again a real vector space with a given basis $\{e_i\}_{1\leq i\leq n}$. A Poisson bivector field π is called linear, if there are $c_k^{ij} \in \mathbb{R}$ with $c_k^{ij} = -c_k^{ji}$, such that

$$\pi = \frac{1}{2} x^k c_k^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Note that the definition of π being linear is also independent of the choice of the basis. Moreover, the constants c_k^{ij} cannot be chosen arbitrary:

Proposition 2.3.2 *A bivector field* $\pi = \frac{1}{2} x^k c_k^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ *for* $c_k^{ij} \in \mathbb{R}$ *is Poisson, iff*

$$c_m^{i\ell} c_\ell^{jk} + c_m^{k\ell} c_\ell^{ij} + c_m^{j\ell} c_\ell^{ki} = 0$$

für alle $i, j, k, m \in \{1, \ldots, n\}$.

PROOF: A tiny computation shows that

$$[\![\pi, \pi]\!]_{\mathrm{S}} = x^m (c_m^{i\ell} c_\ell^{jk} + c_m^{k\ell} c_\ell^{ij} + c_m^{j\ell} c_\ell^{ki}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}$$

□

Corollary 2.3.3 *A bivector field* $\pi = \frac{1}{2} x^k c_k^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ *for* $c_k^{ij} \in \mathbb{R}$ *is Poisson, iff the map*

$$V^* \times V^* \ni (\alpha, \beta) \mapsto \alpha_i \beta_j c_k^{ij} e^k \in V^*$$

is a Lie bracket.

Note that this corollary shows that linear Poisson structures and finite dimensional real Lie algebras are in one-to-one correspondence given by Corollary 2.3.3:

$$\Big\{ \text{linear Poisson structures on } V \Big\} \stackrel{1:1}{\longleftrightarrow} \Big\{ \text{Lie algebra structures on } V^* \Big\}.$$

For a real finitie dimensional Lie algebra $\mathfrak{g}$, we call the associated Poisson structure the KKS-Poisson structure (Kostant-Kirillov-Souriau).

2.3.2 Symplectic Manifolds

Definition 2.3.4 *A symplectic manifold* (M, ω) *is a manifold endowed with a 2-form* $\omega \in \Omega^2(M)$, *such that*

(a) $\mathrm{d}\omega = 0$ *and*

(b) ω *is non-degenerate, i.e.* $\omega^\flat \colon TM \ni v_p \mapsto \omega(p)(v_p, \cdot) \in T^*M$ *is bijective.*

Even though symplectic manifolds play an important role in many branches in math, we will only consider them as particularly nice Poisson manifolds. So let us show that we can define a canonical Poisson bracket: First we define a vector field H_f for each function $f \in \mathscr{C}^\infty(M)$ which is uniquely determined by the equation

$$\iota_{H_f} \omega = \mathrm{d}f, \tag{2.3.1}$$

the we set

$$\{f, g\}_\omega := H_g(f) \tag{2.3.2}$$

for all $f, g \in \mathscr{C}^\infty(M)$.

Proposition 2.3.5 *Let (M, ω) be a symplectic manifold and let $\{\,\cdot\,,\,\cdot\,\}_\omega$ be defined as in Equation* (2.3.2)*, then $\{\,\cdot\,,\,\cdot\,\}_\omega$ is a Poisson bracket. Moreover, the vector fields defined in Equation* (2.3.1) *are exactly the Hamiltonian vector fields of $\{\,\cdot\,,\,\cdot\,\}_\omega$ and we have that*

$$\pi^\sharp \circ \omega^\flat = \mathrm{id}_{TM} \quad \text{as well as } \omega^\flat \circ \pi^\sharp = \mathrm{id}_{T^*M}$$

for the Poisson bivector field π of $\{\,\cdot\,,\,\cdot\,\}_\omega$.

PROOF: We begin showing that $\{\,\cdot\,,\,\cdot\,\}_\omega$ is a Poisson bracket. So let $f, g, h \in \mathscr{C}^\infty(M)$

- $\{f, g\}_\omega = H_g(f) = \mathrm{d}f(H_g) = \iota_{H_f}\omega(H_g) = \omega(H_f, H_g) = -\omega(H_g, H_f) = -\{g, f\}_\omega$.
- $\{f, gh\}_\omega = -H_f(gh) = -H_f(g)h - gH_f(h) = \{f, g\}_\omega h + g\{f, h\}_\omega$.
- We first proof that $[H_f, H_g] = -H_{\{f,g\}_\omega}$:

$$\begin{aligned}
\iota_{[H_f,H_g]}\omega &= [\mathscr{L}_{H_f}, \iota_{H_g}]\omega = \mathscr{L}_{H_f}\iota_{H_g}\omega - \iota_{H_g}\mathscr{L}_{H_f}\omega \\
&= \mathscr{L}_{H_f}\,\mathrm{d}g - \iota_{H_g}\iota_{H_f}\,\mathrm{d}\omega - \iota_{H_g}\,\mathrm{d}\iota_{H_f}\omega \\
&= \mathrm{d}\mathscr{L}_{H_f}g - 0 - \iota_{H_g}\,\mathrm{d}^2 f \\
&= -\,\mathrm{d}\{f, g\}_\omega,
\end{aligned}$$

  where we used that $[\mathscr{L}_X, \iota_Y] = \iota_{[X,Y]}$ holds for all vector fields $X, Y \in \mathfrak{X}(M)$ and Cartan's magic formula. Now we have

$$\begin{aligned}
\{f, \{g, h\}_\omega\}_\omega &= H_{\{g,h\}_\omega}(f) = -[H_g, H_h](f) = -H_g(H_h(f)) + H_h(H_g(f)) \\
&= -\{\{f, h\}_\omega, g\}_\omega + \{\{f, g\}_\omega, h\}_\omega = \{g, \{f, h\}_\omega\}_\omega + \{\{f, g\}_\omega, h\}_\omega.
\end{aligned}$$

  It follows now immediately, that $H_f = X_f$ for all $f \in \mathscr{C}^\infty(M)$, since $X_f(g) = \{g, f\}_\omega = H_f(g)$ for all $g \in \mathscr{C}^\infty(M)$. Moreocer, we have that

$$\mathrm{d}f = \omega^\flat(X_f) = \omega^\flat(\pi^\sharp(\mathrm{d}f))$$

  and since the differentials of functions span the cotangent space, we have the claim. □

Moreover, if we define non-degenerate Poisson bivector field by requiring that the map $\pi^\sharp \colon T^*M \to TM$ is bijective, then we have

$$\Big\{\text{non-degenerate Poisson bivector fields on } M\Big\} \stackrel{1:1}{\longleftrightarrow} \Big\{\text{ symplectic 2-forms on } M\Big\}.$$

We can see now Poisson brackets as a generalization of symplectic structures allowing them to be *singular*.

Let us check in this case how the Casimir functions and the Poisson vector fields behave. In fact, there are very few Casimir functions:

Lemma 2.3.6 *Let (M, ω) be a symplectic manifold. Then*

$$X_f = 0 \quad \Longleftrightarrow \quad f \textit{ is locally constant}$$

PROOF: We have

$$X_f = 0 \quad \stackrel{\omega \text{ is sympl.}}{\Longleftrightarrow} \quad \mathrm{d}f = 0 \quad \Longleftrightarrow \quad f \text{ is locally constant.}$$ □

Note that however the converse statement is not true: we consider

$$\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

as a Poisson structure on $\mathbb{R}^2$. For $\mathscr{C}^\infty(\mathbb{R}^2)$, we have

$$X_f = x\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - x\frac{\partial f}{\partial x}\frac{\partial}{\partial y},$$

and so $X_f = 0$ is equivalent to $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$ and so f has to be constant. On the other hand π is clearly not symplectic.

We also have a nice interpretation of the Poisson vector fields:

Lemma 2.3.7 *Let (M, ω) be a symplectic manifold with Poisson bivector field π. Then*

(a) $\mathscr{L}_X\omega = 0$ iff X is a Poisson vector field

(b) the form $\omega^\flat(X)$ is closed iff X is a Poisson vector field.

(c) the vector field $\pi^\sharp(\alpha)$ is Poisson iff α is closed.

Proof: Exercise. □

Now we want to find examples for symplectic manifolds, which are by the above discussion also Poisson manifolds. Luckily enough there are a lot examples ranging from rather abstract constructions to concrete examples.

Example 2.3.8 We consider $S^2 \subseteq \mathbb{R}^3$ and view the tangent bundle TS^2 as a subbundle of the trivial bundle $T_S\mathbb{R}^3 = S \times \mathbb{R}^3$. We define $\omega \in \Omega^2(S^2)$ pointwise by

$$\omega(p)(v, w) = \langle p, v \times w\rangle$$

This form is closed by dimensional arguments and non-degenerate which can be shown by a tiny computation.

In fact there are two possible generalizations of this example: The first one comes from the observation that $S^2 \simeq \mathbb{C}P^1$ and in fact every complex projective space $\mathbb{C}P^n$ admits a symplectic structure, the so-called Fubini-Study form (Exercise!). The other possble way of generalizng is to go to arbitrary orientable 2-dimensional manifolds:

Lemma 2.3.9 *Every orientable two dimensional manifold admits a symplectic structure.*

Proof: Let M be an orientable 2-dimensional manifolds, then there exists a volume form $\omega \in \Omega^2(M)$. This form is closed by dimensional reasons (i.e. $d\omega \in \Omega^3(M) = 0$) and since it is a volume form it is also symplectic. □

But also beyond two dimension there are plenty of examples. Let us consider an arbitrary manifold M and its cotangent bundle $\pi\colon T^*M \to M$ and let us define the *canonical 1-form* $\theta \in \Omega^1(T^*M)$ by

$$\theta_{\alpha_x}(v_{\alpha_x}) = \alpha_x(T_{\alpha_x}\pi(v_{\alpha_x}))$$

Lemma 2.3.10 *Let (U, q) be a chart of M and (T^*U, q, p) the induced chart of of T^*M, then*

$$\theta\big|_{T^*U} = p_i\, dq^i.$$

PROOF: In general, we know that there exist functions $\alpha_i, \beta^j \in \mathscr{C}^\infty(T^*U)$, such that $\theta\big|_{T^*U} = \alpha_i \, \mathrm{d}q^i + \beta^j \, \mathrm{d}p_j$. In the induced chart the projection π is particualrly easy: $\pi(q,p) = q$ and hence $T\pi(\frac{\partial}{\partial q^i}) = \frac{\partial}{\partial q^i}$ and $T\pi(\frac{\partial}{\partial p_j}) = 0$ for all i, j. This implies that

$$\theta_{(q,p)}(\frac{\partial}{\partial q^i}) = p_i \quad \text{and} \quad \theta_{(q,p)}(\frac{\partial}{\partial p_j}) = 0.$$

If we now compare the coefficients, we see that $\alpha_i = p_i$ and $\beta_j = 0$ for all i, j. □

Proposition 2.3.11 *Let M be a manifold. The two form $\omega_0 \in \Omega^2(T^*M)$ defined by $\omega_0 = -\mathrm{d}\theta$ is symplectic. ω_0 is called the canonical symplectic structure on T^*M.*

PROOF: ω_0 is obviously closed, since $\mathrm{d}^2 = 0$. Using the the local formula from Lemma 2.3.10, we see that $\omega_0\big|_{T^*U} = \mathrm{d}q^i \wedge \mathrm{d}p_i$ and hence it is non-degenerate. □

Remark 2.3.12 Note that symplectic manifolds are by themselves interesting objects to study not only from the geometric, but also from the topological point of view, since the existence of a symplectic structure on a (compact) manifold induces topological constraints on the manifold itself (not only the dimension). There are "easy" arguments that S^2 is the only even dimesnional sphere that admits a symplectic structure.

2.4 Poisson Submanifolds, the Weinstein Splitting Theorem and Symplectic foliations

In this section we will see that symplectic manifolds can be seen as the smallest building blocks of Poisson manifolds, i.e. every Poisson manifold is made up of symplectic manifolds which nicely glued together (we will give this sentence sense throughout this section). In order to show this, we first have to improve Proposition 2.1.6 to a more fundamental local structure theorem:

Theorem 2.4.1 (Weinstein Splitting theorem) *Let (M, π) be a Poisson manifold and let $x \in M$, then there exist local coordinates $(q^1, \dots, q^k, p_1, \dots, p_k, y^1 \dots, y^s)$ centered around x, such that*

$$\pi\big|_U = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\varphi^{ab}(y)\frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b},$$

where $k = \mathrm{rank}(\pi)_x$ and $\varphi^{kl}(y)$ are functions only depending on y with $\varphi^{kl}(0) = 0$.

In order to proof that theorem, we need a standard lemma about commuting vector fields.

Lemma 2.4.2 *Let M be a manifold, and let $V_1, \dots, V_N \in \mathfrak{X}(M)$ be pairwise commuting vector fields, such that $V_1(p), \dots, V_N(p)$ are linear independent for a point $p \in M$, then there exist local coordinates (U, x) around p, such that*

$$V_i\big|_U = \frac{\partial}{\partial x^i}$$

for $i \in \{1, \dots, N\}$.

PROOF: Exercise. □

Proof (of Theorem 2.4.1): Let $x \in M$ and let $k = \text{rank}(\pi)_x > 0$, since otherwise there is nothing to show. We can assume that we have shown the statement for all Poisson structures with $\text{rank}(\pi) < k$. Since the rank is bigger than 0, we can find a Function $p \in \mathscr{C}^\infty(M)$, such that $X_p(x) \neq 0$. With Lemma 2.4.2, we can find local coordinates (U, x) such that

$$X_p\big|_U = \frac{\partial}{\partial x^1}.$$

We set $q = x^1$ and see that

$$X_p(p) = 0, \quad X_q(q) = 0, \quad X_p(q) = 1 \quad \text{and} \quad X_q(p) = -1,$$

which means in particular that $X_p and X_q$ are linearly independent. Moreover, we see that

$$[X_p, X_q] = -X_{\{p,q\}} = -X_{-1} = 0$$

and we can find, by Lemma 2.4.2, new coordinates $(y_1, \ldots, y_n)$, such that

$$X_q = \frac{\partial}{\partial y^1} \quad \text{and} \quad X_p = \frac{\partial}{\partial y^2}.$$

Since

$$\begin{aligned} \mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}y^3 \wedge \cdots \wedge \mathrm{d}y^n &= \Big(\frac{\partial q}{\partial y^1}\frac{\partial p}{\partial y^2} - \frac{\partial p}{\partial y^1}\frac{\partial q}{\partial y^2}\Big)\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n \\ &= \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \end{aligned}$$

we see that $(q, p, y^3, \ldots, y^n)$ is also a coordinate chart. Moreover, we have $\{q, p\} = 1$ and $\{p, y^i\} = \{q, y^i\} = 0$ for all $3 \leq i \leq n$, which means in particular:

$$\pi\big|_U = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} + \frac{1}{2} \sum_{m,n=3}^{s} \tilde{\pi}^{mn}(p, q, y) \frac{\partial}{\partial y^m} \wedge \frac{\partial}{\partial y^n}.$$

Since $\tilde{\pi}^{mn} = \{y^m, y^n\}$, we have that

$$\frac{\partial \tilde{\pi}^{mn}}{\partial p} = \{q, \{y^m, y^n\}\} = \{\{q, y^m\}, y^n\} + \{y^m, \{q, y^n\}\} = 0$$

And hence the functions $\tilde{\pi}^{mn}$ do not depend on the coordinate p. With a similar computation, one sees that they also do not depend on q. One sees that $\tilde{\pi} := \frac{1}{2}\sum_{m,n=3}^{s} \tilde{\pi}^{mn}(y) \frac{\partial}{\partial y^m} \wedge \frac{\partial}{\partial y^n}$ is a Poisson structure as well and having rank strictly smaller than k at x. So inductively we can proceed to get the claim. □

Corollary 2.4.3 (Darboux Theorem) *Let (M, ω) by a symplectic manifold of dimension $2n$, then around every point $x \in M$ there exists coordinates $(U, \{q^1, \ldots, q^n, p_1, \ldots, p_n\})$, such that*

$$\omega\big|_U = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

In this section we are dealing with submanifolds, which means for us always *immersed submanifolds*. If the manifold happens to be embedded, we will always refer to it as embedded submanifold.

Definition 2.4.4 *Let (M, π) be a submanifold. A Poisson submanifold is a is a submaniold $\iota\colon C \hookrightarrow M$ together with a Poisson bivector field $\pi_C \in \mathfrak{X}^2(C)$, such that ι is a Poisson map.*

One special case of Poisson submanifold are level sets of Casimir functions:

Lemma 2.4.5 *Let (M, π) be a Poisson manifold and let $f_i \in \mathrm{H}^0_\pi(M)$ be Casimir functions for $i \in \{1, \dots, N\}$ such that $F = (f_1, \dots, f_N)\colon M \to \mathbb{R}^N$ has regular value 0. Then there is a unique Poisson structure π_C on $C = F^{-1}(\{0\})$, such that (C, π_C) is a(n embedded) Poisson submanifold of (M, π).*

PROOF: First of all, it is clear that C is an embedded submanifold, since $0 \in \mathbb{R}$ is a regular value of F. Let $g, h \in \mathscr{C}^\infty(M)$ be two functions, then we define

$$\{g, h\}_C = \iota^*\{\tilde{g}, \tilde{h}\},$$

where $\tilde{g}, \tilde{h} \in \mathscr{C}^\infty(M)$, such that $\iota^*\tilde{f} = f, \iota^*\tilde{g} = g$. Let us now check that $\{\,\cdot\,,\,\cdot\,\}_C$ is well-defined, therefore it is enough to show that for a function $\tilde{g} \in \mathscr{C}^\infty(M)$ with $\iota^*\tilde{g} = 0$, we also have $\iota^*\{\tilde{g}, \tilde{h}\} = 0$ for all $\tilde{h} \in \mathscr{C}^\infty(M)$. Since $\iota^* g = 0$ we can always find a smooth functions $h^i \in \mathscr{C}^\infty(M)$ for $i \in \{1, \dots, N\}$, such that $\tilde{g} = f_i h^i$, then we see $X_{\tilde{g}} = f_i X_{h^i} + h^i X_{f_i} = f_i X_{h^i}$ which is a vector field vanishing at every point $c \in C \subseteq M$, and moreover it follows by definition, that $\iota\colon C \to M$ is a Poisson map. The Jacobi identity and that $\{\,\cdot\,,\,\cdot\,\}_C$ is a biderivation follows trivially from the one from $\{\,\cdot\,,\,\cdot\,\}$. ☐

In the following we will try to divide a Poisson manifolds into special submanifolds. In the previous Lemma we could see that regular level sets of Casimir functions are indeed a first approximation to that, but we can improve this division drastically. To do so, we introduce so-called distributions.

Definition 2.4.6 *Let M be a manifold. A smooth distribution on M is a subset $D \subseteq TM$, such that*

(a) for each $p \in M$, the set $D_p := D \cap T_pM$ is a subvector space, in particular $D_p \neq \emptyset$.

(b) for $\Gamma^\infty(D) := \{X \in \mathfrak{X}(M) \mid X(p) \in D_p\}$, we have that for each point there exists a open neighbourhood U and $X_1 \dots, X_k \in \Gamma^\infty(D\big|_U)$, such that that $D_y = \operatorname{span}\{X_i(y)\}_{1 \leq i \leq k}$ for all $y \in U$.

Moreover, a smooth distribution $D \subseteq TM$ is called,

- *regular, if $\dim(D_p) = const.$*
- *involutive, if there exists a set of local sections $\mathcal{D} \subset \Gamma^\infty_{loc}(D)$, such that*

$$D_p = \{X(p) \in T_pM \mid X \in \mathcal{D}\} \quad \text{and } [\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}$$

whenever defined.

Corollary 2.4.7 *A smooth distribution $D \subseteq TM$ is regular if and only if it is a subbundle.*

PROOF: Exercise. ☐

Note that not every distribution is regular, in fact the distributions which are induced by Poisson manifolds are usually not. If a distribution is not regular, one can divide it into two disjoint sets:

Definition 2.4.8 *Let $E \subseteq TM$ be a smooth distribution. A point $p \in M$ is called*

(a) regular, if there exists an open neighbourhood U of p, such that $\dim(D_p) = \dim(D_y)$ for all $y \in U$.

(b) singular, if it is not regular.

Proposition 2.4.9 *Let $D \subseteq TM$ be a smooth distribution. Then* *(a) the map* $p \mapsto \dim(D_p)$ *is lower semi-continuous.*

(b) the set of regular points is open and dense.

Proof: Let $p \in M$, then we find an open neighbourhood U of p and $X_1, \dots, X_k \in \Gamma^\infty(D\big|_U)$, such that $D_y = \operatorname{span}\{X_1(y), \dots, X_k(y)\}$. Moreover, we can find a subset $\{X_{i_1}, \dots, X_{i_l}\}$ of $\{X_1, \dots, X_k\}$, such that $\{X_{i_1}(p), \dots, X_{i_l}(p)\}$ is a basis of D_p. Since $\{X_{i_1}(p), \dots, X_{i_l}(p)\}$ is linearly independent at p, there is an open neighbourhood $V \subset U$ of p, where they are still linear independent, because of continuity. In particular, at very point y in V they span a subspace of D_y of dimension $\dim(D_p)$.

For point (b) it is enough to show that every open neighbourhood U of a point p contains a regular point. We define

$$R = \{\dim(D_y) \mid y \in U\},$$

then clearly we have that $R \leq \dim(M)$ hand hence there is a maximum m of R and let $x \in U$ be chosen such that $\dim(D_x) = m$. The fact that x is a regular point follows now with (a), since we can find an open neighbourhood $V \subseteq U$ of x, such that $\dim(D_x) \leq \dim(D_y)$ for all $y \in V$, but

$$m = \dim(D_x) \leq \dim(D_y) \leq m$$

for all $y \in V$ and the claim is proven. □

Before going on with distributions, let us interrupt this with some Poisson geometry:

Theorem 2.4.10 *Let* (M, π) *be a Poisson manifold. Then* $\operatorname{im} \pi^\sharp \subseteq TM$ *is an involutive distribution.*

Proof: As the image of a vector bundle map, we clearly have that $\operatorname{im} \pi^\sharp_p \subseteq T_pM$ is a vector subspace. Moreover, we can choose $\mathcal{D} = \{X_f \in \Gamma^\infty(\operatorname{im} \pi^\sharp) \mid f \in \mathscr{C}^\infty(M)\}$. This is a set of sections for which we have

$$\operatorname{im}(\pi^\sharp)_p = \{X(p) \in T_pM \mid X \in \mathcal{D}\} \quad \text{and } [\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}$$

since for all f, g, we have $[X_f, X_g] = X_{-\{f,g\}} \in \mathcal{D}$ and Hamiltonian vector fields span $\operatorname{im}(\pi^\sharp)$ at every point and hence $\operatorname{im}(\pi^\sharp)$ is involutive. □

In fact, among involutive distributions there are even nicer ones, namely integrable ones.

Definition 2.4.11 *A distribution* $D \subseteq TM$ *is called integrable, if for each point* $p \in M$ *there exists a submanifold* $\iota \colon N \hookrightarrow M$*, such that* $p \in \iota(N)$ *and*

$$T_q\iota(T_qN) = D_{\iota(q)}$$

for all $q \in N$*. A submanifold fulfilling this property is called integral submanifold.*

Note that it clear, that every integrable distribution is involutive. In fact, we can choose all the sections as $\mathcal{E}$ (from Definition 2.4.6). The converse is however not true in general: one has to impose another condition on the distribution in order to get an equivalence. This is what is called *Sefan-Sussmann-Distributions.* This additional condition is always fulfilled, if the distribution is regular.

What follows now are some rather technical facts about integral submanifolds, in fact we want to discuss the global nature of integral submanifolds inside M. One of the difficulties is, that in general integral submanifolds are only injectively immersed, but in fact they behave slightly nicer than just that.

Lemma 2.4.12 *Let $D \subseteq TM$ be a smooth distribution and let $\iota_j \colon N_j \to M$ be integral submanifolds for $j = 1, 2$ of D. Then $\iota_j^{-1}(\iota_1(N_1) \cap \iota_2(N_2))$ is open and if $\iota_1(N_1) \cap \iota_2(N_2)$ is non-empty, then*

$$\iota_2^{-1} \circ \iota\big|_{\iota_1(N_1)\cap\iota_2(N_2)} \colon \iota_1^{-1}(\iota_1(N_1) \cap \iota_2(N_2)) \to \iota_2^{-1}(\iota_1(N_1) \cap \iota_2(N_2))$$

is a diffeomorphism.

PROOF: We assume from the beginning that $\iota_1(N_1) \cap \iota_2(N_2) \neq \emptyset$, since otherwise the statement is trivial. Thus, let $p \in \iota_1(N_1) \cap \iota_2(N_2)$, then there exists an open neighbourhood U of p and vector fields $X_1, \dots, X_k \in \Gamma^\infty(D\big|_U)$, such that they are pointwise linear independent and $k = \dim(D_p)$. Since the N_j are integral submanifolds, we find $X_1^j, \dots, X_k^j \in \mathfrak{X}(\iota_j^{-1}(U))$ such that

$$X_i^j \sim_{\iota_j} X_i.$$

Let us now define the map $\Phi_j \colon V_j \to N_j$ for $0 \in V_j \subseteq \mathbb{R}^k$ by

$$\Phi(t_1, \dots, t_k) = \Phi_{t_1}^{X_1^j} \circ \dots \circ \Phi_{t_k}^{X_k^j}(p_j),$$

where we denote by $p_j \in N_j$ the unique points, such that $\iota_j(p_j) = p$ and by $\Phi_t^{X_i^j}$ the flows of X_i^j. Moreover, V_j is a small enough neighbourhood, such that Φ_j is defined. We have that

$$\begin{aligned} T\Phi_j\Big(\frac{\partial}{\partial t_i}\Big|_0\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_j(0, \dots, t, 0, \dots, 0) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{X_i^j}(p_j) \\ &= X_i^j(p_j) \end{aligned}$$

and hence Φ_j is a local diffeomorphism around 0. Using now that $X_i^j \sim_{\iota_j} X_i$, we know that we have for the flows $\iota_j \circ \Phi_t^{X_i^j} = \Phi_t^{X_i} \circ \iota_j$ whenever defined. It follows that

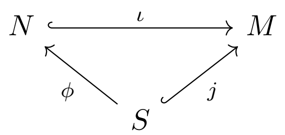
$$\begin{aligned} \iota_1(\Phi_1(t_1, \dots, t_k)) &= \iota_1(\Phi_{t_1}^{X_1^1} \circ \dots \circ \Phi_{t_k}^{X_k^1}(p_1)) \\ &= \Phi_{t_1}^{X_1} \circ \dots \circ \Phi_{t_k}^{X_k}(p) \\ &= \iota_2(\Phi_2(t_1, \dots, t_k)), \end{aligned}$$

and hence on $V = V_1 \cap V_2$ we have that $\iota_1 \circ \Phi_1 = \iota_2 \circ \Phi_2$ is an embedding. This means in particular, that $\Phi_j(V)$ is an open neighbourhood of $p_j \in N_j$, such that $\Phi_j(V) \subseteq \iota_j^{-1}(\iota_1(N_1) \cap \iota_2(N_2))$ and since p was arbitrary, we have that the subsets $\iota_j^{-1}(\iota_1(N_1) \cap \iota_2(N_2))$ are open.

Moreover it follows that $\iota_2^{-1} \circ \iota_1 \circ \Phi_1 = \Phi_2$ and hence $\iota_2^{-1} \circ \iota_1$ is a diffeomorphism around $p_1 = \iota_1^{-1}(p)$. A bijective local diffeoemorphism is a always a diffeomorphism and the claim is proven. □

The preceding lemma is the key to see that we can define a *maximal* integral submanifold through a point.

Theorem 2.4.13 *Let $D \subset TM$ be a smooth distribution, such that there exists an integral submanifold through $p \in M$. Then there is a unique maximal connected integral submanifold $\iota \colon N \hookrightarrow M$, i.e. for all integral submanifolds $j \colon S \to M$ through p, there exists an open embedding $\phi \colon S \to N$, such that*



commutes.

Proof: First we define $N \subseteq M$ to be the subset of points which are contained in a connected integral submanifold through p. We define now a topology on N by declaring $\iota_\alpha(U_\alpha) \subseteq N$ for $U_\alpha \subseteq N_\alpha$ to be a basis for this topology. Note that this is in general finer than the subspace topology and N is by definition connected and Hausdorff. Moreover, using charts for N_α we obtain charts for N which are indeed an atlas by Lemma 2.4.12. Furthermore, $\iota\colon N \to M$ is smooth and an immersion by construction. The last, but most subtle point is the second countability of N. This follows from a theorem which states that if the codomain of an immersion is second countable, then the domain as well, if it is connected. As a last point we notice that by the very definition of the topology of N, that every $\iota_\alpha\colon N_\alpha \to N$ is an open embedding. □

Using this we can apply our new knowledge to Poisson geometry:

Theorem 2.4.14 *Let $(M.\pi)$ be a Poisson manifold.*

(a) The distribution $\operatorname{im}\pi^\sharp$ is integrable.

(b) Each maximal integral submanifold $\iota\colon L \to M$ has a unique symplectic structure ω_L, such that ι is Poisson map.

Proof: Let $p \in M$ be arbitrary such that $\operatorname{rank}(\pi)_p$ using Theorem 2.4.1, we can find a local chart U with coordinates $(q^1, \dots, q^k, p_1 \dots, p_k)$ such that

$$\pi\big|_U = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\varphi^{ab}(y)\frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b},$$

where $\varphi^{kl}(y)$ are functions only depending on y with $\varphi^{kl}(0) = 0$. We use the embedding $\iota(q,p) = (q,p,0)$ (restricted to opens). It is now clear that ι defines an integral submanifold and hence $\operatorname{im}\pi^\sharp$ is integrable.

Let now $\iota\colon L \to M$ be a maximal integral submanifold. And let $f, g \in \mathscr{C}^\infty(L)$ and let $x \in L$, then there exist open neighbourhoods $U \subseteq L$ and $V \subseteq M$, such that $\iota\big|_U\colon U \to V$ is an embedding and hence there there exist $\tilde{f}, \tilde{g} \in \mathscr{C}^\infty(V)$, such that $(\iota\big|_U)^*\tilde{f} = f\big|_U$ and $(\iota\big|_U)^*\tilde{g} = g\big|_U$. We define

$$\{f, g\}_L(x) = \{\tilde{f}, \tilde{g}\}_V(\iota(x)).$$

To check that this is well defined we have to check that if for a function $\tilde{f}$, such that $\iota^* f = 0$ we have $\{\tilde{f}, \tilde{g}\}_V = 0$. This follows from the fact that $X_{\tilde{g}} \in \Gamma^\infty(\operatorname{im}\pi^\sharp\big|_V)$ and hence there exists a vector field $Y \in \mathfrak{X}(U)$ such that $Y \sim_\iota X_{\tilde{g}}$, which implies that

$$\{\tilde{f}, \tilde{g}\}_V(\iota(x)) = \iota^*(X_{\tilde{g}}(\tilde{f}))(x) = Y(\iota^*\tilde{f})(x) = 0,$$

and hence $\{\,\cdot\,,\,\cdot\,\}_L$ is well defined. It is now an easy exercise to see the Jacobi identity, skew symmetry and the derivation property. Moreover we can check the non-degeneracy of $\{\,\cdot\,,\,\cdot\,\}$ in a chart from Theorem 2.4.1 easily. □

So in some sense, we can understand a Poisson manifold as a collection of symplectic submanifolds, such that the symplectic structures glue smoothly in a sense. A maximal integral submanifold is called *symplectic leaf* and the collection of all symplectic leaves is called the symplectic foliation of the Poisson manifold.

Let us discuss some examples. We start with the easiest:

Example 2.4.15 Let (M, ω) be a symplectic manifold then the corresponding symplectic leaves are the connected components of M.

A bit more involved are linear Poisson structures: let us pick a connected Lie group G with Lie algebra $\mathfrak{g}$ and let π the Poisson structure on $\mathfrak{g}^*$ from Section 2.3.1. We denote by $\mathrm{Ad}^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ the coadjoint action, then the symplectic leaves are exactly the coadjoint orbits. We are going to use the following lemma

Lemma 2.4.16 *Let $\Phi\colon M \times G \to M$ be a Lie group action of a connected Lie group G with Lie algebra $\mathfrak{g}$ on a manifold M. The distribution D given by $D_m = \{\xi_M(m) \mid \xi \in \mathfrak{g}\}$ is a smooth integrable distribution. Moreover, the maximal integral submanifolds are exactly the orbits.*

PROOF: Let us fix a point $m \in M$ and edfine the map $\Phi_m\colon G \to M$ by

$$\Phi_m(g) = \Phi(m, g).$$

By definition we have that the orbit through m is the image of Φ_m. We want to show first that we can find a manifold and an injective immersion, such that the orbit is an immersed submanifold. Let us define $G_m = \{g \in G \mid \Phi(m, g) = m\}$, which is a closed Lie subgroup and hence G/G_m is a manifold, such that $p\colon G \to G_m\backslash G$ is a surjective submersion. Thus, there exists a smooth map $\phi_m\colon G_m\backslash G \to M$, such that

$$\begin{array}{ccc} G & \xrightarrow{\Phi_m} & M \\ \downarrow{\scriptstyle p} & \nearrow{\scriptstyle \phi_m} & \\ G_m\backslash G & & \end{array}$$

commutes. Let $[g], [h] \in G/G_m$ be such that $\phi_m([g]) = \phi_m([h])$, which means that $\Phi(m, g) = \Phi(m, h) \iff \Phi(h^{-1}g, m) = m$. This in turn means that $gh^{-1} \in G_m$ and hence $[h] = [gh^{-1}h] = [g]$. Let $v \in T_{[g]}G_m\backslash G$ then there exists a curve $\gamma\colon I \to G$, such that $\gamma(0) = g$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}[\gamma(t)] = 0$. Let us denote $\xi = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\gamma(t)\gamma(0)^{-1} \in \mathfrak{g}$, then

$$T_{[g]}\phi_m(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(\gamma(t), m) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(\gamma(t)\gamma(0)^{-1}, \Phi(\gamma(0), m)) = \xi_M(\Phi(g, m)) = \xi_M(\phi_m([g])),$$

which means in particular that $T\phi_m(TG_m\backslash G) = D$. The last thing to show is that ϕ_m is immersive, so let $v \in \ker T_{[g]}\phi_m$. As above we choose a path such that $\gamma\colon I \to G$, such that $\gamma(0) = g$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}[\gamma(t)] = v$. Without loss of generality we may assume that $\gamma(t) = \exp(t\xi)g$ with $\xi \in \mathfrak{g}$. We have

$$0 = T_{[g]}\phi(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(m, \exp(t\xi)g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(\Phi(m, \exp(t\xi)), g) = T_m\Phi(\,\cdot\,, g)(\xi_M(m))$$

Since $\Phi(\,\cdot\,, g)$ is diffeomorphism, we have that $\xi_M(m) = 0$. We claim now that this implies that $\exp(t\xi) \in G_m$ and in fact this follwos from the exsitence and uniqueness theorem for flows of vector fields. Therefore we have $\gamma(t) = [\exp(t\xi)g] = [g]$ and is therefore constant and hence $v = 0$. □

So we are left to show that the distribution induced by the Poisson structure π is the same as the distribution spanned by the fundemantal vector fields. Let us choose a basis $\{e_i\}_{i\in\{1,\dots,N\}}$, then we have $\pi = \frac{1}{2}x_k C^k_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ and $\xi_{\mathfrak{g}^*} = x_j\xi^k C^j_{kl}\frac{\partial}{\partial x_l}$ and hence we have $\xi_{\mathfrak{g}^*} = \pi^\sharp(-\xi^i\,\mathrm{d}x_i)$ and we get the claim.

2.5 Regular Poisson Structures

There is a big class of Poisson manifolds which behave very nicely in many ways: regular Poisson manifolds. We are mainly interested in them, since they provide a nice enough structure to quantize them rather easily, which we will see in the the next chapter.

Definition 2.5.1 *A Poisson manifold (M, π) is called regular, if* $\operatorname{rank}(\pi)$ *is constant or equivalently if its distribution is regular. Moreover, all the symplectic leaves of a regular Poisson manifold have the same dimension.*

Let us start with motivating examples:

Example 2.5.2 Every symplectic manifold is a regular Poisson manifold.

Example 2.5.3 Let $\mathfrak{so}(3)^*$ be the dual of the Lie algebra of infinitesimal rotations together with the KKS-Poisson structure. If we restrict to $\mathfrak{so}(3)^* \setminus \{0\}$, we get a regular Poisson manifold.

Recall from Corollary 2.4.7 that regularity of the Poisson tensor implies that $\mathcal{F}_\pi := \operatorname{im}(\pi^\sharp) \subseteq TM$ is a subbundle.

Remark 2.5.4 Note that since $\Gamma^\infty(\mathcal{F}_\pi) \subseteq \mathfrak{X}(M)$, we can take the commutator of two sections in $\Gamma^\infty(\mathcal{F})$ and the involutivity implies that we get back a section of $\mathcal{F}_\pi$.

Moreover, we have a short exact sequence of vector bundles and thus we can choose a splitting

$$0 \longrightarrow \ker \pi^\sharp \longrightarrow T^*M \underset{\phi}{\overset{\pi^\sharp}{\rightleftarrows}} \mathcal{F}_\pi \longrightarrow 0,$$

i.e. a vector bundle map $\phi \colon \mathcal{F}_\pi \to T^*M$, such that $\pi^\sharp \circ \phi = \mathrm{id}$ and hence $T^*M = \ker \pi^\sharp \oplus \operatorname{im} \phi$. With this map we define $\omega_\pi \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi^*)$ by

$$\omega_\pi(e, f) = \pi(\phi(e), \phi(f))$$

for all $e, f \in \mathcal{F}_\pi$. ω_π is called the foliated symplectic form associated to π. Moreover, one can show that ω is independent of ϕ (Check!).

Lemma 2.5.5 *The tensor $\omega_\pi \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi^*)$ is non-degenerate.*

Proof: Let $e \in \mathcal{F}_\pi$ such that $\omega_\pi(e, f) = 0$ for all f, then we have $\phi(f)(e) = 0$. Moreover, let $\alpha \in \ker \pi^\sharp$, then we have $\alpha(e) = \alpha(\pi^\sharp \phi(e)) = -\phi(e)(\pi^\sharp(\alpha) = 0$ and hence $e = 0$. □

Since $\mathcal{F}_\pi$ is involutive, the sections $\Gamma^\infty(\mathcal{F}_\pi)$ possess a Lie bracket, which is just the Lie bracket of the vector fields. Therefore, we can define the $\mathbb{R}$-linear map $\mathrm{d}^k_{\mathcal{F}_\pi} : \Gamma^\infty(\Lambda^k \mathcal{F}_\pi^*) \to \Gamma^\infty(\Lambda^{k+1} \mathcal{F}_\pi^*)$ by

$$\begin{aligned} \mathrm{d}_{\mathcal{F}_\pi} \omega_\pi(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^i X_i(\omega(X_1, \ldots, \overset{i}{\wedge}, \ldots, X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \overset{i}{\wedge}, \ldots, \overset{j}{\wedge}, \ldots, X_{k+1}) \end{aligned}$$

for all $\omega \in \Gamma^\infty(\Lambda^k \mathcal{F}_\pi^*)$ and $X_1, \ldots, X_{k+1} \in \Gamma^\infty(\mathcal{F}_\pi)$ and $k \geq 1$. For $k = 0$ we define

$$\Gamma^\infty(\Lambda^0 \mathcal{F}_\pi^*) = \mathscr{C}^\infty(M) \ni f \mapsto (X \mapsto X(f)) \in \Gamma^\infty(\mathcal{F}_\pi^*).$$

$\mathrm{d} = \sum_k \mathrm{d}^k \colon \Gamma^\infty(\Lambda^\bullet \mathcal{F}_\pi^*) \to \Gamma^\infty(\Lambda^{\bullet+1} \mathcal{F}_\pi^*)$ is called foliated de Rham differential. And we obtain

Lemma 2.5.6 *The de Rham differential $\mathrm{d}_{\mathcal{F}_\pi}$ has the following properties*

(a) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$ *for all* $\alpha \in \Gamma^\infty(\Lambda^k \mathcal{F}_\pi^*)$ *and* $\Gamma^\infty(\Lambda^\ell \mathcal{F}_\pi^*)$.

(b) $\mathrm{d}^2 = 0$.

Lemma 2.5.7 *The two form $\omega_\pi \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi^*)$ is closed, i.e. $\mathrm{d}_{\mathcal{F}_\pi}\omega_\pi = 0$ and we have*

$$\{f, g\} = \omega_\pi(X_f, X_g).$$

Moreover, the inverse $\omega^{-1} \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi)$ is given by π under the canonical inclusion $\iota\colon \mathcal{F}_\pi \hookrightarrow TM$.

PROOF: Let $f, g \in \mathscr{C}^\infty(M)$, then we have $X_f, X_g \in \Gamma^\infty(\mathcal{F}_\pi)$ and hence

$$\omega(X_f, X_g) = \phi(X_f)(X_g) = \pi(\phi(X_f), \mathrm{d}g) = -\,\mathrm{d}g(\pi^\sharp(\phi(X_f))) = -\,\mathrm{d}g(X_f) = \{f, g\}.$$

It follows for $f, g, h \in \mathscr{C}^\infty(M)$ that

$$\mathrm{d}_{\mathcal{F}_\pi}\omega_\pi(X_f, X_g, X_h) = 2(\{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\}) = 0,$$

because of the Jacobi identity. Since Hamiltonian vector fields span $\mathcal{F}_\pi$ at every point, the claim is proven. Let us now denote by $\omega_\pi^{-1} \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi)$ denote the tensor, such that $(\omega_\pi^{-1})^\sharp \circ \omega_\pi^\flat = \mathrm{id}_{\mathcal{F}_\pi}$, then we have for $f \in \mathscr{C}^\infty(M)$ that $\mathrm{d}f\big|_{\mathcal{F}_\pi} = \omega^\flat(X_f)$ this means

$$\mathrm{d}f \otimes \mathrm{d}g(\omega_\pi^{-1}) = \mathrm{d}f((\omega_\pi^{-1})^\sharp(\mathrm{d}g)) = \mathrm{d}f(X_g) = \{f, g\}$$

and by the definition of the Poisson bivector field. □

Corollary 2.5.8 *Let $\mathcal{F} \subseteq TM$ be a regular foliation. There is a one-to-one correspondence between foliated symplectic 2-forms $\omega \in \Gamma^\infty(\Lambda^2 \mathcal{F}^2)$ and regular Poisson structures with $\mathcal{F}_\pi = \mathcal{F}$.*

The associated foliated symplectic form has a very nice connection to the symplectic forms on the symplectic leaves:

Lemma 2.5.9 *Let (M, π) be a regular Poisson manifold, let $\omega_\pi \in \Gamma^\infty(\Lambda^2 \mathcal{F}_\pi^*)$ the associated foliated symplectic form and let $\iota\colon S \to M$ be a symplectic leaf. The symplectic structure $\omega_S \in \Omega^2(S)$ is given by*

$$\omega_S(X_p, Y_p) = \omega_\pi(T_p\iota X_p, T_p\iota Y_p)$$

PROOF: We use Theorem 2.4.14: Let $f, g \in \mathscr{C}^\infty(M)$ and let $\{\,\cdot\,,\,\cdot\,\}_S$ be the Poisson structure associated to ω_S. Then we know that

$$\iota^*\{f, g\} = \{\iota^* f, \iota^* g\}_S$$

and thus

$$\begin{aligned} \omega_S(X_{\iota^* f}, X_{\iota^* g})\big|_p &= \{\iota^* f, \iota^* g\}_S(p) = \iota^*\{f, g\}(p) = \iota^*(\omega_\pi(X_f, X_g))\big|_p \\ &= \omega_\pi(X_f(\iota(p)), X_g(\iota(p))) = \omega_\pi(T_p\iota X_{\iota^* f}, T_p\iota X_{\iota^* g}). \end{aligned}$$

Since the Hamiltonian vector fields span the tangent spaces T_pS point-wise, the claim is proven. □

This means, to some extend, we can use symplectic techniques for regular Poisson manifolds in the sense that we can just replace the tangent bundle with $\mathcal{F}_\pi$. Our last aim is to show that there is a partial connection for which the symplectic form is parallel.

Definition 2.5.10 *Let (M, π) be a regular Poisson manifold. A partial connection on $\mathcal{F}_\pi$ is a bilinear map $\nabla^\pi\colon \Gamma^\infty(\mathcal{F}_\pi) \times \Gamma^\infty(\mathcal{F}_\pi) \to \Gamma^\infty(\mathcal{F}_\pi)$, such that* *(a)* $\nabla^\pi_{fX} Y = f \nabla^\pi_X Y$

(b) $\nabla^\pi_X fy = X(f)Y + f \nabla^\pi_X Y$

(c) $\nabla^\pi_X Y - \nabla^\pi_Y X - [X, Y] = 0$

for all $X, Y \in \Gamma^\infty(\mathcal{F}_\pi)$ *and* $f \in \mathscr{C}^\infty(M)$.

Lemma 2.5.11 *There exists a partial connection on* $\mathcal{F}_\pi$.

PROOF: Let us choose a fibre metric g on $\mathcal{F}_\pi$. We define implictely

$$g(\nabla^\pi_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(X,Z))) - Z(g(X,Y)) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)\Big)$$

for $X, Y, Z \in \Gamma^\infty(\mathcal{F}_\pi)$. To prove that this operator is well-defined and has peroperties (a)-(c) is an exercise. □

Having a partial connection ∇^π, we can immediatly extend it to $\Gamma^\infty(\Lambda^\bullet \mathcal{F}_\pi)$ by demanding a Leibniz rule on $\wedge$-product:

$$\nabla^\pi_X (Y \wedge Z) = (\nabla^\pi_X Y) \wedge Z + Y \wedge (\nabla^\pi_X Z)$$

for $X, Y, Z \in \Gamma^\infty(\mathcal{F}_\pi)$. Moreover, it induces always a dual connection by demanding a Leibniz rule on insertions:

$$\nabla^\pi_X \alpha(Y) = X(\alpha(Y)) - \alpha(\nabla^\pi_X Y)$$

for $X, Y \in \Gamma^\infty(\mathcal{F}_\pi)$ and $\alpha \in \Gamma^\infty(\mathcal{F}^*_\pi)$. We also can extend it by

$$\nabla^\pi_X (\alpha \wedge \beta) = (\nabla^\pi_X \alpha) \wedge \beta + \alpha \wedge (\nabla^\pi_X \beta)$$

for $X \in \Gamma^\infty(\mathcal{F}_\pi)$ and $\alpha, \beta \in \Gamma^\infty(\mathcal{F}^*_\pi)$ to $\Gamma^\infty(\Lambda^\bullet \mathcal{F}^*_\pi)$. For a foliated k-form $\gamma \in \Gamma^\infty(\Lambda^k \mathcal{F}^*_\pi)$, we have then

$$\nabla^\pi_X \gamma(X_1, \ldots, X_k) = X(\gamma(X_1, \ldots, X_k)) - \gamma(\nabla^\pi_X X_1, X_2, \ldots, X_k) - \cdots - \gamma(X_1, \ldots, X_{k-1}, \nabla^\pi_X X_k).$$

Lemma 2.5.12 *Let* (M, π) *be a regular Poisson manifold and* $\omega_\pi \in \Gamma^\infty(\Lambda^2 \mathcal{F}^*_\pi)$ *the associated foliated symplectic form. Then there exists a partial connection* ∇^π*, such that*

$$\nabla^\pi_X \omega_\pi = 0 \quad \textit{and} \quad \nabla^\pi_X \pi = 0$$

for all $X \in \Gamma^\infty(\mathcal{F}_\pi)$.

PROOF: Let us choose any partial connection $\hat{\nabla}^\pi$, then we define implicitly

$$\omega_\pi(\nabla^\pi_X Y, Z) = \omega_\pi(\hat{\nabla}^\pi_X Y, Z) + \frac{1}{3}\nabla^\pi_X \omega_\pi(Y, Z) + \frac{1}{3}\nabla^\pi_Y \omega_\pi(X, Z)$$

for $X, Y, Z \in \Gamma^\infty(\mathcal{F}_\pi)$. We first have to check that it is in fact a partial connection: let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \Gamma^\infty(\mathcal{F}_\pi)$, then

$$\begin{aligned}\omega_\pi(\nabla^\pi_X fY, Z) &= \omega_\pi(\hat{\nabla}^\pi_X fY, Z) + \frac{1}{3}\hat{\nabla}^\pi_X \omega_\pi(fY, Z) + \frac{1}{3}\hat{\nabla}^\pi_{fY} \omega_\pi(X, Z) \\ &= \omega_\pi(X(f)Y + f\hat{\nabla}^\pi_X Y, Z) + f(\frac{1}{3}\hat{\nabla}^\pi_X \omega_\pi(Y, Z) + \frac{1}{3}\hat{\nabla}^\pi_{fY} \omega_\pi(X, Z))\end{aligned}$$

$$
\begin{aligned}
&= \omega_\pi(X(f)Y, Z) + f(\omega_\pi(\hat{\nabla}^\pi_X Y, Z) + \frac{1}{3}\hat{\nabla}^\pi_X\omega_\pi(Y, Z) + \frac{1}{3}\hat{\nabla}^\pi_{fY}\omega_\pi(X, Z)) \\
&= \omega_\pi(X(f)Y, Z) + f\omega_\pi(\nabla^\pi_X Y, Z) \\
&= \omega_\pi(X(f)Y + f\nabla^\pi_X Y, Z).
\end{aligned}
$$

Since theis equality holds for all Z and ω is non-degenerate, it follows that $\nabla^\pi_X fY = X(f)Y + f\nabla^\pi_X Y$, similarly we can see that $\nabla^\pi_{fX} Y = f\nabla^\pi_X Y$. Let again be $X, Y, Z \in \Gamma^\infty(\mathcal{F}_\pi)$ given, then

$$
\begin{aligned}
\omega_\pi(\nabla^\pi_X Y - \nabla^\pi_Y X, Z) &= \omega_\pi(\nabla^\pi_X Y, Z) - \omega_\pi(\nabla^\pi_Y X, Z) = \omega_\pi(\hat{\nabla}^\pi_X Y - \hat{\nabla}^\pi_Y X, Z) \\
&= \omega_\pi([X, Y], Z)
\end{aligned}
$$

and henceforth also $\nabla^\pi_X Y - \nabla^\pi_Y X = [X, Y]$. The most important property is that $\nabla^\pi_X \omega_\pi = 0$, which follows since

$$
\begin{aligned}
\nabla^\pi_X\omega_\pi(X, Y) &= X(\omega_\pi(Y, Z)) - \omega_\pi(\nabla^\pi_X Y, Z) - \omega_\pi(Y, \nabla^\pi_X Z) \\
&= X(\omega_\pi(Y, Z)) - \omega_\pi(\hat{\nabla}^\pi_X Y, Z) - \frac{1}{3}\hat{\nabla}^\pi_X\omega_\pi(Y, Z) - \frac{1}{3}\hat{\nabla}^\pi_Y\omega_\pi(X, Z) + \omega_\pi(\hat{\nabla}^\pi_X Z, Y) \\
&+ \frac{1}{3}\hat{\nabla}^\pi_X\omega_\pi(Z, Y) + \frac{1}{3}\hat{\nabla}^\pi_Z\omega_\pi(X, Y) \\
&= \nabla^\pi_X\omega_\pi(Y, Z) - \frac{1}{3}\hat{\nabla}^\pi_X\omega_\pi(Y, Z) - \frac{1}{3}\hat{\nabla}^\pi_Y\omega_\pi(X, Z) \\
&+ \frac{1}{3}\hat{\nabla}^\pi_X\omega_\pi(Z, Y) + \frac{1}{3}\hat{\nabla}^\pi_Z\omega_\pi(X, Y) \\
&= \frac{1}{3}(\nabla^\pi_X\omega_\pi(Y, Z) - \hat{\nabla}^\pi_Y\omega_\pi(X, Z) + \hat{\nabla}^\pi_Z\omega_\pi(X, Y)) \\
&= \frac{1}{3}(\mathrm{d}_{\mathcal{F}_\pi}\omega_\pi(X, Y, Z)) \\
&= 0.
\end{aligned}
$$

The last claim, $\nabla^\pi_X \pi = 0$, follows directly from $\nabla^\pi_X \omega_\pi = 0$ and $\pi = \omega_\pi^{-1}$. □

2.6 Lie group actions, moment maps and phase space reduction

Throughout the whole section we assume that all Lie group actions are free and proper, which means that a reasonable quotient space (as a manifold) exists.

Definition 2.6.1 *Let (M, π) be a Poisson manifold and let $\Phi\colon M \times G \to M$ be a Lie group action. The action is called Poisson, if the map $\Phi_g\colon M \ni m \mapsto \Phi(m, g)$ is a Poisson map for all $g \in G$.*

Lemma 2.6.2 *Let $\Phi\colon M \times G \to M$ be a Poisson action on (M, π). Then there exists a unique Poisson structure π_G on M/G such that the canonical projection is $p\colon M \to M/G$ is a Poisson map.*

PROOF: We can identify $\mathscr{C}^\infty(M/G) = \mathscr{C}^\infty(M)^G := \{f \in \mathscr{C}^\infty(M) \mid \Phi_g^* f = f \text{ for all } g \in G\}$ and thus we get that for $f, h \in \mathscr{C}^\infty(M)^G$

$$
\Phi_g^*\{f, h\} = \{\Phi_g^* f, \Phi_h^* h\} = \{f, h\}.
$$

This we get a Poisson bracket on $\mathscr{C}^\infty(M/G)$. Note that $p^*\colon \mathscr{C}^\infty(M/G) \to \mathscr{C}^\infty(M)$ conicides with with the identification $\mathscr{C}^\infty(M/G) = \mathscr{C}^\infty(M)^G$. □

Note that one can say very few about the structure of π_G even having a full knowledge of π, which is illustrated by the following example

$$R\colon G \times G \ni (h, g) \mapsto hg \in G$$

is a Lie group action. By exercise 4.3 we see that for all $g \in G$ the map $T_*R_g\colon T^*G \to T^*G$ is a symplectic map and hence a Poisson map (since it is a diffeomorphism). Moreover, we have

$$T_*R_e = T_* \operatorname{id}_G = \operatorname{id}_{T^*G}$$

and

$$\begin{aligned} T_*R_g \circ T_*R_h(\alpha_k) &= T_*R_h(\alpha_k) \circ T_{khg}R_{g^{-1}} = \alpha_k \circ T_{kh}R_{h^{-1}} \circ T_{khg}R_{g^{-1}} \\ &= \alpha_k \circ T_{khg}R_{(hg)^{-1}} = T_*R_{hg}(\alpha_k). \end{aligned}$$

and hence T_*R defines a Lie group action (which is free and proper). One can show (Exercise!), that

$$p\colon T^*G \to T^*G/G \cong \mathfrak{g}^*$$

is a Poisson map with respect to the KKS-Poisson structure on $\mathfrak{g}^*$. This means in turn that even starting with a symplectic structure, one can arrive by taking quotients to a Poisson structure with non-constant rank. Also from the physical point of view this does not make too much sense, since if we start with symmetry of the configuration space and our phase space is the cotangent of the latter, we want to arrive the cotangent bundle of a new configuration space.

Definition 2.6.3 *Let (M, π) be a Poisson manifold and let $\Phi\colon M \times G \to M$ be a Poisson action. A map $J\colon M \to \mathfrak{g}^*$ is called moment map, if*

(a) $J \circ \Phi_g = \operatorname{Ad}_g^* \circ J$ for all $g \in G$.

(b) $\xi_M = X_{\hat{J}(\xi)}$, for all $x \in \mathfrak{g}$ where $\hat{J}(\xi) \in \mathscr{C}^\infty(M)$ is given by $\hat{J}(\xi)(p) = J(p)(\xi)$.

An Poisson action admitting a moment map is called Hamiltonian.

Lemma 2.6.4 *Let (M, π) be a Poisson manifold and let $\Phi\colon M \times G \to M$ be a Hamiltonian action with moment map $J\colon M \to \mathfrak{g}^*$.*

(a) $\{\hat{J}(\xi), \hat{J}(\eta)\} = \hat{J}([\xi, \eta])$

(b) $J\colon (M, \pi) \to (\mathfrak{g}^*, \pi_{KKS})$ is a Poisson map.

Example 2.6.5

(a) Let (M, π) be a Poisson manifold and let $H \in \mathscr{C}^\infty(M)$, such that X_H has complete flow $\Phi_t^{X_H}$. Then $\Phi\colon M \times \mathbb{R} \to M$ with $\Phi(m, t) = \Phi_t(m)$ is a Poisson action with moment map H.

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$. The coadjoint action $\operatorname{Ad}^*\colon \mathfrak{g}^* \times G \to \mathfrak{g}^*$ is a Poisson action with respect to the KKS Poisson srtucture. Moreover, it is Hamiltonian with moment map $\operatorname{id}\colon \mathfrak{g}^* \to \mathfrak{g}^*$.

(c) Let $\Phi\colon M \times G \to M$ be a Lie group action, then the cotangent lift $T_*\Phi\colon T^*M \times G \to T^*M$ is a Poisson action. Moreover, the map

$$J\colon T^*M \ni \alpha_p \to \alpha_p((e_i)_M(p))e^i \in \mathfrak{g}^*$$

turns $T_*\Phi$ into a Hamiltonian action.

A Hamiltonian action is compatible with the foliation of a Poisson manifold.

Proposition 2.6.6 *Let (M, π) be a Poisson manifold, let $\Phi\colon M \times G \to M$ be a Hamiltonian action of a connected Lie group with moment map $J\colon M \to \mathfrak{g}^*$ and let $\iota\colon S \to M$ be a symplectic leaf.*

(a) The action Φ restricts to an action $\Phi_S\colon S \times G \to S$, i.e. Φ_S is a Lie group action, such that

$$\iota(\Phi_S(s, g)) = \Phi(\iota(s), g).$$

(b) The map $J_S\colon S \to \mathfrak{g}^*$ given by $J_S(s) = J(\iota(s))$ turns Φ_S into a Hamiltonian action.

PROOF: We consider the fundamental vector fields ξ_M. Since $J\colon M \to \mathfrak{g}^*$ is a moment map, we have $\xi_M = X_{\hat{J}(\xi)}$. We define

$$\xi_S := X_{\hat{J}_S(\xi)} \quad \text{für alle } \xi \in \mathfrak{g}$$

an thus we have $\xi_S \sim_\iota \xi_M$ for all $\xi \in \mathfrak{g}$. This means in particular, that the distribution which is spanned by the fundamental vector fields through $\iota(s)$ is contained in $T\iota(TS)$. Due to Lemma 2.4.16, this means that the orbits are contained in $\iota(S)$. The restriction of the action is also smooth since smoothness is a local property and locally S is embedded. $\square$

Remark 2.6.7 Note that even if we can restrict a Hamiltonian action to a symplectic leaf due to Propostion 2.6.6, the restricted action might not by proper anymore. Nevertheless, it will still be free.

The existence of a moment map to a given action is not always clear, but it is clear that a necessary condition is that the fundamental vector fields have to be tangent to all the symplectic leaves.

Proposition 2.6.8 *Let (M, π) be a symplectic manifold with Poisson action $\Phi\colon M \times G \to M$ of a connected Lie group. If $\mathrm{H}^1_{dR}(M) = 0$, then there exists a moment map. Moreover, the difference J'_J of two moment maps J and J' is locally constant taking values in $(\mathfrak{g}^*)^G$.*

PROOF: Since the fundamental vector fields are in particular Poisson vectorfields, there are, using Lemma 2.3.7, unique closed 1-Forms α_ξ, such that $\xi_M = \pi^\sharp(\alpha_\xi)$. Moreover, since

$$\pi^\sharp(\alpha_{\mathrm{Ad}_g \xi}) = (\mathrm{Ad}_g \xi)_M = \Phi_g^* \xi_M = \Phi_g^* \pi^\sharp(\alpha_\xi) = \pi^\sharp(\Phi_g^* \alpha_\xi),$$

we get $\Phi_g^* \alpha_\xi = \alpha_{\mathrm{Ad}_g \xi}$. Moreover, the map $\mathfrak{g} \ni \xi \mapsto \alpha_\xi \in \Omega^1(M)$ is linear. We choose a basis $\{e_i\}_{i \in I}$ of $\mathfrak{g}$ and since $\mathrm{H}^1_{dR}(M) = 0$ we can find $J_i \in \mathscr{C}^\infty(M)$, such that $\mathrm{d}J_i = \alpha_{e_i}$. We define $J\colon M \ni m \mapsto J_i(m) e^i \in \mathfrak{g}^*$. Note that this map is not equivariant so far, but with standard averaging techniques for proper actions, we can find a J being equivariant having the same derivative.

Let now J and J' be two moment maps, then

$$\pi^\sharp(\mathrm{d}(J_\xi - J'_\xi)) = \xi_M - \xi_M = 0$$

and since π is non-degenerate, we get that $\mathrm{d}(J_\xi - J'_\xi) = 0$ and hence it is locally constant. In particular we have on a connected component of M we define $J - J' = \alpha$ for $\alpha \in \mathfrak{g}^*$. Therefore,

$$\alpha = J(m) - J'(m) = J(\Phi_g(m)) - J(\Phi_g(m)) = \mathrm{Ad}_g^*(J(m) - J'(m)) = \mathrm{Ad}_g^* \alpha,$$

since G is connected, we have that $\Phi_g(m)$ and m are always in the same connected component. $\square$

Remark 2.6.9 It is clear from the proof Propisition 2.6.8 that for general Poisson manifolds it is a non-trivial task to decide if a given Poisson action is Hamiltonian or not.

Theorem 2.6.10 (Noether) *Let (M,π) be a Poisson manifold, let $\Phi\colon M\times G\to M$ be a Hamiltonian action with moment map $J\colon M\to\mathfrak{g}^*$ and let $H\in\mathscr{C}^\infty(M)^G$. The functions $\hat{J}(\xi)$ are constant along the flow lines of X_H.*

PROOF: We have

$$X_H(\hat{J}(\xi)) = -X_{\hat{J}(\xi)}(H) = \xi_M(H) = 0$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^*\hat{J}(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^*(X_H(\hat{J}(\xi)) = 0. \qquad \square$$

A moment map does not only provide conserved quantities, but also allows us to get rid of unnecessary "un-physical" variables. In the geometric setting this was obtained by Marsden and Weinstein in the symplectic case and by Ortega Ratiu in the Poisson case. Both cases are more general than the one presented here.

Theorem 2.6.11 (phase space reduction) *Let (M,π) be a Poisson manifold, let $\Phi\colon M\times G\to M$ be a Hamiltonian action with moment map $J\colon M\to\mathfrak{g}^*$. If $0\in\mathfrak{g}^*$ is a regular value of J, then*

$$M_{\mathrm{red}} := J^{-1}(\{0\})/G$$

is manifold which admits a unique Poisson structure π_{red}, such that

$$\iota^*\{f,g\} = p^*\{\tilde{f},\tilde{g}\}_{\mathrm{red}},$$

for $M \overset{\iota}{\hookleftarrow} J^{-1}(\{0\}) \overset{p}{\twoheadrightarrow} M_{\mathrm{red}}$ and $f,g\in\mathscr{C}^\infty(M)$ and $\tilde{f},\tilde{g}\in\mathscr{C}^\infty(M_{\mathrm{red}})$, such that $\iota^*f = p^*\tilde{f}$ and $\iota^*g = p^*\tilde{g}$.

PROOF: Let $m\in J^{-1}(\{0\})$, then $J(\Phi_g(m)) = \mathrm{Ad}_g^* J(m) = 0$ and hence G restricts to an action on C. G acts free and proper on M and since $J^{-1}(\{0\})\hookrightarrow M$ is embedded, it also acts free and proper on $J^{-1}(\{0\})$ and there is a unique smooth structure on $M_{\mathrm{red}} = J^{-1}(\{0\})/G$ turning the canonical projection $p\colon J^{-1}(\{0\})\to M_{\mathrm{red}}$ into a surjective submersion. Let $f,g\in\mathscr{C}^\infty(M_{\mathrm{red}})$ then $p^*f, p^*g\in\mathscr{C}^\infty(J^{-1}(\{0\}))^G$ morover let $\tilde{f},\tilde{g}\in\mathscr{C}^\infty(M)$ such that $\iota^*\tilde{f} = p^*f$ $\iota^*\tilde{g} = p^*g$. We define

$$\{p^*f,p^*g\}_{\mathrm{red}} = \iota^*\{\tilde{f},\tilde{g}\}$$

and note that for $\iota^*h = 0$, we find $h^i\in\mathscr{C}^\infty(M)$, such that $h = h^i\hat{J}(e_i)$ and hence

$$\iota^*\{h,\tilde{g}\} = \iota^*(\{h^i,\tilde{g}\}\hat{J}(e_i) + h^i\{\hat{J}(e_i),\tilde{g}\}) = \iota^*h^i\{\hat{J}(e_i),\tilde{g}\} = -\iota^*h^i(e_i)_M(\tilde{g}) = \iota^*h^i(e_i)_C(\iota^*\tilde{g}).$$

Since $\iota^*\tilde{g} = p^*g\in\mathscr{C}^\infty(J^{-1}(\{0\}))^G$, we get that $\iota^*\{h,\tilde{g}\} = 0$ and the bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ is well defined. Moreover, we have for all $h\in G$

$$(\Phi_h^C)^*\{p^*f,p^*g\}_{\mathrm{red}} = (\Phi_h^C)^*\iota^*\{\tilde{f},\tilde{g}\} = \iota^*\Phi_h^*\{\tilde{f},\tilde{g}\} = \iota^*\{\Phi_h^*\tilde{f},\Phi_h^*\tilde{g}\}$$

But $\iota^*\Phi_h^*\tilde{f} = (\Phi_h^C)^*\iota^*\tilde{f} = (\Phi_h^C)^*p^*f = p^*f$ and hence we get that $\{p^*f,p^*g\}_{\mathrm{red}}$ is G-invariant for every $f,g\in\mathscr{C}^\infty(M_{\mathrm{red}})$, which means in particular that using the isomorphism $\mathscr{C}^\infty(J^{-1}(\{0\})^G = \mathscr{C}^\infty(M_{\mathrm{red}})$ we get a bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ on M_{red}. One can chek, that using the definition that it is indeed a Poisson bracket and is also the only possible choice by construction. $\square$

Lemma 2.6.12 *Let (M,π) be a regular Poisson manifold, let $\Phi\colon M\times G\to M$ be a Hamiltonian action with moment map $J\colon M\to\mathfrak{g}^*$. Then the Poisson structure π_{red} on M_{red} is also regular and moreover, we have that* $\mathrm{rank}(\ker\pi^\sharp_{\mathrm{red}}) = \mathrm{rank}(\ker\pi^\sharp)$. *In particular, if π is symplectic then so is π_{red}.*

PROOF: Let $c \in J^{-1}(\{0\})$, then we can choose a splitting $T_c^*M = E \oplus \mathrm{Ann}(TC)$, such that $\ker \pi^\sharp\big|_c \subseteq E$, since $\mathrm{Ann}(T_cC) = \langle \mathrm{d}\hat{J}(\xi)\big|_c \rangle$ and $\pi^\sharp(\hat{J}(\xi)\big|_c) = \xi_M(c)$ and since Φ is a free action we get that $\pi^\sharp\big|_{\mathrm{Ann}(T_cC)}$ is injective. Moreover, we have that $E \cong T_c^*C$ Let $\alpha \in \ker \pi^\sharp\big|_c \subseteq E$, then we have that

$$\alpha(\xi_M(c)) = \alpha(\pi^\sharp(\mathrm{d}\hat{J}(\xi)) = -\,\mathrm{d}\hat{J}(\xi)(\pi^\sharp(\alpha)) = 0.$$

This means in particular that there exists a unique $\beta \in T^*_{p(c)}M_{\mathrm{red}}$, such that $\alpha = \beta \circ T_cp$. We choose an $f \in \mathscr{C}^\infty(M_{\mathrm{red}})$ such that $\mathrm{d}f\big|_{p(c)} = \beta$, and moreover we can find $\tilde{f} \in \mathscr{C}^\infty(M)$, such that $\iota^*\tilde{f} = p^*f$, moreover we can choose $\tilde{f}$ in such a way that $\mathrm{d}\tilde{f}\big|_c = \alpha$. Now we find that for $g \in \mathscr{C}^\infty(M_{\mathrm{red}})$, we have

$$-\,\mathrm{d}g\big|_{p(c)}(\pi^\sharp_{\mathrm{red}}(\mathrm{d}f\big|_{p(c)}) = \{f, g\}_{\mathrm{red}}(p(c)) = \iota^*\{\tilde{f}, \tilde{g}\}(c) = -\,\mathrm{d}\tilde{g}\big|_c(\pi^\sharp(\mathrm{d}\tilde{f}\big|_c)) = 0$$

for an arbitrary $\tilde{g} \in \mathscr{C}^\infty(M)$ with $\iota^*\tilde{g} = p^*g$. Since $\mathrm{d}f\big|_{p(c)} = \beta$, we get that $\beta \in \ker \pi^\sharp_{\mathrm{red}}$. Moreover, we have shown that the map $\chi\colon \ker \pi^\sharp_{\mathrm{red}}\big|_{p(c)} \ni \beta \mapsto \beta \circ T_cp \in T_c^*C \cong E$ is injective with image $\ker \pi^\sharp\big|_c$. And the claim is proven. $\square$

Remark 2.6.13 There is a slightly more involved proof of Lemma 2.6.12, which roughly speaking goes as follows: one considers the subbundle $\mathcal{F} = \mathrm{im}\,\pi^\sharp$ and defines

$$\mathcal{F}_C = TC \cap \mathcal{F}\big|_C \subseteq TC,$$

where $C := J^{-1}(\{0\})$ This is a regular involutive distribution on C together with morphism

$$\begin{array}{ccc} \mathcal{F}_C & \xrightarrow{\ I\ } & \mathcal{F} \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{\ \iota\ } & M \end{array}$$

such that $I^*\colon \Gamma^\infty(\Lambda^\bullet\mathcal{F}^*) \to \Gamma^\infty(\Lambda^\bullet\mathcal{F}_C^*)$ is a cochain map. The one shows that there is a regular involutive distribution $\mathcal{F}_{\mathrm{red}} \subseteq TM_{\mathrm{red}}$ and a morphism

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ P\ } & \mathcal{F}_{\mathrm{red}} \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{\ p\ } & M_{\mathrm{red}} \end{array}$$

such that $P^*\colon \Gamma^\infty(\Lambda^\bullet\mathcal{F}_C^*) \to \Gamma^\infty(\Lambda^\bullet\mathcal{F}_{\mathrm{red}}^*)$ ic a cochain map. The foliated symplectic structure on M_{red} is now a non-degenerate closed foliated 2-Form $\omega_{\mathrm{red}} \in \Gamma^\infty(\Lambda^\bullet\mathcal{F}_{\mathrm{red}}^*)$, such that $P^*\omega_{\mathrm{red}} = I^*\omega$, where ω is the foliated symplectic 2-Form induced by π.

Chapter 3

Formal Deformation Quantization

After having discussed the basics of Poisson geometry, we want to understand deformation quantization. But before we want to motivate the idea behind and to do so we have to clarify what a quantization is. Quantum mechanics describes the world on a very small scale very well, but in daily life we do not see quantum effects directly and it is for the motion hardly of importance of macroscopic objects. This is what is called a classical limit, i.e. the quantum mechanical description of a macroscopic system should be *close* to its classical description. The physical parameter which "measures" the ratio of the difference of classical and quantum description is the Planck constant $\hbar$. To summarize, admittedly over-simplified, we have

$$\text{Quantum theory} \xrightarrow{\hbar \to 0} \text{Classical theory}.$$

A quantization Q is now a right inverse to the classical limit $\hbar \to 0$. We argued already in Chapter 2, that classical theories are linked with Poisson geometry and this will be our starting point:

$$\text{Poisson geometry} \xrightarrow{Q} ???.$$

To understand what we should expect, we list now the important issues of classical and quantum mechanical despription (we choose the Heisenberg picture):

	Classical	Quantum
Observables	Poisson subalgebra $\mathscr{A}_{cl} \subseteq \mathscr{C}^\infty(M)$ on a Poisson manifold (M, π)	subalgebra $\mathscr{A}_{QM} \subseteq$ Operators on a Hilbert space $\mathcal{H}$
Time evolution	Hamiltonian function $H \in \mathscr{A}_{cl}$: $\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \{f(t), H\}$	Hamilton operator $\hat{H} \in \mathscr{A}_{QM}$: $\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{1}{\mathrm{i}\hbar}[A(t), \hat{H}]$

With this in mind we can formulate what we expect from a quantization, which is summarized in a wishlist:

- Q should be a linear map from $\mathscr{A}_{cl}$ (usually we have to take $\mathscr{C}^\infty(M)$, sicnce there is no distinguished Poisson subalgebra on a general Poisson manifold) to $\mathscr{A}_{QM}$.
- $\lim_{\hbar \to 0} Q(f) = f$
- $Q(H) = \hat{H}$
- $[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f, g\}) + \mathcal{O}(\hbar^2)$

Where we added the term $\mathcal{O}(\hbar^2)$, since one can show that in easy examples this map can otherwise not exist (Exercise!).

Formal deformation quantization was developed by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in their seminal work [1]. The idea is following the motto:

> " *We suggest that quantization be understood as a deformation of the structure of the algebra of classical observables,rather than as a radical change in the nature of the observables.*"
>
> – Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer

We will discuss their ideas in this chapter and make precise what this quote means.

3.1 Differential Operators on manifolds

Differential operators are actually algebraic objects definable for any (commutative) algebra over a field $\Bbbk$. We define them in full generality and show afterwards, that in our setting they look exactly how one thinks of differential operators. To do so let us start with an obvious operation on a commutative algebra $\mathscr{A}$: let $a \in \mathscr{A}$, then we define

$$l_a \colon \mathscr{A} \ni b \mapsto ab \in \mathscr{A}.$$

Definition 3.1.1 *Let $\mathscr{A}$ be a commutative algebra. Then the vector space $\operatorname{DiffOp}^{(k)}(\mathscr{A}) \subseteq \operatorname{End}_{\Bbbk}(\mathscr{A})$ is recursively defined for $k \geq -1$ by $\operatorname{DiffOp}^{(-1)}(\mathscr{A}) = \{0\}$ and*

$$\operatorname{DiffOp}^{(k+1)}(\mathscr{A}) := \{D \in \operatorname{End}_{\Bbbk}(\mathcal{A}) \mid [l_a, D] = l_a D - D l_a \in \operatorname{DiffOp}^{(k)}(\mathscr{A})\}.$$

We have $\operatorname{DiffOp}^{(k+1)}(\mathscr{A}) \supseteq \operatorname{DiffOp}^{(k)}(\mathscr{A})$ for all $k \geq -1$ and hence we set

$$\operatorname{DiffOp}(\mathscr{A}) = \bigcup_{k=0}^{\infty} \operatorname{DiffOp}^{(k)}(\mathscr{A}).$$

We know now from this definition, that for $D \in \operatorname{DiffOp}(\mathscr{A})$, there is a $k \in \mathbb{N}$, such that $D \in \operatorname{DiffOp}^{(k)}(\mathscr{A})$. We call

$$o(D) = \min\{k \in \mathbb{N} \mid D \in \operatorname{DiffOp}^{(k)}(\mathscr{A})\}$$

the order of k.

Corollary 3.1.2 *Let $\mathscr{A}$ be a commutative unital algebra, then*

(a) $\mathscr{A} \ni a \mapsto l_a \in \operatorname{DiffOp}^0(\mathscr{A})$ is surjective.

(b) for $D_1, D_2 \in \operatorname{DiffOp}(\mathscr{A})$ their concatenation $D_1 \circ D_2 \in \operatorname{DiffOp}(\mathscr{A})$. Moreover, $o(D_1 \circ D_2) \leq o(D_1) + o(D_2)$.

PROOF: Exercise. □

For a later use we introduce

Definition 3.1.3 *Let $\mathscr{A}$ be a commutative $\mathscr{A}$. A bilinear map $B \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is called bidifferential operator of order (r, s), if for every $a \in \mathscr{A}$, we have that*

- *the map $L_a \colon \mathscr{A} \ni b \mapsto B(a, b) \in \mathscr{A}$ is a differential operator of order s, and*
- *the map $R_a : \mathscr{A} \ni b \mapsto B(b, a) \in \mathscr{A}$ is a differential operator of order r.*

We denote the set of bidifferential operators of order (r, s) *on* $\mathscr{A}$ *by* $\mathrm{BiDiffop}^{(r,s)}(\mathscr{A})$.

We are mainly interested in the algebra of smooth functions on a manifold, so let us prove a local structure theorem for differential operators, which justifies the nomenclature:

Theorem 3.1.4 *Let* M *be a manifold. For* $D \in \mathrm{DiffOp}^k(M) := \mathrm{DiffOp}^k(\mathscr{C}^\infty(M))$, *we have*

(a) $\operatorname{supp} D(f) \subseteq \operatorname{supp} f$ *and for every open subset* $U \subseteq M$ *there exists* $D_U \in \mathrm{DiffOp}^k(U)$, *such that for all* $f \in \mathscr{C}^\infty(M)$ *we have*

$$D(f)\big|_U = D_U(f\big|_U).$$

(b) that for a coordinate chart (U, x), *we have for* $f \in \mathscr{C}^\infty(U)$

$$D_U(f) = \sum_{r=0}^{k} \frac{1}{r!} D_{U,r}^{i_1,\dots,i_r} \frac{\partial^r f}{\partial x^{i_1} \cdots \partial x^{i_r}}$$

for local functions $D_{U,r}^{i_1,\dots,i_r} \in \mathscr{C}^\infty(U)$.

PROOF: We proof (a) with induction: for $k = 0$ the statement is clear, since $\mathrm{DiffOp}^0(\mathscr{C}^\infty(M))$ are exactly the left multiplications with functions. So let us assume that the statement is true for all $k < N$ and let $f \in \mathscr{C}^\infty(M)$ and let $x_0 \in M \setminus \operatorname{supp} f$, then there is a function $\rho \in \mathscr{C}^\infty(M)$ with $\rho(x_0) = 0$ and $\rho\big|_{\operatorname{supp} f} = 1$ and it follows $\rho f = f$. We have for $D \in \mathrm{DiffOp}^N(M)$

$$D(f)(x_0) = D(\rho f)(x_0) = \rho(x_0)D(f)(x_0) - [\rho, D](f)(x_0) = -[\rho, D](f)(x_0) = 0,$$

where the last equality follows from the fact that $[\rho, D]$ is a differential operator of order $N - 1$ and the first statement fromn (a) follows. The proof of the second statement follows the same lines as the proof of Proposition 2.1.6. The proof of part (b) is an exercise. □

Example 3.1.5 (a) One first immediate example are the vector fields on a manifold. In fact, the Leibniz rule shows that they are differential operators of order one.

(b) The Laplace operator Δ_g induced by a Riemannian metric g is a differential Operator of order 2.

Theorem 3.1.4 showed that differential operators are local operators, which is not very surprising concerning how one would imagine a differential operator. There is also a remarkable theorem by Peetre, which states that every local operator looks locally like a differential operator (Note that this does not imply it is a differential operator.)

Theorem 3.1.6 *(Peetre) Let* $D \colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ *be a linear local operator. Then for each point* $p \in M$ *there exists an open neighbourhood* U, *such that* $D\big|_U$ *is a differential operator.*

We are not going to show this theorem in this course, it is just helpful to justify late choices.

As a last part of this section and a first application of Theorem 3.1.4, we prove a local structure result for Bidifferential operatorsd on a manifold.

Theorem 3.1.7 *Let* M *be a manifold. For* $B \in \mathrm{BiDiffop}^{(r,s)}(\mathscr{C}^\infty(M))$, *we have*

(a) $\operatorname{supp} B(f,g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g$ *and for every open subset* $U \subseteq M$ *there exists* $B_U \in \mathrm{BiDiffop}^{(r,s)}(U)$, *such that for all* $f, g \in \mathscr{C}^\infty(M)$ *we have*

$$B(f,g)\big|_U = B_U(f\big|_U, g\big|_U).$$

(b) that for a coordinate chart (U, x), we have for $f, g \in \mathscr{C}^\infty(U)$

$$B_U(f,g) = \sum_{n=0}^{r}\sum_{m=0}^{s} \frac{1}{n!m!} B_{U,n,m}^{i_1,\dots,i_n;j_1,\dots,j_m} \frac{\partial^n f}{\partial x^{i_1}\cdots\partial x^{i_n}} \frac{\partial^m g}{\partial x^{j_1}\cdots\partial x^{j_m}}$$

for local functions $B_{U,n,m}^{i_1,\dots,i_n;j_1,\dots,j_m} \in \mathscr{C}^\infty(U)$.

Proof: Let $f, g \in \mathscr{C}^\infty(M)$, then we have that $B(f,g) = L_f(g)$, which is a differential operator and thus we get by Theorem 3.1.4, that $\operatorname{supp} B(f,g) \subseteq \operatorname{supp} f$. Since we also have that $B(f,g) = R_g(f)$, we get also $\operatorname{supp} B(f,g) \subseteq \operatorname{supp} g$. We can now use the same arguments as in Theorem 3.1.4 in order to show the existence of B_U for open subsets U.

We work now in a chart (U, x), we denote by L_f and R_f the corresponding operators of the restricted bidifferential operator B_U. Using again Theorem 3.1.4, we see that for $f \in \mathscr{C}^\infty(U)$ we have that L_f is a differential operator of order s and hence

$$L_f = \sum_{m=0}^{s} \frac{1}{m!} (L_f)_m^{j_1,\dots j_m} \frac{\partial^m}{\partial x^{j_1}\dots\partial x^{j_m}}$$

in a chart (U, x). With an induction one can see that

$$\begin{aligned} L_f(x^{i_1}\cdots x^{i_k}) &= \sum_{m=0}^{k}\sum_{\sigma\in S_k} \frac{1}{m!(k-m)!} x^{i_{\sigma(1)}}\cdots x^{i_{\sigma(m)}} (L_f)_{k-m}^{i_{\sigma(m+1)},\dots i_{\sigma(k)}} \\ &= (L_f)_k^{i_1,\dots i_k} + \sum_{m=1}^{k}\sum_{\sigma\in S_k} \frac{1}{m!(k-m)!} x^{i_{\sigma(1)}}\cdots x^{i_{\sigma(m)}} (L_f)_{k-m}^{i_{\sigma(m+1)},\dots i_{\sigma(k)}}, \end{aligned}$$

which means, that we can write all $(L_f)_k^{i_1,\dots i_k}$ as $\mathscr{C}^\infty(U)$-linear combinations of $L_f(x^{i_1}\cdots x^{i_\ell})$, such that the coefficient functions are independent of f. We have that the map

$$\mathscr{C}^\infty(U) \ni f \mapsto L_f(x^{i_1}\cdots x^{i_\ell}) = R_{x^{i_1}\dots x^{i_\ell}}(f) \in \mathscr{C}^\infty(U)$$

is a differential operator of order r and hence we get the claim. □

3.2 Formal Deformations and Star Products on Poisson Manifolds

The idea of deformations goes back to Gerstenhaber in a series of papers [5–7] where he discusses deformations of algebraic structures. For an algebra the precise definition of a deformation is the following:

Definition 3.2.1 *Let $\mathscr{A}^\bullet$ be a unital (graded *) commutative algebra over a field $\Bbbk$. A formal deformation is a formal power series $\sum_k \hbar^k \mu_k$ of $\Bbbk[[\hbar]]$-bilinear maps $\mu_k \colon \mathscr{A}[[\hbar]] \times \mathscr{A}[[\hbar]] \to \mathscr{A}[[\hbar]]$, such that*

(a) $a \star b := \sum_k \hbar^k \mu_k(a,b)$ is an associative product on $\mathscr{A}[[\hbar]]$.

(b) $a \star b = ab \pmod{\hbar}$ for all $a, b \in \mathscr{A}$

(c) $1 \star a = a \star 1 = a$ for $a \in \mathscr{A}[[\hbar]]$

(d) $a \star b \in \mathscr{A}^{k+\ell}[[\hbar]]$ for all $a \in \mathscr{A}^k[[\hbar]]$ and $b \in \mathscr{A}^\ell[[\hbar]]$ (*)

Remark 3.2.2 For the moment we can ignore the graded version of Definition 3.2.1 and consider only trivially graded commutative algebras. This graded version is not much more difficult than the trivially graded version and it will be needed in a later stage of this lecture.

Let us, before we go towards the differential geometric picture, give a first important example:

Example 3.2.3 (Commuting derivations) Let $\mathscr{A}$ be a commutative algebra over $\Bbbk$, let $\{D_i\}_{i\in\{1,\dots,N\}}$ be a finite set of commuting derivations and let $\pi^{ij} \in \Bbbk$ for $1 \leq i, j \leq N$. We define:

$$\Pi\colon \mathscr{A} \otimes \mathscr{A} \ni a \otimes b \mapsto \pi^{ij} D_i(a) \otimes D_j(b) \in \otimes\mathscr{A} \otimes \mathscr{A}$$

and finally

$$a \star b = \mu(\mathrm{e}^{\frac{\hbar\Pi}{2}}(a \otimes b)) = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} \pi^{i_1 j_1} \dots \pi^{i_k j_k} D_{i_1} \dots D_{i_k}(a) D_{j_1} \dots D_{j_k}(b).$$

This is a formal deformation of the algebra $\mathscr{A}$. (The proof is an Exercise!)

Let us show the significance of formal deformations with respect to Chapter 2:

Lemma 3.2.4 *Let $\mathscr{A}$ be a commutative algebra and let $\star$ be a formal deformation, then*

$$\{a, b\}_\star := \frac{a \star b - b \star a}{\hbar}\Big|_{\hbar=0}$$

for $a, b \in \mathscr{A}$ is a Poisson bracket on $\mathscr{A}$. We call $\{\,\cdot\,,\,\cdot\,\}_\star$ the classical limit of $\star$.

PROOF: $\{\,\cdot\,,\,\cdot\,\}$ is obviously skew-symmetric. Let us denote by $[\,\cdot\,,\,\cdot\,]_\star$ the commutator with respect to $\star$. Since $\star$ is an associative product, we know that

$$[a, b \star c]_\star = [a, b]_\star \star c + b \star [a, c]_\star$$

for $a, b, c \in \mathscr{A}$. Evaluating this in order 1 (of $\hbar$), we get that $\{a, bc\}_\star = \{a, b\}_\star c + b\{a, c\}_\star$. Moreover, again since $\star$ is associative, we see that

$$[a, [b, c]_\star]_\star = [[a, b]_\star, c]_\star + [b, [a, c]_\star]_\star,$$

for $a, b, c \in \mathscr{A}$. evaluating this in order 2, we obtain the Jacobi identity for $\{\,\cdot\,,\,\cdot\,\}_\star$ and the claim is proven. □

If one has a formal deformation $\sum_k \hbar^k \mu_k$ of an associative algebra $\mathscr{A}$, the associativity of the product is encoded in order $\hbar^k$ by

$$\sum_{i=0}^{k} \mu_i(\mu_{k-i}(a, b), c) = \sum_{i=0}^{k} \mu_i(a, \mu_{k-i}(b, c)).$$

for all $a, b, c \in \mathscr{A}$. If this equation is fulfilled for all $k \leq N$, we say that $\sum_k \hbar^k \mu_k$ is a formal deformation up to order N.

So one can try to make the following Ansatz: starting with a Poisson algebra $(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$, we define

$$a \star b = ab + \frac{\hbar}{2}\{a, b\}.$$

this is a formal deformation up to order 1, because of the fact that $\{\,\cdot\,,\,\cdot\,\}$ is a biderivation. so we can try to find μ_2 in order to make it a formal deformation up to order 2, which means that

$$\mu_2(ab,c) + \frac{1}{4}\{\{a,b\},c\} + \mu_2(a,b)c = \mu_2(a,bc) + \frac{1}{4}\{a,\{b,c\}\} + a\mu_2(b,c)$$

has to hold for all $a,b,c \in \mathscr{A}$ and already in order 2, we see that this might be a highly non-trivial task. Note that a necessary condition for such a μ_2 to exist, is the Jacobi identity of $\{\,\cdot\,,\,\cdot\,\}$, see the proof of Lemma 3.2.4, but this condition is not for every algebra sufficient and in fact, it is not sufficient for the algebras we are interested in this course: we already argued that the algebra we want to deform for quantization is $\mathscr{C}^\infty(M)$ for a manifold M. In fact, this step-by-step procedure works here only if $\dim(M) \leq 2$.

We do not only want arbitrary deformations, we want to make the structure of our specific algebra visible, i.e. we impose differentiability and hence also locality:

Definition 3.2.5 *A star product $\star$ on a manifold is a is a formal deformation $\sum_k \hbar^k \mu_k$ of $\mathscr{C}^\infty(M)$, such that μ_k is a bidifferential operator for all k.*

Remark 3.2.6 The products defined in 3.2.5 are sometimes called *differential star products* to emphasise that they are series of bidifferential operators. In fact, the only known general constructions produce differential star products. Since we are only dealing with these kinds of star products we omit the additonal term "differential". Note that a star product induces now a Poisson bracket on a manifold and this does not depend on the differentiability of the star product anyway.

As a first example of a star product, we can use Example 3.2.3: let p^{ij} be an $n \times n$-matrix, we define

$$P\colon \mathscr{C}^\infty(\mathbb{R}^n) \otimes \mathscr{C}^\infty(\mathbb{R}^n) \to \mathscr{C}^\infty(\mathbb{R}^n) \otimes \mathscr{C}^\infty(\mathbb{R}^n)$$

by

$$P(f \otimes g) = p^{ij} \frac{\partial f}{\partial x^i} \otimes \frac{\partial g}{\partial x^j}$$

then we go on to define the product $\star_p$

$$f \star_p g = \mu(\mathrm{e}^{\frac{\hbar P}{2}}(f \otimes g)) = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} p^{i_1 j_1} \dots p^{i_k j_k} \frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}} \frac{\partial^k g}{\partial x^{j_1} \dots \partial x^{j_k}}, \tag{3.2.1}$$

where μ is just the point-wise product of $\mathscr{C}^\infty(\mathbb{R}^n)$. Additionally, it is immediate that the induced Poisson structure is given by

$$\{f,g\}_{\star_p} = \frac{1}{2}(p^{ij} - p^{ji}) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

which is the Poisson structure associated to a constant skew-symmetric matrix from 2.3.1. Obviously we used a lot of properties of the flat $\mathbb{R}^n$, for example the exisence of non-trivial global derivations, in order to define this product which are not available in the general geometric picture on a manifold, so there is no hope to just globalize this product for smooth manifolds, since Poisson structures can have rather wild local behaviour.

Inducing a Poisson structure via a formal deformation of the smooth functions on a manifold can be understood as the classical limit of a quantum system, as we already discussed in the introduction of this chapter. Starting from a classical system, i.e. a Poisson manifold, we want to make precise what we mean by quantization:

Definition 3.2.7 *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson manifold. A star product $\star$ is called formal deformation quantization of $(M, \{\cdot\,,\cdot\})$, if $\{\cdot\,,\cdot\}_\star = \{\cdot\,,\cdot\}$.*

Remark 3.2.8 Assume that $\star$ is a formal deformation quantization of $(M, \{\cdot\,,\cdot\})$, then we have trivially that the canonical map

$$Q\colon \mathscr{C}^\infty(M) \ni f \hookrightarrow f \in \mathscr{C}^\infty(M)[[\hbar]]$$

is a quantization map in the sense of the wish-list in the introduction.

The question which arises now is: Does every Poisson manifold admit a deformation quantization? In fact, the answer is yes and was fully answered by Kontsevich in his seminal paper [9]. His methods use a lot of different techniques from various areas of math and is not easy to understand. A more conceptual proof followed by Tamarkin in [11], but this proof uses even more techniques and is harder to understand.

Nevertheless, there are constructions of star products which are formal quantizations of certain non-trivial Poisson brackets which do not use a lot of machinery, which we will see in the next two sections.

3.3 Lie algebras and the Gutt Product

The Gutt product is a quantization of the KKS-Poisson structure associated to a real finite dimensional Lie algebra. It was one of the first star products ever to be constructed in [8], besides the star product for constant Poisson structures.

Its construction is rather algebraic, so we need beforehand some construction from algebra. Note that to every associative algebra $\mathscr{A}$, we can associate a Lie algebra by taking commutators, i.e.

$$[a, b] := ab - ba$$

for $a, b \in \mathscr{A}$ defines a Lie bracket on $\mathscr{A}$. Let us denote this Lie algebra by $\mathscr{A}_L$.

Definition 3.3.1 *Let $\mathfrak{g}$ be a Lie algebra. A universal enveloping algebra for $\mathfrak{g}$ is a pair (U, ι) consisting of an associative algebra U and a Lie algebra morphism map $\iota\colon \mathfrak{g} \to U_L$ with the property that for every associative algebra $\mathscr{A}$ with a Lie algebra morphism $\phi\colon \mathfrak{g} \to \mathscr{A}_L$ there exists a unique algebra morphism $\Phi\colon U \to \mathscr{A}$, such that*

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathscr{A} \\ {\scriptstyle\iota}\uparrow & \nearrow_{\phi} & \\ \mathfrak{g} & & \end{array}$$

commutes. This property is called the universal property of the universal enveloping Lie algebra.

Note that it is not clear that this object exists, but with the definition we can already prove its uniqueness:

Proposition 3.3.2 *Let $\mathfrak{g}$ be a Lie algebra and let (U, ι) and $(\hat{U}, \hat{\iota})$ be two universal enveloping algebras, then $U \simeq \hat{U}$ as algebras.*

Proof: We see from the universal properties of U and $\hat{U}$, that we get the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\hat{I}} & \hat{U} & \xrightarrow{I} & U \\ {\scriptstyle\iota}\uparrow & \nearrow^{\hat{\iota}} & & \nearrow_{\iota} & \\ \mathfrak{g} & & & & \end{array} .$$

Bu this means, that we have an algebra morphism $I \circ \hat{I}$, such that $\iota = I \circ \hat{I} \circ \iota$, which is clearly also fulfilled by $\mathrm{id}\colon U \to U$ and by the uniqueness of the morphism from Definition 3.3.1, we get $I \circ \hat{I} = \mathrm{id}$. Exchanging the roles of U and $\hat{U}$, we also get $\hat{I} \circ I = \mathrm{id}$. □

This proposition shows that the universal enveloping algebra is unique and we can speak of *the* universal enveloping algebra. Nevertheless, we still have to show that it exists

Theorem 3.3.3 *Let $\mathfrak{g}$ be a Lie algebra, then*

$$\mathscr{U}(\mathfrak{g}) := \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x, y] \rangle_{x,y \in \mathfrak{g}}}$$

together with the canonical map $\iota\colon \mathfrak{g} \to T\mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ is a universal enveloping algebra.

PROOF: By definition $\mathscr{U}(\mathfrak{g})$ is an associative algebra and $\iota(x) = \overline{x}$ (here $\overline{\cdot}$ means taking the equivalence class) and hence

$$\iota([x, y]) = \overline{[x, y]} = \overline{x \otimes y - y \otimes x} = \overline{xy} - \overline{yx} = \iota(x)\iota(y) - \iota(y)\iota(x).$$

Let now $\mathscr{A}$ be an associative algebra and $\phi\colon \mathfrak{g} \to \mathscr{A}_L$ be a Lie algebra morphism, then we define $\Phi\colon Tg \to A$

$$\tilde{\Phi}(x_1 \otimes \cdots \otimes x_k) = \phi(x_1) \ldots \phi(x_k)$$

which is clearly an algebra morphism. Moreover, since $\phi\colon \mathfrak{g} \to \mathscr{A}_L$ is a Lie algebra morphism, we get that $\langle x \otimes y - y \otimes x - [x, y] \rangle_{x,y \in \mathfrak{g}} \subseteq \ker \tilde{\Phi}$ and hence we get an algebra morphism $\Phi\colon \mathscr{U}(\mathfrak{g}) \to \mathscr{A}$, such that

$$\begin{array}{ccc} T\mathfrak{g} & \xrightarrow{\tilde{\Phi}} & \mathscr{A} \\ \downarrow & \nearrow_{\Phi} & \\ \mathscr{U}(\mathfrak{g}) & & \end{array}$$

commutes and hence also

$$\begin{array}{ccc} \mathscr{U}(\mathfrak{g}) & \xrightarrow{\Phi} & \mathscr{A} \\ {\scriptstyle\iota}\uparrow & \nearrow_{\phi} & \\ \mathfrak{g} & & \end{array}.$$

Let us now prove that Φ is unique: we assume that there is Φ' fulfilling the same property than Φ, then we have

$$\Phi'(\overline{x_1 \otimes \cdots \otimes x_k}) = \Phi'(\overline{x_1} \cdots \overline{x_k}) = \Phi'(\overline{x_1}) \cdots \Phi'(\overline{x_k}) = \Phi(\overline{x_1}) \cdots \Phi(\overline{x_k}) = \Phi(\overline{x_1 \otimes \cdots \otimes x_k})$$

and since elements of the form $\overline{x_1 \otimes \cdots \otimes x_k}$ generate $\mathscr{U}(\mathfrak{g})$ as a vector space, we have that $\Phi = \Phi'$.□

We can use from now on $\mathscr{U}(\mathfrak{g})$ as a model for the universal enveloping algebra, i.e. if we write $\mathscr{U}(\mathfrak{g})$ we refer to the quotient construction above and not just its isomorphism class. This precise model has a canonical filtration: we define

$$\mathscr{U}(\mathfrak{g})^{(k)} = \operatorname{span}\{\overline{x_1 \otimes \cdots \otimes x_\ell} \mid x_i \in \mathfrak{g},\ 0 \leq \ell \leq k\} \subseteq \mathscr{U}(\mathfrak{g}).$$

We clearly have that $\mathscr{U}(\mathfrak{g})^{(k)} \subseteq \mathscr{U}(\mathfrak{g})^{(m)}$, whenever $k \leq m$ and moreover $\mathscr{U}(\mathfrak{g}) = \bigcup_k \mathscr{U}(\mathfrak{g})^{(k)}$. Note that this is even compatible with the algebra structure, i.e. $\mathscr{U}(\mathfrak{g})^{(k)} \cdot \mathscr{U}(\mathfrak{g})^{(m)} \subseteq \mathscr{U}(\mathfrak{g})^{(k+m)}$. Let us now prove the so-called Poincaré-Birkoff-Witt theorem which relates the universal enveloping algebra to an algebra we know already

Theorem 3.3.4 (Poincaré-Birkoff-Witt) *Let $\mathfrak{g}$ be a Lie algebra. The map*

$$q_k \colon \mathrm{S}^k\mathfrak{g} \ni x_1 \vee \cdots \vee x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \overline{x_{\sigma(1)}} \ldots \overline{x_{\sigma(k)}} \in \mathscr{U}(\mathfrak{g})^{(k)}$$

sums up to an isomorphism $q = \sum_{k=0}^{\infty} q_k \colon \mathrm{S}\mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ of vector spaces. Note that we set q_0 to be the identity.

PROOF: The key point in the proof is to see that for

$$\overline{x_1 \otimes \cdots \otimes x_k} - \overline{x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \ldots \otimes x_k} \in \mathscr{U}(\mathfrak{g})^{(k-1)}$$

for all $i \in \{1, \ldots, k-1\}$. This implies that $q_k(x_1 \vee \cdots \vee x_k) - \overline{x_1 \otimes \cdots \otimes x_k} \in \mathscr{U}(\mathfrak{g})^{(k-1)}$. We show now by induction that

$$\sum_{k=0}^{n} q_k \colon \bigoplus_{k=0}^{n} \mathrm{S}^k\mathfrak{g} \to \mathscr{U}(\mathfrak{g})^{(k)}$$

is an isomorphism for all n. For $n = 0$ the claim is true by definition. Let $u \in \mathscr{U}(\mathfrak{g})^{(n+1)}$, then we can find $x_j^i \in \mathfrak{g}$ for $i \in \{1, \ldots, N\}$ for some N and and $j \in \{1, \ldots, n+1\}$, such that

$$u - \sum_i \overline{x_1^i \otimes \cdots \otimes x_{n+1}^i} \in \mathscr{U}(\mathfrak{g})^{(n)}$$

and thus

$$\begin{aligned} X &= u - \sum_i q_{n+1}(x_1^i \vee \cdots \vee x_{n+1}^i) \\ &= u - \sum_i \overline{x_1^i \otimes \cdots \otimes x_{n+1}^i} + \sum_i \overline{x_1^i \otimes \cdots \otimes x_{n+1}^i} - q_{n+1}(x_1^i \vee \cdots \vee x_{n+1}^i) \in \mathscr{U}(\mathfrak{g})^{(n)}. \end{aligned}$$

By Induction hypothesis we can find $V \in \bigoplus_{k=0}^{n} \mathrm{S}^k\mathfrak{g}$ such that $\sum_{k=0}^{n} q_k(V) = X$ and $\sum_{k=0}^{n+1} q_k$ is surjective. To show that the map is injective, one shows that $\mathrm{S}^n\mathfrak{g} \simeq \frac{\mathscr{U}(\mathfrak{g})^{(n)}}{\mathscr{U}(\mathfrak{g})^{(n-1)}}$ (Exercise!).

Let us now denote the canonical projections $p_k \colon \mathrm{S}\mathfrak{g} \to \mathrm{S}^k\mathfrak{g}$.

Theorem 3.3.5 *The $\Bbbk[[t]]$-bilinear extension of*

$$\star_G \colon \mathrm{S}^k\mathfrak{g} \times \mathrm{S}^\ell\mathfrak{g} \ni (f, g) \mapsto \sum_{n=0}^{k+\ell-1} \hbar^n p_{k+\ell-n} q^{-1}(q(f) \cdot q(g)) \in \mathrm{S}\mathfrak{g}[[t]]$$

is a formal deformation of $\mathrm{S}\mathfrak{g}$ with the symmetric product. We call $\star_G$ the Gutt product.

PROOF: We have $1 \star_G f = f \star_G 1 = f$ for all $f \in \mathrm{S}\mathfrak{g}$ by definition. Moreover, with the same arguments from the proof of Theorem 3.3.4, one see that $q(f) \cdot q(g) - q(f \vee g) \in \mathscr{U}(\mathfrak{g})^{(k+\ell-1)}$ and hence we have $f \star_G g = f \vee g + \mathcal{O}(\hbar)$. Let us now show associativity: we choose $f_i \in \mathrm{S}^{k_i}\mathfrak{g}$ for $i = 1, 2, 3$, then

$$\begin{aligned} f_1 \star_G (f_2 \star_G f_3) &= \sum_{i=0}^{k_2+k_3-1} \hbar^i f_1 \star_G (p_{k_2+k_3-i} \circ q^{-1})(q(f_2) \cdot q(f_3)) \\ &= \sum_{i=0}^{k_2+k_3-1} \sum_{j=0}^{k_1+k_2+k_3-i-1} \hbar^{i+j} p_{k_1+k_2+k_3-i-j} \circ q^{-1}(q(f_1) \cdot q(p_{k_2+k_3-i} \circ q^{-1})(q(f_2) \cdot q(f_3)))) \end{aligned}$$

$$
\begin{aligned}
&\overset{*}{=} \sum_{i=0}^{k_2+k_3-1} \sum_{j=-i}^{k_1+k_2+k_3-i-1} \hbar^{i+j} p_{k_1+k_2+k_3-i-j} \circ q^{-1}(q(f_1) \cdot q(p_{k_2+k_3-i} \circ q^{-1})(q(f_2) \cdot q(f_3)))) \\
&= \sum_{i=0}^{k_2+k_3-1} \sum_{j=0}^{k_1+k_2+k_3-1} \hbar^{j} p_{k_1+k_2+k_3-j} \circ q^{-1}(q(f_1) \cdot q(p_{k_2+k_3-i} \circ q^{-1})(q(f_2) \cdot q(f_3)))) \\
&= \sum_{j=0}^{k_1+k_2+k_3-1} \hbar^{j} p_{k_1+k_2+k_3-j} \circ q^{-1}(q(f_1) \cdot q(\sum_{i=0}^{k_2+k_3-1} p_{k_2+k_3-i} \circ q^{-1})(q(f_2) \cdot q(f_3)))) \\
&= \sum_{j=0}^{k_1+k_2+k_3-1} \hbar^{j} p_{k_1+k_2+k_3-j} \circ q^{-1}(q(f_1) \cdot q(f_2) \cdot q(f_3))
\end{aligned}
$$

where we used in $*$ that $p_\ell \circ q^{-1} \colon \mathscr{U}(\mathfrak{g})^{(k)} \to \mathrm{S}^\ell\mathfrak{g}$ vanishes whenever $\ell > k$. We get the same result when we compute $(f_1 \star_G f_2) \star_G f_3$. □

Note that by construction of $\star_G$, it is clear that

$$
x_1 \vee \cdots \vee x_k = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \star_G \cdots \star_G x_{\sigma(k)}
$$

for all $x_1, \ldots, x_k \in \mathfrak{g}$.

Let us denote the series expansion of the Gutt product by

$$
\star_G = \sum_{k \geq 0} \hbar^k C_r,
$$

where the C_rs are bilinear maps. The Gutt product has some remarkable properties with respect to the adjoint action. Let us first extend the adjoint action $\mathrm{ad}\, \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as a derivation of the symmetric product to $\mathrm{S}\mathfrak{g}$, i.e.

$$
\mathrm{ad} \colon \mathfrak{g} \times \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}
$$

ba declaring $\mathrm{ad}_\xi(P \vee Q) = \mathrm{ad}_\xi(P) \vee Q + P \vee \mathrm{ad}_\xi(Q)$.

Lemma 3.3.6 *Let $\xi \in \mathfrak{g}$ and let $g \in \mathrm{S}\mathfrak{g}$, then*

$$
[\xi, g]_{\star_G} = \hbar\, \mathrm{ad}_\xi(g).
$$

PROOF: Let $\xi \in \mathfrak{g}$ and $x_1 \vee \cdots \vee x_\ell \in \mathrm{S}^\ell\mathfrak{g}$, then we have

$$
\begin{aligned}
q(\xi)q(x_1 \vee \cdots \vee x_\ell) &= \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \overline{\xi \otimes x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(\ell)}} \\
&= \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \overline{x_{\sigma(1)} \otimes \xi \otimes x_{\sigma(2)} \otimes \otimes x_{\sigma(\ell)}} + \overline{[\xi, x_{\sigma(1)}] \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(\ell)}]} \\
&= \ldots \\
&= \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \overline{x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(\ell)} \otimes \xi} \\
&+ \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \sum_{j=1}^{\ell} \overline{x_{\sigma(1)} \otimes \ldots x_{\sigma(j-1)} \otimes [\xi, x_{\sigma(j)}] \otimes x_{\sigma(j+1)} \otimes \ldots \otimes x_{\sigma(\ell)}}
\end{aligned}
$$

$$
\begin{aligned}
&= q(x_1 \vee \dots \vee x_\ell) q(\xi) \\
&+ \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \sum_{j=1}^{\ell} \overline{x_{\sigma(1)} \otimes \dots x_{\sigma(j-1)} \otimes [\xi, x_{\sigma(j)}] \otimes x_{\sigma(j+1)} \otimes \dots \otimes x_{\sigma(\ell)}} \\
&= q(x_1 \vee \dots \vee x_\ell) q(\xi) \\
&+ \frac{1}{\ell!} \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \sum_{\sigma \in S_\ell,\ \sigma(j)=i} \overline{x_{\sigma(1)} \otimes \dots x_{\sigma(j-1)} \otimes [\xi, x_{\sigma(j)}] \otimes x_{\sigma(j+1)} \otimes \dots \otimes x_{\sigma(\ell)}} \\
&= q(x_1 \vee \dots \vee x_\ell) q(\xi) \\
&+ \sum_{i=1}^{\ell} \frac{1}{\ell!} \sum_{j=1}^{\ell} \sum_{\sigma \in S_\ell,\ \sigma(j)=i} \overline{x_{\sigma(1)} \otimes \dots x_{\sigma(j-1)} \otimes [\xi, x_{\sigma(j)}] \otimes x_{\sigma(j+1)} \otimes \dots \otimes x_{\sigma(\ell)}} \\
&= q(x_1 \vee \dots \vee x_\ell) q(\xi) \\
&+ \sum_{i=1}^{\ell} q(x_1 \vee \dots, \vee x_{i-1} \vee [\xi, x_i] \vee x_{i+1} \vee \dots \vee x_\ell) \\
&= q(x_1 \vee \dots \vee x_\ell) q(\xi) \\
&+ q(\mathrm{ad}_\xi (x_1 \vee \dots \vee x_\ell))
\end{aligned}
$$

and the claim is proven by the eplicit formula of the Gutt product. □

Lemma 3.3.7 *The Gutt product is a series out of bidifferential operators. Moreover, we have that C_r is of order (r, r).*

Proof: One can show (see [10] for a detailed proof), that the Gutt product has the formula

$$
\eta \star_G (\xi_1 \vee \dots \vee \xi_k) = \sum_{j=0}^{k} \frac{\hbar^j}{k!} \binom{k}{j} B_j \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\dots, [\xi_{\sigma(j)}, \eta] \dots] \vee \xi_{\sigma(j+1)} \vee \dots \vee \xi_{\sigma(k)},
$$

where B_j are the Taylor coefficients of $\frac{x}{\mathrm{e}^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$ (or equivalently the *Bernoulli numbers*). This means in particular, that

$$
C_r(\eta, \cdot \,) \colon \mathrm{S}\mathfrak{g} \ni \xi_1 \vee \dots \vee \xi_k \mapsto \frac{1}{k!} \binom{k}{r} B_r \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\dots, [\xi_{\sigma(r)}, \eta] \dots] \vee \xi_{\sigma(r+1)} \vee \dots \vee \xi_{\sigma(k)} \in \mathrm{S}\mathfrak{g}
$$

which is a differential operator of order r (Exercise!). We can now use that

$$
(\xi_1 \vee \dots \vee \xi_k) \star_G \eta = \eta \star_G (\xi_1 \vee \dots \vee \xi_k) - \hbar \, \mathrm{ad}_\eta (\xi_1 \vee \dots \vee \xi_k)
$$

in order to show that $C_r(\,\cdot\,, \eta) \colon \mathrm{S}\mathfrak{g} \to \mathfrak{g}$ is a differential operator of order r for $\eta \in \mathfrak{g}$. Let now $\eta_1 \vee \dots \vee \eta_k \in$
$Sym^k \mathfrak{g}$, then we have

$$
\begin{aligned}
(\eta_1 \vee \dots \vee \eta_k) \star_G P &= \frac{1}{k!} \sum_{\sigma \in S_k} \eta_{\sigma(1)} \star_G \dots \star_G \eta_{\sigma(k)} \star P \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{i_1, \dots, i_k = 0}^{\infty} \hbar^{i_1 + \dots + i_k} C_{i_1}(\eta_{\sigma(1)}, C_{i_2}(\eta_{\sigma(2)}, \cdots C_{i_k}(\eta_{\sigma(k)}, P) \cdots)
\end{aligned}
$$

and the claim is proven for general elements in $\mathrm{S}\mathfrak{g}$ and hence C_r is a bidifferential operator of order (r, r). □

We are not yet at the point where we have obtain a star product. We can see $\mathrm{S}\mathfrak{g}$ as the polynomial functions on $\mathfrak{g}^*$ (Exercise!) so the next aim is to extend the product to $\mathscr{C}^\infty(\mathfrak{g}^*)$.

Apparently, this has nothing to do with the actual shape of the Gutt product:

Lemma 3.3.8 *Let V be a finite dimensional real vector space and let D be a differential operator on $\mathrm{S}V = \mathrm{Pol}(V^*)$ of order k, then we can extend it to a differential operator of $\mathscr{C}^\infty(V^*)$ of order k.*

PROOF: Let us choose a basis $\{e_i\}_{i\in\{1,\dots,N\}}$ of V. With the same idea of the proof of Theorem 3.1.4 (we only have one chart!), we can show that

$$D = \sum_{r=0}^{k} \frac{1}{r!} D_r^{i_1,\dots,i_r} \frac{\partial^r}{\partial x^{i_1}\cdots\partial x^{i_r}}$$

where we see that all $D_r^{i_1,\dots,i_r}$ have to be polynomial functions. Now, we are able to extend it trivially by letting the partial derivatives act on every smmoth function, i.e. for $f \in \mathscr{C}^\infty(V^*)$ we set

$$D(f) := \sum_{r=0}^{k} \frac{1}{r!} D_r^{i_1,\dots,i_r} \frac{\partial^r f}{\partial x^{i_1}\cdots\partial x^{i_r}}$$

which is clearly a differential operator. □

Using this Lemma, we get immediately

Theorem 3.3.9 *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and let $\star_G$ be the Gutt product extended to $\mathscr{C}^\infty(\mathfrak{g}^*)$, then it is a quantization of the KKS Poisson structure associated to $\mathfrak{g}$.*

PROOF: Let us choose a basis $\{e_i\}_{i\in\{1,\dots,N\}}$ with corresponding coordinates $(x = x_1, \dots, x_N)$. We have

$$\{x_i, x_j\}_{\star_G} = \mathcal{J}(p_1 q^{-1}(q(e_i)q(e_j) - q(e_j)q(e_i))) = \mathcal{J}(p_1 q^{-1}(q([e_i, e_j]))) = \mathcal{J}([e_i, e_j]) = x_k C_{ij}^k$$

where we denote by C_{ij}^k the structure constants of the Lie algebra $\mathfrak{g}$ and $\mathcal{J}\colon \mathrm{S}\mathfrak{g} \to \mathscr{C}^\infty(\mathfrak{g}^*)$ and hence

$$\{f, g\}_{\star_G} = x^k C_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

and the claim is proven. □

The Gutt product shows already that it is non trivial to find star product even in the easy situation of a vector space with a linear Poisson structure. For quadratic and/or higher polynomial Poisson structures there is no general way of quantizing them (without using the results of Kontsevich or Tamarkin), but however recently it was discovered that at least some star products can be obtained combinatorially in a similar way as the Gutt product.

3.4 Fedosov's Construction

The original construction of Fedosov included the construction of star products only for symplectic manifolds (see [3]), but his techniques are so flexible that one can use it for regular Poisson manifolds as well. It was not the first proof of the existence of star products on symplectic manifolds, this was given in [2] by de Wilde and Lecomte, but Fedosov's proof is very constructive, allows even to classify all symplectic star products and in fact, his techniques where used to globalize the existence of star products for the case of $\mathbb{R}^d$ with an arbitrary Poisson structure from Kontsevich to an arbitrary

Poisson manifold. It is very much inspired by Gel'fand and Fuks's *formal geometry* from [4], which has many applications in differential geometry.

Let us discuss the basic idea behind his construction for $M = \mathbb{R}^{2n}$ with coordiantes $\{x^i\}_{i\in\{1,\ldots,2n\}} = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ together with the standard Poisson structure

$$\pi = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

This is clearly a Poisson structure with constant coefficients and thus can be quantized with Equation (3.2.1). This idea has no hope to be globalized to manifolds, but there is a way out: let us introduce formal variables $y^1, \ldots, y^{2n}$ and consider $\mathscr{C}^\infty(M)[[y^1, \ldots, y^{2n}]]$, i.e. formal power series in $2n$. In particular, elements of $\mathscr{C}^\infty(M)[[y^1, \ldots, y^{2n}]]$ are of the form

$$\sum_{k=0}^{\infty} a_{i_1\ldots i_k}(x) y^{i_1} \ldots y^{i_k}.$$

We define now a product $\circ$ by

$$a \circ b = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}\pi^{i_1 j_1} \ldots \pi^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \ldots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \ldots \partial y^{j_k}}.$$

Note that this product is $\mathscr{C}^\infty(M)$-linear since the derivatives are only in y-directions and hence far away from what we want. But we have a canonical map $\mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)[[y^1, \ldots, y^{2n}]]$ given by the Taylor expansion:

$$T(f) = \mathrm{e}^{y^i \frac{\partial}{\partial x^i}}(f) = \sum_{k=0}^{\infty} \sum_{I\in\mathbb{N}_0^{2n}, |I|=k} \frac{1}{I!}\frac{\partial^k f}{\partial x^I} y^I.$$

We can now define

$$f \star g = p(T(f) \circ T(g)), \tag{3.4.1}$$

but in order to show that this product is associative, we have to make sure that $T(f) \circ T(g) \in \operatorname{im} T$ for all $f, g \in \mathscr{C}^\infty(M)$. Taylor series can by characterized by the following equation

$$a = T(f) \quad \Longleftrightarrow \quad (\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i})a = 0 \;\text{ for all } i \in \{1, \ldots, 2n\}.$$

And in fact, one can show that

$$(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i})(a \circ b) = ((\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i})a) \circ b + a \circ (\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i})b$$

holds for all $i \in \{1, \ldots, 2n\}$ and hence Formula (3.4.1) defines an associative product and coincides with the one we defined in Equation (3.2.1). The difference is now that this ansatz can be formalized and applied to arbitrary regular Poisson manifolds.

Throughout the whole section we fix a regular Poisson manifold (M, π) and we denote by $\mathcal{F} \subseteq TM$ the associated involutive subbundle and by ω the associated foliated symplectic 2-form.

3.4.1 The Weyl algebra bundle

We consider the $\mathscr{C}^\infty(M)$-module

$$\mathcal{W} := \prod_{i=0}^{\infty} \Gamma^\infty(\mathrm{S}^i\mathcal{F}^*).$$

Note that there are definitions of pro-finite dimensional vector bundles in differential geometry, where the above $\mathscr{C}^\infty(M)$-module is the space of smooth sections of a vector bundle in this sense. Nevertheless, for us this doe not matter at the moment.

We have an obvious commutative multiplication on $\mathcal{W}$, which is given by the symmetric product

$$\mathcal{W} \times \mathcal{W} \ni (P, Q) \mapsto P \vee Q \in \mathcal{W},$$

which is compatible with the $\mathscr{C}^\infty(M)$-module structure Moreover, we consider

$$\mathcal{W} \otimes \Lambda^\bullet := \prod_{i=0}^{\infty} \Gamma^\infty(\mathrm{S}^i\mathcal{F}^* \otimes \Lambda^\bullet\mathcal{F}^*)$$

with the product

$$\mu\colon \mathcal{W} \otimes \Lambda^k \times \mathcal{W} \otimes \Lambda^\ell \ni (P \otimes \alpha, Q \otimes \beta) \mapsto P \vee Q \otimes \alpha \wedge \beta \in \mathcal{W} \otimes \Lambda^{k+\ell}.$$

This product is graded commutative with respect to the anti symmetric degree, i.e.

$$a \cdot b := \mu(a, b) = (-1)^{k\ell}\mu(b, a) = (-1)^{k\ell} b \cdot a$$

for $a \in \mathcal{W} \otimes \Lambda^k$ and $b \in \mathcal{W} \otimes \Lambda^\ell$. Let us denote for a section $X \in \Gamma^\infty(\mathcal{F})$ the insertions

$$\mathrm{i_s}(X)(P \otimes \alpha) := \iota_X(P) \otimes \alpha \quad \text{and} \quad \mathrm{i_a}(X)(P \otimes \alpha) := P \otimes \iota_X\alpha$$

for $P \otimes \alpha \in \mathcal{W} \otimes \Lambda^\bullet$. We want to keep track of different gradings and hence, we introduce:

$$\deg_s, \deg_a\colon \mathcal{W} \otimes \Lambda \to \mathcal{W} \otimes \Lambda^\bullet$$

by

$$\deg_s(P \otimes \alpha) = k(P \otimes \alpha) \quad \text{and}$$
$$\deg_a(P \otimes \alpha) = \ell(P \otimes \alpha)$$

for $P \otimes \alpha \in \Gamma^\infty(\mathrm{S}^k\mathcal{F}^* \otimes \Lambda^\ell\mathcal{F}^*)$.

Lemma 3.4.1 *The maps* $\deg_s, \deg_a\colon \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^\bullet$ *can be realized in a local trivialization by*

$$\deg_s = (e^i \otimes 1) \cdot \mathrm{i_s}(e_i) \quad \textit{and} \quad \deg_s = (1 \otimes e^i) \cdot \mathrm{i_a}(e_i),$$

where $\{e_i\}_{i\in I}$ *are local basis sections of* $\mathcal{F}$ *with dual* $\{e^i\}_{i\in I}$.

PROOF:

We introduce the linear map $\delta\colon \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ by

$$\delta(P \otimes \alpha) = (1 \otimes e^i) \cdot \mathrm{i_s}(e_i)(P \otimes \alpha) = \iota_{e_i}P \otimes e^i \wedge \alpha$$

for a basis sections $\{e_i\}_{i\in I}$ with dual $\{e^i\}_{i\in I}$ of $\mathcal{F}$. Note that the definition of δ is independent of the chosen basis section (check!) and hence it is globally defined.

Lemma 3.4.2 *The following identities hold*

(a) $\delta^2 = 0$

(b) $\delta(a \cdot b) = (\delta a) \cdot b + (-1)^k a \cdot (\delta b)$ *for* $a \in \mathcal{W} \otimes \Lambda^k$ *and* $b \in \mathcal{W} \otimes \Lambda^\bullet$.

PROOF: The first result is an easy consequences of the fact that $\mathrm{i_s}(X)\,\mathrm{i_s}(Y) = \mathrm{i_s}(Y)\,\mathrm{i_s}(X)$ for all $X, Y \in \Gamma^\infty(\mathcal{F})$. And the second result is a consequence of the fact that $\mathrm{i_s}(X)$ is a derivation of the symmetric product. ☐

The previous Lemma showed that the map δ is a differential and hence we have a canonical cohomology attached to it. In this case it is rather simple: let us define the map $\delta^* \colon \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet-1}$ by

$$\delta^*(P \otimes \alpha) = (e^i \otimes 1) \cdot \mathrm{i_a}(e_i)(P \otimes \alpha) = e^i \vee P \otimes \iota_{e_i}\alpha$$

for a basis sections $\{e_i\}_{i\in I}$ with dual $\{e^i\}_{i\in I}$ of $\mathcal{F}$. Note that the definition of δ^* is also independent of the chosen basis section (check!) and hence it is globally defined.

Lemma 3.4.3 *The following equation holds*

$$[\delta^*, \delta] = \delta^*\delta + \delta\delta^* = \deg_s + \deg_a.$$

PROOF: Let $P \otimes \alpha \in \mathcal{W} \otimes \Lambda$, then

$$\begin{aligned} \delta\delta^*(P \otimes \alpha) &= \delta(e^i \vee P \otimes \iota_{e_i}\alpha) = \iota_{e_j}(e^i \vee P) \otimes e^j \wedge \iota_{e_i}\alpha \\ &= P \otimes e^i \wedge \iota_{e_i}\alpha + e^i \vee \iota_{e_j} P \otimes e^j \wedge \iota_{e_i}\alpha \\ &= \deg_a(P \otimes \alpha) - e^i \vee \iota_{e_j} P \otimes \iota_{e_i}(e^j \wedge \alpha) + e^i \vee \iota_{e_i} P \otimes \alpha \\ &= \deg_a(P \otimes \alpha) - e^i \vee \iota_{e_j} P \otimes \iota_{e_i}(e^j \wedge \alpha) + \deg_s(P \otimes \alpha) \\ &= \deg_a(P \otimes \alpha) + \deg_s(P \otimes \alpha) - \delta^*\delta(P \otimes \alpha). \end{aligned}$$ ☐

Let us now define for $P \otimes \alpha \in \Gamma^\infty(\mathrm{S}^k\mathcal{F}^* \otimes \Lambda^\ell\mathcal{F}^*)$

$$\delta^{-1}(P \otimes \alpha) = \begin{cases} \frac{1}{k+\ell}\delta^*(P \otimes \alpha), & \text{for } k + \ell \neq 0 \\ 0, & \text{for } k + \ell = 0 \end{cases}$$

and with Lemma 3.4.3 we see that

$$\delta^{-1}\delta + \delta\delta^{-1} + \sigma = \mathrm{id}$$

where $\sigma \colon \mathcal{W} \otimes \Lambda^\bullet \to \mathscr{C}^\infty(M) = \Gamma^\infty(\mathrm{S}^0\mathcal{F}^* \otimes \Lambda^0\mathcal{F}^*)$ is the projection to symmetric and anti-symmetric degree 0.

The idea is now to deform the (graded) commutative algebra $\mathcal{W} \otimes \Lambda^\bullet$ into a non-commutative one using the foliated symplectic form, or better said the Poisson structure: We define

$$\Pi((P \otimes \alpha) \otimes (Q \otimes \beta)) := \pi^{ij}(\iota_{e_i} P \otimes \alpha) \otimes (\iota_{e_j} Q \otimes \beta)$$

as an operation on $(\mathcal{W} \otimes \Lambda^\bullet)^{\otimes 2}$ and define

$$a \circ_F b = \mu \circ \mathrm{e}^{\frac{\hbar}{2}\Pi}(a \otimes b) \in \mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$$

for $a, b \in \mathcal{W} \otimes \Lambda^\bullet$ similar to Example 3.2.1.

Lemma 3.4.4 *The product $\circ_F$ is an associative deformation of the graded product μ and δ is a derivation of degree 1 of $\circ_F$ with respect to the anti-symmetric degree.*

PROOF: The product is an associative deformation, since the symmetric insertions $\mathrm{i_s}(e_i)$ are commuting derivations and this is also the reason, why δ is a derivation. □

Proposition 3.4.5 *Let $a \in \mathcal{W} \otimes \Lambda^k[[\hbar]]$ such that*

$$a \circ b - (-1)^{k\ell} b \circ a = 0$$

for all $\mathcal{W} \otimes \Lambda^\ell[[\hbar]]$ and all $\ell \in \mathbb{N}_0$, then $a \in \Gamma^\infty(\Lambda^k \mathcal{F}^*)$. Moreover, every $a \in \Gamma^\infty(\Lambda^k \mathcal{F}^*)[[\hbar]]$ fulfills Equation (3.4.5).

PROOF: Let $s \in \Gamma^\infty(\mathcal{F}^*) \subseteq \mathcal{W} \otimes \Lambda^0$, then we have

$$[a, s]_{\circ_F} = a \circ s - (-)^{k\ell} s \circ a = \hbar\, \mathrm{i_s}(\pi^\sharp(s))\alpha$$

and if this is 0, α only have trivial symmetric degree, since π is non-degenerate on $\mathcal{F}$. □

3.4.2 The Fedosov Derivation and the Fedov Star Product

We choose now a partial connection ∇ with $\nabla_X \omega = 0$ for all $X \in \Gamma^\infty(\mathcal{F})$, which exists due to Lemma 2.5.12 and introduce the map

$$D \colon \mathcal{W} \otimes \Lambda^\bullet \ni P \otimes \alpha \mapsto \nabla_{e_i} P \otimes e^i \wedge \alpha + P \otimes \mathrm{d}_\mathcal{F} \alpha \in \mathcal{W} \otimes \Lambda^{\bullet+1}.$$

Note that this is equivalent to

$$D = (1 \otimes e^i) \cdot \nabla_{e_i},$$

where ∇ is extended to $\mathcal{W} \otimes \Lambda^\bullet$ by $\nabla_X(P \otimes \alpha) = \nabla_X P \otimes \alpha + P \otimes \nabla_X \alpha$ and the reason for this is that ∇ fulfills $\nabla_X Y - \nabla_Y X - [X, Y] = 0$. Let us we introduce the curvature

$$\hat{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \Gamma^\infty(\mathcal{F})$. Note that we actually have

$$\hat{R} \in \Gamma^\infty(\Lambda^2 \mathcal{F}^* \otimes \mathrm{End}(\mathcal{F})) = \Gamma^\infty(\Lambda^2 \mathcal{F}^* \otimes \mathcal{F}^* \otimes \mathcal{F}),$$

so it is $\mathscr{C}^\infty(M)$-linear. We now contract this to R defined by

$$R(Z, U, X, Y) = \omega(R(X, Y)Z, U)$$

for $Z, U, X, Y \in \Gamma^\infty(\mathcal{F})$ Using the fact that $\nabla_X \omega = 0$ for all $X \in \Gamma^\infty(\mathcal{F})$, we see that $R(Z, U, X, Y) = R(U, Z, X, Y)$ and by the very definition of $\hat{R}$ we get $R(U, Z, X, Y) = -R(U, Z, Y, X)$. And thus we have

$$R \in \Gamma^\infty(\mathrm{S}^2 \mathcal{F}^* \otimes \Lambda^2 \mathcal{F}^*) \subseteq \mathcal{W} \otimes \Lambda^\bullet$$

Proposition 3.4.6 *The identities*

$$[\delta, D] = 0 \quad \textit{and} \quad D^2 = \frac{1}{2}[D, D] = \frac{1}{\hbar}[R, \cdot\,]_{\circ_F}$$

hold. Moreover, D is a derivation of $\circ_F$ of degree 1 with respect to the anti-symmetric degree.

PROOF: Let $P \otimes \alpha \in \mathcal{W} \otimes \Lambda^k$ be given, then we have:

$$\begin{aligned}\delta D(P \otimes \alpha) &= \delta(\nabla_{e_i} P \otimes e^i \wedge \alpha + P \otimes \mathrm{d}_{\mathcal{F}}\alpha)\\ &= \iota_{e_j}\nabla_{e_i} P \otimes e^j \wedge e^i \wedge \alpha + \iota_{e_i} P \otimes e^i \wedge \mathrm{d}_{\mathcal{F}}\alpha\\ &= [\iota_{e_j}, \nabla_{e_i}]P \otimes e^j \wedge e^i \wedge \alpha + \nabla_{e_i}\iota_{e_j} P \otimes e^j \wedge e^i \wedge \alpha - \iota_{e_i} P \otimes \mathrm{d}_{\mathcal{F}}(e^i \wedge \alpha) + \iota_{e_i} P \otimes (\mathrm{d}_{\mathcal{F}} e^i) \wedge \alpha\\ &= -D\delta(P \otimes \alpha) + [\iota_{e_j}, \nabla_{e_i}]P \otimes e^j \wedge e^i \wedge \alpha + \iota_{e_i} P \otimes (\mathrm{d}_{\mathcal{F}} e^i) \wedge \alpha\end{aligned}$$

So let us check what the two terms are: for a local basis there are local functions C^k_{ij}, such that $[e_i, e_j] = C^k_{ij} e_k$, thereforewe have

$$\mathrm{d}_{\mathcal{F}} e^i(e_m, e_n) = \overbrace{e_m(e^i(e_n))}^{e_m(\delta^i_n)=0} - e_n(e^i(e_m)) - e^i([e_m, e_n]) = -C^i_{mn} = -\frac{1}{2} C^i_{jk} e^j \wedge e^k(e_m, e_n).$$

Moreover, we have for $\alpha \in \Gamma^\infty(\mathcal{F}^*)$, that

$$\iota_X \nabla_Y \alpha = \nabla_Y \alpha(X) = Y\alpha(X) - \alpha(\nabla_Y X) = \nabla_Y \iota_X - \iota_{\nabla_Y X}\alpha.$$

So if we denote by Γ^k_{ij} the coeeficients $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, we find

$$\begin{aligned}[\iota_{e_j}, \nabla_{e_i}]P \otimes e^j \wedge e^i \wedge \alpha &= -\iota_{\nabla_{e_i} e_j} P \otimes e^j \wedge e^i \wedge \alpha = -\Gamma^k_{ij}\iota_{e_k} P \otimes e_j \wedge e^i \wedge \alpha\\ &= -\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji})\iota_{e_k} P \otimes e^j \wedge e^i \wedge \alpha.\end{aligned}$$

With $\Gamma^k_{ij} - \Gamma^k_{ji} = C^k_{ij}$, since $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, we get $[\delta, D] = 0$. We proceed showing that D is a derivation of $\circ_F$. Note that by definition it is a derivation of the undeformed product. We see that for $X \in \Gamma^\infty(\mathcal{F})$

$$\begin{aligned}&(\nabla_X \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X)(\Pi((P \otimes \alpha) \otimes (Q \otimes \beta)) = (\nabla_X \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X)(\pi^{ij}(\iota_{e_i} P \otimes \alpha) \otimes (\iota_{e_j} Q \otimes \beta))\\ &\qquad = X(\pi^{ij})(\iota_{e_i} P \otimes \alpha) \otimes (\iota_{e_j} Q \otimes \beta) + \pi^{ij}((\nabla_X \iota_{e_i} P \otimes \alpha)(\otimes \iota_{e_j} Q \otimes \beta))\\ &\qquad + \pi^{ij}((\iota_{e_i} P \otimes \alpha)(\nabla_X \iota_{e_j} Q \otimes \beta))\\ &\qquad = \ldots\\ &\qquad = (\nabla_X \pi)^{ij}(\iota_{e_i} P \otimes \alpha) \otimes (\iota_{e_j} Q \otimes \beta) + \Pi(\nabla_X \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X)((P \otimes \alpha) \otimes (Q \otimes \beta)),\end{aligned}$$

but since $\nabla_X \pi = 0$ we get that $(\nabla_X \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X) \circ \Pi = \Pi \circ (\nabla_X \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_X)$ and hence we get that ∇_X is a derivation of $\circ_F$ by the construction of $\circ_F$. This means in particular for $a \in \mathcal{W} \otimes \Lambda^k$ and $b \in \mathcal{W} \otimes \Lambda^\bullet$:

$$\begin{aligned}D(a \circ_F b) = (1 \otimes e^i) \cdot \nabla_{e_i}(a \circ_F b) &= (1 \otimes e^i) \cdot \nabla_{e_i} a \circ_F b + (1 \otimes e^i) \cdot a \circ_F \nabla_{e_i} b\\ &= ((1 \otimes e^i) \cdot \nabla_{e_i} a) \circ_F b + (-1)^k a \circ_F (1 \otimes e^i) \cdot \nabla_{e_i} b,\end{aligned}$$

where we used that $(1 \otimes e^i) \cdot a = (1 \otimes e^i) \circ_F a$ and $(1 \otimes e^i) \cdot a = (-1)^k a \cdot (1 \otimes e^i)$.

For the last part we take $P \otimes \alpha \in \mathcal{W} \otimes \Lambda^k$ and see

$$\begin{aligned}D^2(P \otimes \alpha) &= D(\nabla_{e_i} P \otimes e^i \wedge \alpha + P \otimes \mathrm{d}_{\mathcal{F}}\alpha)\\ &= \nabla_{e_j}\nabla_{e_i} P \otimes e^j \wedge e^i \wedge \alpha + \nabla_{e_i} P \otimes \mathrm{d}_{\mathcal{F}}(e^i \wedge \alpha) + \nabla_{e_i} P \otimes e^i \wedge \mathrm{d}_{\mathcal{F}}\alpha\\ &= \frac{1}{2}[\nabla_{e_j}, \nabla_{e_i}]P \otimes e^j \wedge e^i \wedge \alpha + \nabla_{e_i} P \otimes \mathrm{d}_{\mathcal{F}}(e^i) \wedge \alpha\\ &= \frac{1}{2}[\nabla_{e_i}, \nabla_{e_j}]P \otimes e^i \wedge e^j \wedge \alpha - \frac{1}{2} C^k_{ij}\nabla_{e_k} P \otimes e^i \wedge e^j \wedge \alpha\end{aligned}$$

$$= \frac{1}{2}\big([\nabla_{e_i}, \nabla_{e_j}]P \otimes e^i \wedge e^j \wedge \alpha - \nabla_{[e_i,e_j]}P \otimes e^i \wedge e^j \wedge \alpha\big)$$
$$= \frac{1}{2}\big(([\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i,e_j]})P \otimes e^i \wedge e^j \wedge \alpha\big)$$

If we define the coefficients $\hat{R}^{\ell}_{kij}$ by $\hat{R}(e_i, e_j)e_k = \hat{R}^{\ell}_{ijk}e_\ell$, we see

$$\begin{aligned} D^2(P \otimes \alpha) &= -\frac{1}{2}\hat{R}^{\ell}_{ijk}e^k \vee \iota_{e_\ell}P \otimes e^i \wedge e^j \wedge \alpha \\ &= -\frac{1}{2}\pi^{m\ell}\omega_{mn}\hat{R}^{n}_{ijk}e^k \vee \iota_{e_\ell}P \otimes e^i \wedge e^j \wedge \alpha \\ &= \frac{1}{2}\pi^{m\ell}R_{ijkm}e^k \vee \iota_{e_\ell}P \otimes e^i \wedge e^j \wedge \alpha \\ &= \frac{1}{4}\pi^{m\ell}(\iota_{e_m}R_{ijko}e^k \vee e^o) \vee \iota_{e_\ell}P \otimes e^i \wedge e^j \wedge \alpha \end{aligned}$$

were $R_{ijk\ell}$ are the coefficients $R_{k\ell ij} = R(e_k, e_\ell, e_i, e_j) = \omega_{km}\hat{R}^m_{\ell ij}$ of the contracted cuvature tensor from Equation (3.4.2). Since $R = \frac{1}{4}R_{k\ell ij}e^k \vee e^\ell \otimes e^i \wedge e^j$, we have that

$$[R, \cdot\,]_{\circ_F} = \hbar D^2 + \mathcal{O}(\hbar^2).$$

Since R has only two symmetric degrees, we see immediatly that $[R, \cdot\,]_{\circ_F} - \hbar D^2$ has order $\hbar^2$, but for $\circ_F$, we have that for all $a \in \mathcal{W} \otimes \Lambda^\bullet$, that $[R, a]$ has only odd orders of $\hbar$ and hence $[R, \cdot\,]_{\circ_F} = \hbar D^2$ and the claim is proven. □

The idea is now to consider the sum $-\delta + D$, which is a derivation of $\circ_F$ of degree 1 and thus we have

$$(-\delta + D)^2 = \frac{1}{\hbar}[R, \cdot\,]$$

and so it is not a differential. To correct this, we make the following Ansatz

$$\mathcal{D} = -\delta + D + \frac{1}{\hbar}[r, \cdot\,]$$

for $r \in \mathcal{W} \otimes \Lambda^1[[\hbar]]$. Using the graded commutator, we see that

$$\mathcal{D}^2 = \frac{1}{2}[\mathcal{D}, \mathcal{D}] = \frac{1}{\hbar}[R - \delta r + Dr + \frac{1}{2\hbar}[r, r], \cdot\,].$$

So if we can find $r \in \mathcal{W} \otimes \Lambda^1[[\hbar]]$, such that $R - \delta r + Dr + \frac{1}{2\hbar}[r, r]$ is central, we have that $\mathcal{D}^2 = 0$. This means in particular that

$$R - \delta r + Dr + \frac{1}{2\hbar}[r, r] = \Omega$$

for $\Omega \in \Gamma^\infty(\Lambda^2\mathcal{F}^*)[[\hbar]]$ using Proposition 3.4.5.

Lemma 3.4.7 *For every $r \in \mathcal{W} \otimes \Lambda^1[[\hbar]]$ we get for $\mathcal{D} = -\delta + D + \frac{1}{\hbar}[r, \cdot\,]$ the equation*

$$\mathcal{D}(-\delta r + R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F}) = 0.$$

PROOF: We have

$$\mathcal{D}(-\delta r + R + Dr + \frac{1}{2\hbar}[r, r]) = -[\delta, D]r + DR - \delta R - \frac{1}{2\hbar}\delta[r, r]_{\circ_F} + \frac{1}{2\hbar}D[r, r]_{\circ_F} - \frac{1}{\hbar}[r, \delta r]_{\circ_F} +$$

$$\begin{aligned}
&\frac{1}{\hbar}[r, R]_{\circ_F} + D^2 r + \frac{1}{\hbar}[r, Dr]_{\circ_F} + \frac{1}{2\hbar^2}[r, [r, r]_{\circ_F}]_{\circ_F} \\
&= -\delta R + DR + \frac{1}{2\hbar^2}[r, [r, r]_{\circ_F}]_{\circ_F},
\end{aligned}$$

where we used that δ and D are graded derivations of degree 1 in the last step as well as $D^2 = \frac{1}{\hbar}[R, \cdot\,]_{\circ_F}$. Note that $[r, [r, r]_{\circ_F}]_{\circ_F} = 0$ for every degree 1 element. Moreover, we get

$$[\delta R, \cdot\,]_{\circ_F} = [\delta, [R, \cdot\,]_{\circ_F}]] = \hbar[\delta, [D, D]] = \hbar([[\delta, D], D] - [D, [\delta, D]]) = 0.$$

So δR is central and by Proposition 3.4.5, it has to have symmetric degree 0, but it has also symmetric degree 1, by the form of R. The only possibility is that $\delta R = 0$. Similarly, we proceed with DR, we have

$$[DR, \cdot\,]_{\circ_F} = [D, [R, \cdot\,]_{\circ_F}] = \hbar[D, [D, D]] = 0$$

and since DR has symmetric degree 2, we get, again by Propisiton 3.4.5, that $DR = 0$. □

If we want now that $\mathcal{D} = -\delta + D + \frac{1}{\hbar}[r, \cdot\,]$ squares to 0 or equivalently

$$R - \delta r + Dr + \frac{1}{2\hbar}[r, r] = \Omega$$

for $\Omega \in \Gamma^\infty(\Lambda^2 \mathcal{F}^*)[[\hbar]]$, we know already that the only possibility is $\mathcal{D}\Omega = \mathrm{d}_{\mathcal{F}}\Omega = 0$ by Lemma 3.4.7. Let us make our ansatz now a bit more precise: we introduce the total degree $\mathrm{Deg}\colon \mathcal{W} \otimes \Lambda^\bullet[[\hbar]] \to \mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$ by

$$\mathrm{Deg} = \deg_{\mathrm{s}} + 2\hbar\frac{\partial}{\partial \hbar}.$$

This is a derivation of degree 0 of $\circ_F$, i.e.

$$\mathrm{Deg}(a \circ_F b) = \mathrm{Deg}(a) \circ_F b + a \circ_F \mathrm{Deg}(b),$$

which can be seen by the observation that $\frac{\hbar}{2}\Pi$ kills two symmetric degrees and adds one $\hbar$-degree, i.e.

$$[\mathrm{Deg} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{Deg}, \tfrac{\hbar}{2}\Pi] = 0.$$

Note that neither $\deg_{\mathrm{s}}$ nor $2\hbar\frac{\partial}{\partial\hbar}$ are derivations of $\circ_F$ separately. We want to understand $\mathcal{D} = -\delta + D + \frac{1}{\hbar}[r, \cdot\,]$ now as a series of operators with respect to the total degree: δ lowers the total degree by 1, D keeps it, and hence we want we want $\frac{1}{\hbar}[r, \cdot\,]$ to be a series which does not decrease the total degree. Note that if an element a has total degree k, then $\frac{1}{\hbar}[a,]$ increases the total degree by $k - 2$.

Let us define $\mathcal{W}_{(k)} \otimes \Lambda^\bullet[[\hbar]] := \{a \in \mathcal{W} \otimes \Lambda^\bullet[[\hbar]] \mid \mathrm{Deg}(a) = ka\}$ and the canonical associated filtration

$$\mathcal{W}_k \otimes \Lambda^\bullet[[\hbar]] = \prod_{i=k}^{\infty} \mathcal{W}_{(i)} \otimes \Lambda^\bullet[[\hbar]].$$

Proposition 3.4.8 *We have*

$$\mathcal{W}_0 \otimes \Lambda^\bullet[[\hbar]] = \mathcal{W} \otimes \Lambda^\bullet[[\hbar]] \quad \textit{and} \quad \bigcap_{i=0}^{\infty} \mathcal{W}_i \otimes \Lambda^\bullet[[\hbar]] = 0.$$

PROOF: The proof is almost a tautology. Let us write for $a^{(k)} = \sum_{2i+j=k}^{k} a_j^{(k)}\hbar^i$ where a_j has symmetric degree j. So for a general element in $a = (a^{(1)}, \dots) \in \prod_{k=0}^{\infty} \mathcal{W}_{(k)} \otimes \Lambda^\bullet[[\hbar]]$, we want to show that

$$a = \sum_{i=0}^{\infty}(\sum_{k=0}^{\infty} a_{k-2i}^{(k)})\hbar^i = \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_j^{(j+2i)})\hbar^i$$

is a well-defined element in $\mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$. So we just have to make sure it is a well-defined formal power series in each $\hbar$-degree(so no infinite series of fixed symmetric degree appear). But the only symmetric term in degree j appearing in $\hbar$-degree m is $a_j^{(j+2m)}$. □

This proposition shows that we can decompose every $a \in \mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$ into a series $\sum_{k=0}^{\infty} a_{(k)}$, where $a^{(k)} \in \mathcal{W}_{(k)} \otimes \Lambda^\bullet[[\hbar]]$. Moreover, every series $\sum_k a^{(k)}$ with $a^{(k)} \in \mathcal{W}_{(k)} \otimes \Lambda^\bullet[[\hbar]]$ converges to an Element in $\mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$. This allows us to prove:

Theorem 3.4.9 *For each $\Omega \in \hbar\Gamma^\infty(\Lambda^2\mathcal{F}^*)[[\hbar]]$ with $\mathrm{d}_\mathcal{F}\Omega = 0$ there is a unique $r \in \mathcal{W}_3 \otimes \Lambda^1[[\hbar]]$, such that*

$$\delta r = R + Dr + \frac{1}{2\hbar}[r, r] + \Omega \quad \textit{and} \quad \delta^{-1} r = 0.$$

PROOF: We define a recursive formula of elements of homogeneous total degree by $r^{(3)} = \delta^{-1}(R+\Omega^{(2)})$ and

$$r^{(k+1)} = \delta^{-1}\big(Dr^{(k)} + \frac{1}{2\hbar}\sum_{i=3}^{k-1}[r^{(i)}, r^{(k+2-i)}] + \Omega^{(k)}\big)$$

for $k \geq 3$. Then we know by Proposition 3.4.8, that $r = \sum_{i=3}^{\infty} r^{(i)}$ an element in $\mathcal{W} \otimes \Lambda^1[[\hbar]]$. This element fulfills, by taking sums over the total degrees, the necessary equation:

$$r = \delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega).$$

Since $(\delta^{-1})^2 = 0$ and $\delta^{-1}r^{(k)} = 0$ for all k, we also have that $\delta^{-1}r = 0$. Let us denote by $A = \delta r - R - Dr - \frac{1}{2\hbar}[r, r]_{\circ_F} - \Omega$, then we have

$$\mathcal{D}A = 0 \iff \delta A = (D + \frac{1}{\hbar}[r, \cdot]_{\circ_F})(A),$$

since $\mathcal{D}\Omega = \mathrm{d}_\mathcal{F}\Omega = 0$ by Lemma 3.4.7. Using $\sigma(r) = \sigma(\delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega)) = 0$, we get

$$\begin{aligned}
\delta^{-1}A &= \delta^{-1}(\delta r - R - Dr - \frac{1}{2\hbar}[r, r]_{\circ_F} - \Omega) \\
&= \delta{-}1\delta r - \delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega) \\
&= r - \delta\delta^{-1}r - \sigma(r) - \delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega) \\
&= r - \delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega) = 0.
\end{aligned}$$

Using again $\delta\delta^{-1} + \delta^{-1}\delta + \sigma = \mathrm{id}$, we get

$$A = \delta^{-1}\delta A = \delta^{-1}(DA + \frac{1}{\hbar}[r, A]).$$

The operator $a \mapsto \delta^{-1}(Da + \frac{1}{\hbar}[r, a])$ is a linear operator which increases the total degree and hence the only fixed point can be 0. So we found an r fulfilling the required equation. The uniqueness follows from the fact that every r fulfilling the required equations has to fulfill

$$r = \delta^{-1}(R + Dr + \frac{1}{2\hbar}[r, r]_{\circ_F} + \Omega)$$

and hence the same recursion. □

Let us pick a closed $\Omega \in \hbar\Gamma^\infty(\Lambda^2\mathcal{F}^*)[[\hbar]]$, denote by r the corresponding solution to Equations (3.4.9) and let us denote by

$$\mathcal{D} = -\delta + D + \frac{1}{\hbar}[r, \cdot\,]_{\circ_F}$$

the associated differential. Our interpretation was that $\mathcal{D}$ is a series of operators, which increase the symmetric degree, so in particular it is a *pertubation* of the differential $-\delta$ which aleady has an homotopy δ^{-1}:

$$\mathscr{C}^\infty(M)[[\hbar]] \underset{\sigma}{\overset{i}{\rightleftarrows}} (\mathcal{W} \otimes \Lambda^\bullet[[\hbar]], -\delta) \circlearrowright_{-\delta^{-1}}$$

where we denote by i the canonical inclusion $\mathscr{C}^\infty(M) = \Gamma^\infty(\mathrm{S}^0\mathcal{F}^* \otimes \Lambda^0\mathcal{F}^*)$ and by σ the projection to symmetric and anti-symmetric degree 0. Note that we have the canonical identities $(\delta^{-1})^2 = 0$, $\delta \circ i = \delta^{-1} \circ i = 0$, $\sigma \circ \delta^{-1} = \sigma \circ \delta = 0$ and $\sigma \circ i = \mathrm{id}$. The idea is now to deform the whole diagram (3.4.2), and not only $-\delta$ to $\mathcal{D}$ to do so we use *homological pertubation theory*:

Theorem 3.4.10 *For* $O = [\delta^{-1}, D + \frac{1}{\hbar}[r, \cdot\,]]$ *we have that* $\mathrm{id} - O$ *is invertible and the operator*

$$\mathcal{D}^{-1} := -\delta^{-1}(\mathrm{id} - O)^{-1} = -(\mathrm{id} - O)^{-1}\delta^{-1}$$

is an endomorphism of degree -1 *and fulfills*

$$\mathcal{D}^{-1}\mathcal{D} + \mathcal{D}\mathcal{D}^{-1} + (id - O)^{-1}i\sigma = \mathrm{id} \tag{3.4.2}$$

PROOF: First of all we notice that O increases the total degree by at least 1 and hence we have

$$\sum_{k=0}^{\infty} O^k$$

is a well-defined map and hence we have that $\mathrm{id} - O$ is invertible. Moreover, O is of anti-symmetric degree 0 and so is $\mathrm{id} - O$ and also $(\mathrm{id} - O)^{-1}$. Let $a \in \mathcal{W} \otimes \Lambda^\bullet[[\hbar]]$, then we have

$$\begin{aligned}
&-\mathcal{D}\delta^{-1}a - \delta^{-1}\mathcal{D}a + \sigma(a) \\
&\quad = \delta\delta^{-1}a - (D + \frac{1}{\hbar}[r, \cdot\,]_{\circ_F})\delta^{-1}a + \delta\delta^{-1}a - \delta^{-1}(D + \frac{1}{\hbar}[r, \cdot\,]_{\circ_F})a + \sigma(a) \\
&\quad = a - [\delta^{-1}, D + \frac{1}{\hbar}[r, \cdot\,]_{\circ_F}]a = (\mathrm{id} - O)a.
\end{aligned}$$

If we apply $\delta{-1}$ once from the right and once from the left to this equation and then subtract the equations from each other, we have

$$\delta^{-1}O = O\delta^{-1}.$$

We apply now $\mathcal{D}$ to the same equation, once from the right and once from the left and get $\mathcal{D}(\mathrm{id} - O) = \mathcal{D}\sigma + (\mathrm{id} - O)\mathcal{D}$ and hence $\mathcal{D}(\mathrm{id} - O)\delta^{-1} = (\mathrm{id} - O)\mathcal{D}\delta^{-1}$ and finally also

$$\mathcal{D}(\mathrm{id} - O)^{-1}\delta^{-1} = (\mathrm{id} - O)^{-1}\mathcal{D}\delta^{-1}$$

and this already proves the claim. □

Let us denote from now on

$$\tau\colon \mathscr{C}^\infty(M)[[\hbar]] \ni f \mapsto (\mathrm{id} - O)f = \sum_{k=0}^{\infty}[\delta^{-1}, D + \frac{1}{\hbar}[r, \cdot]_{\circ_F}]^k f \in \mathcal{W} \otimes \Lambda^0[[\hbar]]$$

and call it the *Fedosov-Taylor series.*

Corollary 3.4.11 $\mathcal{D}\tau = 0$, $(\mathcal{D}^{-1})^2 = 0$, $\sigma\mathcal{D}^{-1} = 0$, $\mathcal{D}\tau = 0$ *and* $\sigma \circ \tau = \mathrm{id}$

PROOF: The proof consists of a careful counting of degrees and Equation (3.4.2). □

Proposition 3.4.12 *The map*

$$\tau\colon \mathscr{C}^\infty(M)[[\hbar]] \to \ker \mathcal{D} \cap \mathcal{W} \otimes \Lambda^0[[\hbar]]$$

is an ismorphism with inverse σ.

PROOF: The map τ is clearly injective, since $\sigma \circ \tau = \mathrm{id}$. Let now $a \in \ker \mathcal{D} \cap \mathcal{W} \otimes \Lambda^0[[\hbar]]$, then we have

$$a = \mathcal{D}\mathcal{D}^{-1}a + \mathcal{D}^{-1}\mathcal{D}a + \tau(\sigma(a)) = \tau(\sigma(a))$$

and thus τ is also surjective. □

Note that $\mathcal{W} \otimes \Lambda^0[[\hbar]]$ is a subalgebra with respect to the product $\circ_F$ by definition and that for $a, b \in \ker \mathcal{D} \cap \mathcal{W} \otimes \Lambda^0[[\hbar]]$, we have

$$\mathcal{D}(a \circ_F b) = \mathcal{D}(a) \circ_F b + a \circ_F \mathcal{D}(b) = 0$$

and hence also $\ker \mathcal{D} \cap \mathcal{W} \otimes \Lambda^0[[\hbar]]$ is a subalgebra. We therefore can define the associative product

$$f \star g = \sigma(\tau(f) \circ_F \tau(g)).$$

Theorem 3.4.13 *The product* $\star$ *is a formal star product on* M *with associated Poisson structure* π. $\star$ *is called the Fedosov star product.*

PROOF: If we write $\tau(f)$ in a series of elements in the total degree, we see that

$$\tau(f) = \sum_{i=0}^{\infty} \tau(f)^{(i)} = f + \mathrm{d}f \otimes 1 + h.o.t.$$

we have that $\tau(f \star g) = \tau(f) \circ_F \tau(g) = \sum_{k=0}^{\infty}(\tau(f) \circ_F \tau(g))^{(k)} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \tau(f)^{(i)} \circ_F \tau(g)^{(k-i)}$ and hence

$$\sigma(\tau(f)^{(i)} \circ_F \tau(g)^{(k-i)}) \in \hbar^{\frac{k}{2}}\mathscr{C}^\infty(M) \tag{3.4.3}$$

if k is even and 0 otherwise. This means in particular that $f \star g = fg + \mathcal{O}(\hbar)$ and hence $\star$ is a formal deformation of the commutative product on $\mathscr{C}^\infty(M)$, since $\tau(1) = 1$. This means in particular that

$$\begin{aligned} f \star g &= fg + \sigma(\sum_{i=0}^{2} \tau(f)^{(i)} \circ_F \tau(g)^{(k-i)}) + \mathcal{O}(\hbar^2) \\ &= fg + \sigma(\tau(f)^{(1)} \circ_F \tau(g)^{(1)}) + \mathcal{O}(\hbar^2) \\ &= fg + \frac{\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2) \end{aligned}$$

by the explicit formula of $\circ_F$. The last thing we have to discuss, is the differentiability of $\star$, but this is clear since τ is differential Operator with values in $\mathcal{W}$. □

To construct the Fedosov star product we used two ingredients: a formal series of foliated 2-forms $\Omega \in \hbar\Gamma^\infty(\Lambda^2\mathcal{F}^*)$ and a partial symplectic connection ∇. To emphasise this dependence, we write

$$(\nabla, \Omega) \to \star_{(\nabla,\Omega)}.$$

Note that this is not very well-behaved from the functorial point of view, but this is another story.

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