Albert-Ludwigs-Universität Freiburg

Habilitationsschrift

Derivator six-functor-formalisms

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Abstract

The theory of derivators enhances and simplifies the theory of triangulated categories. In this work a notion of fibered (multi)derivator is developed, which similarly enhances fibrations of (monoidal) triangulated categories. We present a theory of cohomological as well as homological descent in this language. The main motivation is a (co)descent theory for six-functor-formalisms. We develop the theory of (op)fibrations of 2-multicategories and use it to define abstract six-functor-formalisms. We also give axioms for Wirthmüller and Grothendieck formalisms (where either $f^! = f^*$ or $f_! = f_*$) or intermediate formalisms where we have e.g. a natural morphism $f_! \to f_*$. We observe that a fibered multiderivator (in particular, a closed monoidal derivator) can be interpreted itself as a six-functor-formalism on diagrams (small categories). Finally a theory of a *derivator version* of six-functor-formalisms is developed, using an extension of the notion of fibered multiderivator. Using the language of (op)fibrations of 2-multicategories this has (like a usual fibered multiderivator) a very neat definition. This definition not only encodes all compatibilities among the six functors but also their interplay with homotopy Kan extensions. One could say: a nine-functor-formalism. Finally, it is shown that every fibered multiderivator (for example encoding any kind of derived four-functor formalism $(f_*, f^*, \otimes, \mathcal{HOM})$ occurring in nature) satisfying base-change and projection formula *formally* gives rise to such a derivator six-functor-formalism in which "$f_! = f_*$", i.e. a derivator Grothendieck context.

Contents

This is a cumulative habilitation thesis. The contents of the various chapters have been submitted for publication as follows: Chapter 4, 7, and 8 in [Hör15], Chapter 2, 3, and 6 in [Hör16], and Chapter 5 and 9 in [Hör17b]. Some 1-categorical versions of parts of Chapter 5 are already contained in [Hör16].

1 Introduction

1.1 Preamble

Grothendieck, Verdier, and Deligne in the 60's observed that classical duality theorems like Poincaré, or Serre duality for the (co)homology of manifolds and algebraic varieties can be most elegantly expressed, and vastly generalized, by a formalism of the *six functors*. This makes essential use of derived categories. The latter are, however, not sufficient for the purpose of *descent*. The notion of descent is ubiquitous in mathematics. An object satisfies descent whenever its nature is determined by local conditions. The easiest example is the "glueing" of functions which are locally defined. The (co)homology of a space can be glued as well from the (co)homology of local pieces of the space; this is essential to define equivariant (co)homology and for equivariant duality theorems, and more generally to extend six-functor-formalisms to stacks, which is very important in applications. Recently six-functor-formalisms have been constructed in many more contexts, including D-modules and motives. The problem with (co)homological descent is that the "glueing data" has a higher-categorical nature. In contrast, classical descent theory can be expressed nicely by the theory of monadic descent (Bénabou-Roubaud Theorem [BR70]). In this work we develop the theory of **fibered derivators** based on the idea of *derivator* due to Grothendieck and Heller, which solves the problem of (higher-categorical, or (co)homological) descent in a way closely related to the classical theory of cohomological descent (due to Deligne [SGA72b]). However, it is, in contrast, completely self-dual, making it very suitable for the descent of six-functor-formalisms. (There is also an infinity categorical version of the theory of monadic descent due to Lurie [Lur09].) The main achievements in this work are:

1. A *precise* definition of abstract **six-functor-formalisms** using the theory of (op)-fibrations of 2-multicategories (developed in this work as well). Results and constructions regarding the six functors were previously restricted to specific settings because no abstract definition was available.

2. Development of a theory of **fibered multiderivators** which enriches "collections of monoidal derived categories" in such a way that (co)descent can be formulated.

3. Main theorems of **(co)homological descent** which give easy criteria under which a fibered derivator satisfies (co)descent.

4. Development of a theory of **derivator six-functor-formalisms**, also using the theory of (op)fibrations of 2-multicategories.

5. The *construction* of fibered multiderivators and derivator six-functor-formalisms from bifibrations of model categories.

1.2 The six functors

A formalism of the "six functors" lies at the core of many different theories in mathematics, as for example the theory of Abelian sheaves on topological spaces, etale, l-adic, or coherent sheaves on schemes, D-modules, representations of (pro-)finite groups, motives, and many more. Given a base category of "spaces" $\mathcal{S}$, for instance, the category of schemes, topological spaces, analytic manifolds, etc. such a formalism roughly consists of a collection of (derived) categories $\mathcal{D}_S$ of "sheaves", one for each "base space" S in $\mathcal{S}$, and the following six types of functors between those categories:

f^*	f_*	*for each* f *in* $\mathrm{Mor}(\mathcal{S})$
$f_!$	$f^!$	*for each* f *in* $\mathrm{Mor}(\mathcal{S})$
$\otimes$	$\mathcal{HOM}$	*in each fiber* $\mathcal{D}_S$

The functors on the left hand side are left adjoints of the functors on the right hand side. The functor $f_!$ is "the dual of f_*" and is called **push-forward with proper support**, because in the topological setting (Abelian sheaves over topological spaces) this is what it is derived from. Its right adjoint $f^!$ is called the **exceptional pull-back**. These functors come along with a bunch of compatibilities between them.

1.2.1. More precisely, part of the datum of the six functors are the following natural isomorphisms in the "left adjoints" column:

	isomorphisms between left adjoints	isomorphisms between right adjoints
$(*,*)$	$(fg)^* \xrightarrow{\sim} g^*f^*$	$(fg)_* \xrightarrow{\sim} f_*g_*$
$(!,!)$	$(fg)_! \xrightarrow{\sim} f_!g_!$	$(fg)^! \xrightarrow{\sim} g^!f^!$
$(!,*)$	$g^*f_! \xrightarrow{\sim} F_!G^*$	$G_*F^! \xrightarrow{\sim} f^!g_*$
$(\otimes,*)$	$f^*(-\otimes-) \xrightarrow{\sim} f^*-\otimes f^*-$	$f_*\mathcal{HOM}(f^*-,-) \xrightarrow{\sim} \mathcal{HOM}(-,f_*-)$
$(\otimes,!)$	$f_!(-\otimes f^*-) \xrightarrow{\sim} (f_!-)\otimes-$	$f_*\mathcal{HOM}(-,f^!-) \xrightarrow{\sim} \mathcal{HOM}(f_!-,-)$
		$f^!\mathcal{HOM}(-,-) \xrightarrow{\sim} \mathcal{HOM}(f^*-,f^!-)$
$(\otimes,\otimes)$	$(-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-)$	$\mathcal{HOM}(-\otimes-,-) \xrightarrow{\sim} \mathcal{HOM}(-,\mathcal{HOM}(-,-))$

Here f,g,F,G are morphisms in $\mathcal{S}$, which in the $(!,*)$-row, are related by a *Cartesian* diagram:

$$\begin{tikzcd} \cdot \arrow[r, "G"] \arrow[d, "F"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

In the right hand side column the corresponding adjoint natural transformations have been inserted. In each case the left hand side natural isomorphism uniquely determines the right hand side one and vice versa. (In the $(\otimes,!)$-case there are two versions

of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two). The $(!,*)$-isomorphism (between left adjoints) is called **base change**, the $(\otimes,!)$-isomorphism is called the **projection formula**, and the $(*,\otimes)$-isomorphism is usually part of the definition of a **monoidal functor**. The $(\otimes,\otimes)$-isomorphism is the associativity of the tensor product and part of the definition of a monoidal category. The $(*,*)$-isomorphism, and the $(!,!)$-isomorphism, express that the corresponding functors arrange as a pseudo-functor with values in categories.
Furthermore part of the datum are isomorphisms

$$f^* \xrightarrow{\sim} f^!$$

for all isomorphisms f[1]. Of course, there have to be compatibilities among those natural isomorphisms. Some of them are listed in figures 1–4. Instead of trying to give a complete list of them (or even only a generating list from which all of them would follow) we proceed in a more abstract way (like in the ideas of fibered category or multicategory) and get a *precise* definition of a **six-functor-formalism** without having to specify any of these compatibilities explicitly. The natural isomorphisms of 1.2.1 will be derived from a composition law in a **2-multicategory** and all compatibilities will be just a consequence of the associativity of this composition law.
The six functors are the right framework to study duality theorems like Serre duality, Poincaré duality, various (Tate) dualities for the (co)homology of groups, etc.

Example 1.2.2 (Serre duality). *Let k be a field. If $\mathcal{S}$ is the category of k-varieties, we have a six-functor-formalism in which $\mathcal{D}_S$ is the derived category of (quasi-)coherent sheaves[2] on S. Let $\pi : S \to \mathrm{spec}(k)$ be a proper and smooth k-scheme of dimension n. Consider a locally free sheaf $\mathcal{E}$ on S and consider the following isomorphism (one of the two adjoints of the projection formula):*

$$\pi_* \mathcal{HOM}(\mathcal{E}, \pi^! k) \xrightarrow{\sim} \mathcal{HOM}(\pi_! \mathcal{E}, k)$$

In this case, we have $\pi_! \mathcal{E} = \pi_* \mathcal{E}$ because π is proper, and $\pi^! k = \Omega_S^n[n]$. Taking i-th homology of complexes we arrive at

$$H^{i+n}(S, \mathcal{E}^\vee \otimes \Omega_S^n) \cong H^{-i}(S, \mathcal{E})^*.$$

This is the classical formula of Serre duality.

Example 1.2.3 (Poincaré duality). *Let k be a field. If $\mathcal{S}$ is a category of nice topological spaces, we have a six-functor-formalism in which $\mathcal{D}_S$ is the derived category of sheaves of k-vector spaces on S. Let X be an n-dimensional topological manifold. Consider a local system $\mathcal{E}$ of k-vector spaces on X and consider the isomorphism (again one of the two adjoints of the projection formula):*

$$\pi_* \mathcal{HOM}(\mathcal{E}, \pi^! k) \xrightarrow{\sim} \mathcal{HOM}(\pi_! \mathcal{E}, k)$$

[1] There are more general formalisms, which we call proper or etale six-functor-formalisms where there is a morphism $f^* \to f^!$ or a morphism $f_! \to f_*$ for *certain* morphisms f (cf. Section 6.2)

[2] Neglecting here for a moment the fact that $f_!$ exists in general only after passing to pro-coherent sheaves.

We have $\pi^! k = \mathcal{L}_{or}[n]$, where $\mathcal{L}_{or}$ is the orientation sheaf of X over k. Taking i-th homology of complexes we arrive at

$$H^{i+n}(X, \mathcal{E}^\vee \otimes \mathcal{L}_{or}) \cong H_c^{-i}(X, \mathcal{E})^*.$$

This is the classical formula of Poincaré duality.

Example 1.2.4 (Group (co)homology). *The six functor formalism of Example 1.2.3 extends to stacks. Let G be a group and consider the classifying stack $[\cdot/G]$ and the projection $\pi : [\cdot/G] \to \cdot$. Note: Abelian sheaves on $[\cdot/G]$ = G-representations in Abelian groups. The extension of the six-functor-formalism encodes duality theorems like Tate duality. In this case π_* yields group cohomology and $\pi_!$ yields group homology. If G is finite, we also have a natural morphism $\pi_! \to \pi_*$ whose cone (homotopy cokernel) is Tate cohomology.*

1.3 History

Recently there has been increasing interest in six-functor-formalisms in various contexts. To indicate some of these developments, we mention some related works without any aim whatsoever towards completeness:

1960's	Grothendieck, Verdier, Deligne	schemes	coherent sheaves
	[Ver77, SGA72a, SGA72b, SGA73]	top. spaces	Abelian sheaves
		schemes	etale sheaves
1980's	Bernstein	char 0 varieties	D-modules
⋮			
2001	Voevodsky	abstract theory	
2003	Fausk, Hu, May [FHM03]	abstract theory	
2006	Ayoub [Ayo07a, Ayo07b]	schemes	motives
2008	Lazlo, Olsson [LO08a, LO08b]	(classical) stacks	etale and ℓ-adic sheaves
2009	Cisinski, Deglise [CD09]	schemes	motives
2009	Lipman, Hashimoto [LH09]	schemes	coherent sheaves
		diag. of schemes	coherent sheaves
2012	Zheng, Liu [LZ12]	(higher) stacks	etale sheaves
			(∞-categorical methods)
2013	Zheng [Zhe10]	DM stacks	constructible sheaves
2015	Schnürer [Sch15]	top. spaces	sheaves of vector spaces
			(dg-categorical methods)
2016	Gaitsgory, Rozenblyum [GR16]	derived schemes	coherent sheaves
		and stacks	(∞-categorical methods)

$$
\begin{tikzcd}[column sep=small]
 & ((E \otimes F) \otimes G) \otimes H \arrow[dl, dash] \arrow[dr, dash] & \\
(E \otimes (F \otimes G)) \otimes H \arrow[d, dash] & & (E \otimes F) \otimes (G \otimes H) \arrow[d, dash] \\
E \otimes ((F \otimes G)) \otimes H) \arrow[rr, dash] & & E \otimes (F \otimes (G \otimes H))
\end{tikzcd}
$$

Figure 1: Pentagon axiom (compatibility of $(\otimes, \otimes)$ iso's).

$$
\begin{tikzcd}
(f^*A \otimes f^*B) \otimes f^*C \arrow[r, dash] \arrow[d, dash] & f^*(A \otimes B) \otimes f^*C \arrow[r, dash] & f^*((A \otimes B) \otimes C) \arrow[d, dash] \\
f^*A \otimes (f^*B \otimes f^*C) \arrow[r, dash] & f^*A \otimes f^*(B \otimes C) \arrow[r, dash] & f^*(A \otimes (B \otimes C))
\end{tikzcd}
$$

Figure 2: Definition of monoidal functor (compatibility between $(*, \otimes)$ and $(\otimes, \otimes)$ iso's).

$$
\begin{tikzcd}
(g_!(A \otimes g^*B)) \otimes C \arrow[r, dash] \arrow[d, dash] & g_!((A \otimes g^*B) \otimes g^*C) \arrow[d, dash] \\
((g_!A) \otimes B) \otimes C \arrow[d, dash] & g_!(A \otimes (g^*B \otimes g^*C)) \arrow[d, dash] \\
(g_!A) \otimes (B \otimes C) \arrow[r, dash] & g_!(A \otimes g^*(B \otimes C))
\end{tikzcd}
$$

Figure 3: Example of compatibility between $(!, \otimes)$ and $(*, \otimes)$ and $(\otimes, \otimes)$ iso's.

$$
\begin{tikzcd}[column sep=small]
G_!F^*(A \otimes g^*B) \arrow[rr, dash] \arrow[d, dash] & & f^*g_!(A \otimes g^*B) \arrow[d, dash] \\
G_!((F^*A) \otimes F^*g^*B) \arrow[d, dash] & & f^*(g_!A \otimes B) \arrow[d, dash] \\
G_!((F^*A) \otimes G^*f^*B) \arrow[dr, dash] & & (f^*g_!A) \otimes f^*B \arrow[dl, dash] \\
 & (G_!F^*A) \otimes f^*B &
\end{tikzcd}
$$

Figure 4: Example of compatibility between $(!, *)$ and $(*, \otimes)$ and $(!, \otimes)$ and $(*, *)$ iso's.

1.4 (Co)homological descent

Our main motivation for defining a derivator enhancement of six-functor-formalisms has been the question of (co)homological descent. For instance, given an abstact six-functor-formalism, we would like to be able to extend it to stacks w.r.t. to a given Grothendieck topology on the base category $\mathcal{S}$.

More precisely, consider a space $S \in \mathcal{S}$ and a hypercover $S_\bullet$ of S, i.e. a simplicial object $S_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{S})$

$$S_\bullet := \ \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} S_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} S_1 \substack{\longrightarrow\\ \longrightarrow} S_0$$

with a map $p: S_\bullet \to S$ which is, in a certain sense, locally trivial. For example $S_\bullet$ could be associated with a Čech cover $U \to S$, that is, the simplicial object

$$S_i := \underbrace{U \times_S \cdots \times_S U}_{i+1\text{-times}}$$

with the obvious maps. Let π be the projection to the final object of $\mathcal{S}$ (assumed here to exist).

1.4.1 (Cohomological descent)**.** We would like to say that the six-functor-formalism satisfies **cohomological descent** whenever, given an object $\mathcal{E}$ over S, we have an isomorphism of the form

$$\pi_*\mathcal{E} \cong \mathrm{Tot}^{\Pi}\left(\cdots \longleftarrow \pi_{2,*}p_2^*\mathcal{E} \longleftarrow \pi_{1,*}p_1^*\mathcal{E} \longleftarrow \pi_{0,*}p_0^*\mathcal{E} \right) \tag{1}$$

for each (finite) hypercover as above. The complex appearing in brackets is the complex associated with the cosimplicial object $\Delta_i \mapsto \pi_{i,*}p_i^*\mathcal{E}$ where the morphisms are given by the various units. Note that this question is *not really well defined* because, when working with derived categories, Tot^{Π} does not make sense at all, and if we work with complexes (on-the-nose, not up to quasi-isomorphism) instead, a *coherent* simplicial diagram of complexes can not be constructed because the $\pi_{i,*}$, and the p_i^* are only derived functors. Usually the reader is used to the situation where the S_i have trivial cohomology and accordingly the $\pi_{i,*}$ come from an *exact* functor between Abelian categories. So working with complexes on-the-nose and taking Tot^{Π} of those, the right hand side becomes well-defined. Then the question is made precise if we understand on the left hand side the derived π_*. The task in general is therefore

1. to give a meaning to the right hand side of (1) in the derived setting (when the $\pi_{i,*}$ are just derived functors),

2. to find criteria under which (1) is an isomorphism (preferably for every finite hypercover at least).

The formalism of derivators is very suitable to adress these problems. The total complex appearing in (1) is just the *homotopy limit* over the cosimplicial object $\Delta_i \mapsto \pi_{i,*}p_i^*\mathcal{E}$. Hence, pursuing the idea of derivators, we have to find means of constructing this diagram in a coherent way (i.e. as an object in the value of the associated derivator at Δ) and not only as a diagram in the derived category.

1.4.2 (Homological descent)**.** Dually, given an object $\mathcal{E}$ over S, we might ask whether we have a identity of the form

$$\pi_!\mathcal{E} \cong \mathrm{Tot}^{\oplus}\left(\cdots \longrightarrow \pi_{2,!}p_2^!\mathcal{E} \longrightarrow \pi_{1,!}p_1^!\mathcal{E} \longrightarrow \pi_{0,!}p_0^!\mathcal{E} \right) \tag{2}$$

The complex is associated with the simplicial object $\Delta_i \mapsto \pi_{2,!}p_2^!\mathcal{E}$ where the morphisms are given by the various counits. Now the question makes even less sense because $p_2^!$ is, in most cases, only constructed as a morphism in the derived category. Again, the question of **homological descent** amounts

1. to give meaning to the right hand side of (2) in the derived setting (when the $\pi_{i,!}$ and $p_i^!$ are classically just functors between derived categories) and
2. to find criteria under which (2) is an isomorphism (preferably for every finite hypercover at least).

1.4.3. As an easy topological example for homological descent (in which we can give meaning to the question) consider a real C^∞-manifold X of dimension n (in general non-compact) and the constant sheaf $\mathbb{R}$ on it. Let $\{U_i\}_i$ be a finite Cech cover. We can compute $\pi_!\mathbb{R}$ as the the total complex of

$$E_{p,q} := \bigoplus_{i_1,\dots,i_p} H_c(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{E}^q)$$

where $\mathcal{E}^q$ is the sheaf of C^∞-differential forms of degree q. Here a section with compact support on a smaller open set is mapped to a section with compact support on a larger open set. If the $U_{i_1} \cap \cdots \cap U_{i_p}$ are sufficiently nice, $H_{p,q}^{vert}$ is zero unless $q = n$, hence $\pi_!\mathbb{R}$ is represented by the complex

$$\cdots \to \bigoplus_{i_1,i_2} H_c^n(U_{i_1} \cap U_{i_2}, \mathbb{R}) \to \bigoplus_i H_c^n(U_i, \mathbb{R}).$$

This example works because we have representations of the $\pi_{i,!}\mathbb{R}$ for which a coherent double complex can be constructed.

1.4.4. More generally, if $S_\bullet$ is any simplicial object in $\mathcal{S}$, for example the presentation of a stack (or even a higher stack), can we *define* a functor π_* by

$$\mathrm{Tot}^{\Pi}\left(\cdots \longleftarrow \pi_{2,*}\mathcal{E}_2 \longleftarrow \pi_{1,*}\mathcal{E}_1 \longleftarrow \pi_{0,*}\mathcal{E}_0 \right)$$

where the $\mathcal{E}_i$ are objects over S_i and we are given (quasi-)isomorphisms $S(\delta)^*\mathcal{E}_i \to \mathcal{E}_j$ for all $\delta : \Delta_j \to \Delta_i$ in a compatible way? We call the collection $\{\mathcal{E}_i\}_i$ together with these quasi-isomorphisms a **coCartesian object** over $\mathcal{S}_\bullet^{\mathrm{op}}$.
And can we *define* a functor $\pi_!$ by

$$\mathrm{Tot}^{\oplus}\left(\cdots \longrightarrow \pi_{2,!}\mathcal{E}_2 \longrightarrow \pi_{1,!}\mathcal{E}_1 \longrightarrow \pi_{0,!}\mathcal{E}_0 \right)$$

where the $\mathcal{E}_i$ are objects over S_i and we are given (quasi-)isomorphisms $\mathcal{E}_i \to S(\delta)^! \mathcal{E}_j$ for all $\delta : \Delta_j \to \Delta_i$ in a compatible way? We call the collection $\{\mathcal{E}_i\}_i$ together with these quasi-isomorphisms a **Cartesian object** over $\mathcal{S}_\bullet$.
In both cases the same coherence problem as in 1.4.1–1.4.2 arises. Without a suitable enhancement of the situation the "definitions" above do not make sense. Assuming that this can be solved by a derivator enhancement, the main question, which will be addressed in Section 1.8, then becomes:

> "Given a morphism between simplicial objects $S_\bullet \to T_\bullet$, when are the two corresponding categories of Cartesian (resp. coCartesian) objects equivalent?"

This question concerns only the two functors f^*, f_* or the two functors $f_!, f^!$ at a time. Another, more involved question related to a full four-functor-formalism is:

> "When are the categories of Cartesian objects over $S_\bullet$ and coCartesian objects over $S_\bullet^{\mathrm{op}}$ equivalent?"

It will be addressed in Section 1.13.

Remark 1.4.5. *From the point of view of ∞-categories, the questions of 1.4.1 (resp. 1.4.2) on the one hand and of 1.4.4 on the other hand are related as follows. In this world a bifibration $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ can be given equivalently as a functor $F : \mathcal{S}^{\mathrm{op}} \to \infty\!-\!\mathcal{CAT}$ such that the functors in the image are right adjoints. Given $S \in \mathcal{S}$, a resolution $\pi : U_\bullet \to S$, and an object $\mathcal{E} \in F(S)$, the first question of cohomological descent asks whether the natural map*

$$\mathcal{E} \cong \lim_{\Delta} \pi_{i,*} \pi_i^* \mathcal{E},$$

is an isomorphism (or maybe whether it becomes an isomorphism after applying a further push-forward to a suitable base), where lim *is the (homotopy) limit of the diagram $\Delta \to F(S)$ given by $\Delta_i \mapsto \pi_{i,*}\pi_i^* \mathcal{E}$.*
The second question of cohomological descent asks whether the functor F itself satisfies a similar property. If we consider it, neglecting non-invertible morphisms, as a functor $F : \mathcal{S}^{\mathrm{op}} \to \infty - \mathcal{GRP}$ to ∞-groupoids the question becomes whether the natural map

$$F(S) \cong \lim_{\Delta} F(U_i),$$

is an isomorphism, where lim *is the (homotopy) limit of the diagram $F \circ U_\bullet : \Delta \to \infty - \mathcal{GRP}$. From this point of view it is already clear that the property of the second question is stronger and implies the first. Both questions cannot be formulated within the realm of classical derivators. Although those nicely encode the occurring homotopy limit functors, there is no way to obtain the diagrams in the argument starting from, say, any kind of pseudo-functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{DER}$ to the 2-category of derivators. However, the language of* fibered *derivators proposed in this work constitutes a nice solution, albeit the similarity between the two questions becomes slightly obscured.*

1.5 The precise definition of six-functor-formalisms

Before proceeding, let us go back to the question of making precise what an abstract six-functor-formalism is. Whenever we are given a compatible bunch of functors like, for example, a tensor product, or a collection of pull-back morphisms, there is a general procedure: There is always a "better structure" in which the compatibility between functors gets encoded in a composition law and its associativity.

Example 1.5.1. *The first example is the one of a collection of pull-back functors (i.e. a* pseudo-functor with values in categories*):*

<table>
<tr><th>collection of functors and compatibilities</th><th>better structure</th></tr>
<tr><td>

$$\boxed{f^*}$$

$$(gf)^* \xrightarrow{\sim} g^* f^*$$

$$\begin{tikzcd} (fgh)^* \arrow[r, dash] \arrow[d, dash] & (gh)^* f^* \arrow[d, dash] \\ h^*(fg)^* \arrow[r, dash] & h^* g^* f^* \end{tikzcd}$$

</td><td>**opfibered category**
$\mathcal{C} \to \mathcal{S}^{\mathrm{op}}$</td></tr>
<tr><td>$\mathrm{Hom}(f^*A, B)$ =</td><td>$\mathrm{Hom}_f(A, B)$</td></tr>
</table>

Here we use the notation $\mathrm{Hom}_f(A, B)$ *to denote the preimage of* f *under the opfibration* $\mathcal{C} \to \mathcal{S}^{\mathrm{op}}$.

Example 1.5.2. *The second example is a tensor product (i.e. a* monoidal category*):*

<table>
<tr><th>collection of functors and compatibilities</th><th>better structure</th></tr>
<tr><td>

$$\boxed{\otimes}$$

$$(-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-)$$

$$\begin{tikzcd}[column sep=small] (-\otimes-)\otimes(-\otimes-) \arrow[rr, dash] \arrow[d, dash] & & ((-\otimes-)\otimes-)\otimes- \arrow[d, dash] \\ -\otimes(-\otimes(-\otimes-)) \arrow[dr, dash] & & (-\otimes(-\otimes-))\otimes- \arrow[dl, dash] \\ & -\otimes((-\otimes-)\otimes-) & \end{tikzcd}$$

</td><td>**representable multicategory**
$\mathcal{C}$</td></tr>
<tr><td>$\mathrm{Hom}((A_1 \otimes (A_2 \otimes (\cdots))), B)$ =</td><td>$\mathrm{Hom}(A_1, \ldots, A_n; B)$</td></tr>
</table>

Example 1.5.3. *The third example is a combination of the first two (i.e. a* monoidal categegory together with a monoidal pseudo-functor *[LH09, (3.6.7) b)]):*

collection of functors and compatibilities		better structure
$\boxed{f^*, \otimes}$ $\cdots$ $f^*(- \otimes -) \xrightarrow{\sim} (f^* -) \otimes (f^* -)$ $\cdots$		**opfibered multicategory** $\mathcal{C} \to \mathcal{S}^{\mathrm{op}}$
$\mathrm{Hom}((f_1^* A_1 \otimes (f_2^* A_2 \otimes (\cdots))), B)$	$=$	$\mathrm{Hom}_f(A_1, \dots, A_n; B)$

Here $\mathcal{S}^{\mathrm{op}}$ is turned into a multicategory in the following way: A multimorphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is a collection of morphims $f_i : T \to S_i$ for each i. This multicategory is representable (i.e. opfibered over $\cdot$, see below), i.e. it is a monoidal category with the monoidal product being given by $\times$.

Example 1.5.4.

collection of functors and compatibilities		better structure
$\boxed{g^*, f_!, \otimes}$ *with all 6 "compatibility" isomorphisms as in 1.2.1* *plus the adjunction of $f_!$ and f^* for isomorphisms* $\cdots$		**opfibered 2-multicategory**[3] $\mathcal{C} \to \mathcal{S}^{\mathrm{cor}}$
$\mathrm{Hom}(f_!(g_1^* A_1 \otimes (g_2^* A_2 \otimes (\cdots))), B)$	$=$	$\mathrm{Hom}_\xi(A_1, \dots, A_n; B)$

Here $\mathcal{S}^{\mathrm{cor}}$ is the following 2-multicategory: Its objects are the objects of $\mathcal{S}$ and a 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is a multicorrespondence

$$\begin{tikzcd}
 & & & A \arrow[dlll, "g_1"'] \arrow[dl, "g_n"] \arrow[dr, "f"] & \\
S_1 & \cdots & S_n & ; & T
\end{tikzcd} \tag{3}$$

The composition of 1-morphisms is given by forming fiber products (here depicted for two 1-ary 1-morphisms, cf. Definition 3.1.1 for the general case):

$$\begin{tikzcd}
 & & A \times_Y B \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

[3] more precisely, a 1-opfibration and 2-bifibration of 2-multicategories with 1-categorical fibers

The 2-morphisms are the isomorphisms of such multicorrespondences. This multicategory is representable (i.e. opfibered over $\cdot$), closed (i.e. fibered over $\cdot$), every object is self-dual, with tensor product and internal hom both *given by $\times$ and having as unit the final object.*

1.5.5. We now proceed to give the precise definition of an (op)fibered multicategory. In the sequel we will also need the definition of 2-multicategories and their fibrations (as was already needed for the definition of $\mathcal{S}^{\mathrm{cor}}$ above). We refer to Sections 2.2 and 2.4 for the definitions. Here, for the sake of simplicity, we will neglect the 2-categorical aspect. The reader should keep in mind that a multicategory abstracts the properties of multilinear maps, and indeed every monoidal category gives rise to a multicategory setting

$$\mathrm{Hom}(A_1, \dots, A_n; B) := \mathrm{Hom}((A_1 \otimes (A_2 \otimes (\cdots))), B). \tag{4}$$

Definition 1.5.6. *A* **multicategory** $\mathcal{D}$ *consists of*

- *a class of objects* $\mathrm{Ob}(\mathcal{D})$*;*
- *for all $n \in \mathbb{Z}_{\geq 0}$, and for all objects $X_1, \dots, X_n, Y$ in* $\mathrm{Ob}(\mathcal{D})$*, a class*

$$\mathrm{Hom}(X_1, \dots, X_n; Y);$$

- *a composition, i.e. for all objects $X_1, \dots, X_n$, $Y_1, \dots, Y_m$, Z in* $\mathrm{Ob}(\mathcal{D})$ *and for all $i \in \{1, \dots, m\}$ a map:*

$$\begin{array}{rcl} \mathrm{Hom}(X_1, \dots, X_n; Y_i) \times \mathrm{Hom}(Y_1, \dots, Y_m; Z) & \to & \mathrm{Hom}(Y_1, \dots, X_1, \dots, X_n, \dots, Y_m; Z) \\ f, g & \mapsto & g \circ_i f; \end{array}$$

- *for all $X \in \mathrm{Ob}(\mathcal{D})$ an identity* $\mathrm{id}_X \in \mathrm{Hom}(X; X)$*;*

satisfying strict associativity and identity laws. The composition w.r.t. independent slots is commutative, i.e. for $1 \leq i < j \leq m$ if $f \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ and $f' \in \mathrm{Hom}(X'_1, \dots, X'_k; Y_j)$ and $g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ we have

$$(g \circ_i f) \circ_{j+n-1} f' = (g \circ_j f') \circ_i f.$$

A symmetric (braided) multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms

$$\alpha : \mathrm{Hom}(X_1, \dots, X_n; Y) \to \mathrm{Hom}(X_{\alpha(1)}, \dots, X_{\alpha(n)}; Y)$$

for $\alpha \in S_n$ (symmetric group), resp. $\alpha \in B_n$ (braid group), forming an action which is strictly compatible with composition in the obvious way (in the braided case: substitution of strings).

In some references the composition is defined in a seemingly more general way; in the presence of identities these descriptions are, however, equivalent.

1.5.7. We leave the obvious definition of a functor between multicategories to the reader. Similarly there is a definition of a **opmulticategory**, in which we have classes

$$\mathrm{Hom}(X; Y_1, \dots, Y_n)$$

and similar data. If $\mathcal{D}$ is a multicategory $\mathcal{D}^{\mathrm{op}}$ is an opmulticategory in a natural way. The trivial category is considered to be a multicategory setting all $\mathrm{Hom}(\{\cdot\}, \dots, \{\cdot\}; \{\cdot\})$ to be the 1 element set. It is the final object in the category of multicategories.

To clarify the precise relation between multicategories and monoidal categories we have to define Cartesian and coCartesian morphisms. It turns out that we can actually give a definition which is a common generalization of coCartesian morphisms in usual opfibered categories (cf. Section 2.1) and those morphisms expressing the existence of a tensor product:

Definition 1.5.8 (cf. the more general Definitions 2.4.3–2.4.5)**.** *Consider a functor of multicategories $p : \mathcal{D} \to \mathcal{S}$.*

- *A morphism*
  $$\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$
  in $\mathcal{D}$ is called **coCartesian** *w.r.t. p, if for all i, for all $\mathcal{F}_1, \dots, \mathcal{F}_m, \mathcal{G}$ with $\mathcal{F}_i = \mathcal{F}$, and for all $f \in \mathrm{Hom}(p(\mathcal{F}_1), \dots, p(\mathcal{F}_m); p(\mathcal{G}))$ the map*
  $$\begin{array}{rcl} \mathrm{Hom}_f(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) & \to & \mathrm{Hom}_{f \circ p(\xi)}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G}) \\ \alpha & \mapsto & \alpha \circ \xi \end{array}$$
  is bijective.

- *A morphism*
  $$\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$
  in $\mathcal{D}$ is called **Cartesian** *w.r.t. p at the i-th slot, if for all $\mathcal{G}_1, \dots, \mathcal{G}_m$, and for all $f \in \mathrm{Hom}(p(\mathcal{G}_1), \dots, p(\mathcal{G}_m); p(\mathcal{E}_i))$ the map*
  $$\mathrm{Hom}_f(\mathcal{G}_1, \dots, \mathcal{G}_m; \mathcal{E}_i) \to \mathrm{Hom}_{p(\xi) \circ f}(\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, \mathcal{G}_1, \dots, \mathcal{G}_m, \mathcal{E}_{i+1}, \dots, \mathcal{E}_n; \mathcal{F}).$$
  $$\alpha \mapsto \xi \circ \alpha$$
  is bijective.

- *The functor $p : \mathcal{D} \to \mathcal{S}$ is called an* **opfibered multicategory** *if for every $g \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$, and for all objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ with $p(\mathcal{E}_i) = S_i$ there is some object $\mathcal{F}$ over T and some coCartesian morphism $\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ with $p(\xi) = g$.*

- *The functor $p : \mathcal{D} \to \mathcal{S}$ is called a* **fibered multicategory** *if for every $1 \le j \le n$, for all $g \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$, and for all objects $\mathcal{E}_1, \overset{\widehat{j}}{\dots}, \mathcal{E}_n$ with $p(\mathcal{E}_i) = S_i$, and $\mathcal{F}$ over T there is some object $\mathcal{E}_j$ over S_j and some Cartesian morphism w.r.t. the j-th slot $\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ with $p(\xi) = g$.*

- *The functor $p : \mathcal{D} \to \mathcal{S}$ is called a* **bifibered multicategory** *if it is both fibered and opfibered.*

It turns out that the composition of Cartesian morphisms is Cartesian (and similarly for coCartesian morphisms)[4].
The following Lemma follows from Corollary 2.4.17 and Proposition 2.4.25 (which deals with a 2-categorical generalization of 2.).

Lemma 1.5.9. *1. An opfibered multicategory $p : \mathcal{D} \to \{\cdot\}$ is a monoidal category in a natural way. Conversely any monoidal category gives rise to an opfibered multicategory $p : \mathcal{D} \to \{\cdot\}$ via (4). A multicategory $\mathcal{D}$ is a* closed *category if and only if it is fibered over $\{\cdot\}$. In particular, the fibers of an (op)fibered multicategory $p : \mathcal{D} \to \mathcal{S}$ are always closed/monoidal in the following sense: Given any functor of multicategories[5] $s : \{\cdot\} \to \mathcal{S}$, the category $\mathcal{D}_s$ of objects over s is closed/monoidal.*

2. Given (op)fibered multicategories $p : \mathcal{C} \to \mathcal{D}$ and $q : \mathcal{D} \to \mathcal{S}$ also the composition $q \circ p$ is an (op)fibered multicategory. In particular, if we have an opfibered multicategory $p : \mathcal{D} \to \mathcal{S}$ and if $\mathcal{S} \to \{\cdot\}$ is opfibered (i.e. $\mathcal{S}$ is monoidal) then also $\mathcal{D} \to \{\cdot\}$ is opfibered (i.e. $\mathcal{D}$ is monoidal). The same holds dually. A morphism α is (co)Cartesian for $q \circ p$ if and only if α is (co)Cartesian for q and $q(\alpha)$ is (co)Cartesian for p.

In the case of an opfibration $p : \mathcal{D} \to \{\cdot\}$, the tensor product $\mathcal{E} \otimes \mathcal{F}$ is reobtained as the target of a coCartesian morphism in $\mathrm{Hom}(\mathcal{E}, \mathcal{F}; \mathcal{E} \otimes \mathcal{F})$ which exists for any $\mathcal{E}$, $\mathcal{F}$ by definition. Similarly, the unit 1 is just the target of a coCartesian morphism in $\mathrm{Hom}(;1)$ which exists by definition (the existence is also required for the empty set of objects).
The second part of the Lemma encapsulates the distinction between internal and external tensor product in a four- (or six-) functor-formalism, see Section 6.1.

Example 1.5.10. *Let $\mathcal{S}$ be a usual category. If $\mathcal{S}$ has coproducts then it may be turned into a symmetric multicategory setting*

$$\mathrm{Hom}(S_1, \dots, S_n; T) := \mathrm{Hom}(S_1; T) \times \cdots \times \mathrm{Hom}(S_n; T).$$

Let $p : \mathcal{D} \to \mathcal{S}$ be an opfibered (usual) category. Any object S induces a canonical functor of multicategories $s : \{\cdot\} \to \mathcal{S}$ with image S hence the fibers of an opfibered multicategory $p : \mathcal{D} \to \mathcal{S}$ are monoidal and the datum p is equivalent to giving a pseudo-functor such that the push-forwards $f_\bullet$ are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a covariant monoidal pseudo-functor in e.g. [LH09, (3.6.7)].

[4] As with fibered categories there are weaker notions of Cartesian which still uniquely determine a Cartesian morphism (up to isomorphism) from given objects over a given multimorphism, however, do not imply that they are stable under composition. Similarly for coCartesian morphisms.

[5] This specifies also morphisms in $\mathrm{Hom}(\underbrace{S, \dots, S}_{n}; S)$, for all n, compatible with composition.

Example 1.5.11. *Similarly, if $\mathcal{S}$ has products, $\mathcal{S}^{\mathrm{op}}$ may be turned into a symmetric multicategory (or $\mathcal{S}$ into a symmetric opmulticategory) setting*

$$\mathrm{Hom}(S_1, \dots, S_n; T) := \mathrm{Hom}(T; S_1) \times \dots \times \mathrm{Hom}(T; S_n).$$

Let $p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be a opfibered (usual) category. Then an opfibered multicategory structure on p is equivalent to giving a monoidal structure on the fibers such that the pull-backs f^* (along morphisms in $\mathcal{S}$) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a contravariant monoidal pseudo-functor in e.g. [LH09, (3.6.7)].

The notion of multicategory has an easy 2-categorical analogue (a multicategory enriched in usual categories) which will be defined precisely in Section 2.2. The notion of (op)fibration generalizes as well, although one has to distinguish two notions of (op)fibrations regarding 1- and 2-morphisms (cf. Section 2.4). We arrive at the sought-for precise definition of a six-functor-formalism:

Definition 1.5.12 (cf. Definition 3.1.4)**.** *Let $\mathcal{S}$ be a category with fiber products, and let $\mathcal{S}^{\mathrm{cor}}$ be the symmetric 2-multicategory defined in 1.5.4. A* **(symmetric) six-functor-formalism** *on $\mathcal{S}$ is a 1-bifibered and 2-bifibered (symmetric) 2-multicategory with 1-categorical fibers*

$$\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}.$$

While a 1-opfibration and 2-fibration of 2-multicategories with 1-categorical fibers over $\mathcal{S}^{\mathrm{cor}}$ is also essentially the same as a pseudo-functor (cf. Proposition 2.4.16)

$$\mathcal{S}^{\mathrm{cor}} \to \mathcal{CAT}$$

where $\mathcal{CAT}$ is equipped with the 2-multicategory structure in which 1-multimorphisms are functors of several variables, the latter description has two disadvantages. Firstly, it only describes three of the six functor types (the left adjoints) and it is not so easy to make a similar definition encoding the right adjoints (among other problems, in the unsymmetric case, there are several internal Hom functors ...) Secondly, it involves choices and compatibilities again, although the definition of a pseudo-functor between 2-multicategories (cf. Definition 2.2.3) is already *much* more manageable than 1.2.1 and its unclear list of compatibilities. We prefer, however, to use the language of (op)fibrations in any case.

1.5.13. We have a morphism of opfibered (over $\{\cdot\}$) 2-multicategories $\mathcal{S}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{cor}}$ where $\mathcal{S}^{\mathrm{op}}$ is equipped with the multicategory structure as in 1.5.11. However there is no reasonable morphism of opfibered multicategories $\mathcal{S} \to \mathcal{S}^{\mathrm{cor}}$ (There is no compatibility involving only $\otimes$ and !). From a six-functor-formalism, we get the operations g_*, g^* as the pullback and the push-forward along the correspondence

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, equal] & \\ Y & ; & X \end{tikzcd}$$

We get $f^!$ and $f_!$ as the pullback and the pushforward along the correspondence

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & ; & Y \end{tikzcd}$$

We get $\mathcal{E} \otimes \mathcal{F}$ for objects $\mathcal{E}, \mathcal{F}$ above S as the target of any coCartesian morphism

$$\otimes \in \mathrm{Hom}_{\xi_S}(\mathcal{E}, \mathcal{F}; \mathcal{E} \otimes \mathcal{F})$$

over the multicorrespondence

$$\xi_S = \left(\begin{tikzcd} & & S \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ S & S & ; & S \end{tikzcd} \right)$$

Alternatively, we could define

$$\mathcal{E} \otimes \mathcal{F} := \Delta^*(\mathcal{E} \boxtimes \mathcal{F}).$$

Here Δ^* is the pushforward along the correspondence

$$\left(\begin{tikzcd} & S \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ S \times S & ; & S \end{tikzcd} \right)$$

induced by the canonical multicorrespondence $\xi_S \in \mathrm{Hom}(S, S; S)$, and $\boxtimes$ is the absolute monoidal product which exists because the composition $\mathcal{D} \to \mathcal{S} \to \{\cdot\}$ is opfibered as well, i.e. $\mathcal{D}$ is monoidal.

1.5.14. It is easy to derive from the definition of bifibered multicategory over $\mathcal{S}^{\mathrm{cor}}$ that the absolute monoidal product $\mathcal{E} \boxtimes \mathcal{F}$ can be reconstructed from the fiber-wise product as $\mathrm{pr}_1^* \mathcal{E} \otimes \mathrm{pr}_2^* \mathcal{F}$ on $S \times T$, whereas the absolute $\mathbf{HOM}(\mathcal{E}, \mathcal{F})$ is given by $\mathcal{HOM}(\mathrm{pr}_1^* \mathcal{E}, \mathrm{pr}_2^! \mathcal{F})$ on $S \times T$. In particular the absolute duality $D\mathcal{E} := \mathbf{HOM}(\mathcal{E}, 1)$ is given by $\mathcal{HOM}(\mathcal{E}, \pi^! 1)$ where $\pi : S \to \cdot$ is the final morphism.

Proposition 1.5.15 (cf. Proposition 3.2.3)**.** *Given a six-functor-formalism on $\mathcal{S}$*

$$\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$$

for the six operations as extracted in 1.5.13 there exist naturally all compatibility isomophisms listed in 1.2.1.

The important point is, that also all compatibilities between these isomorphisms can be derived, like e.g. those in figures 1–4, cf. 3.2.5 for an example. Each of these compatibilities corresponds to an associativity relation in $\mathcal{S}^{\mathrm{cor}}$. If we generalize the definition of $\mathcal{S}^{\mathrm{cor}}$ and allow more morphisms of multicorrespondences as 2-morphisms, we easily axiomatize a morphism $f_! \to f_*$ that often accompanies a six-functor-formalism, and its properties (cf. Section 6.2).

1.6 Fibered derivators

The notion of triangulated category developed by Grothendieck and Verdier in the 1960's, as successful as it has been, is not sufficient for many purposes, for both practical reasons (certain natural constructions cannot be performed) as well as for theoretical reasons (the axioms are rather involved and lack conceptual clarity). Grothendieck much later [Gro91] and Heller independently, with the notion of **derivator**, proposed a marvelously simple remedy to both deficiencies. The basic observation is that all problems mentioned above are based on the following fact: Consider a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ (quasi-isomorphisms, weak equivalences, etc.) which one would like to become isomorphisms. Then *homotopy limits and colimits* w.r.t. $(\mathcal{C}, \mathcal{W})$ cannot be reconstructed once passed to the localization (or homotopy category) $\mathcal{C}[\mathcal{W}^{-1}]$, for example a derived category, or the homotopy category of a model category. Examples of homotopy (co)limits are the cone and the total complex of a complex of complexes. While the existence of the cone is rescued in a triangulated category in a brute-force way (but it is not functorial anymore), the total complex is totally lost in a derived category. Furthermore, very basic and intuitive properties of homotopy limits and colimits, and more general Kan extensions, not only determine the additional structure (triangles, shift functors) on a triangulated category but also *imply* all of its rather involved axioms. This idea has been successfully worked out by Cisinski, Groth, Grothendieck, Heller, Maltsiniotis, and others. We refer to the introductory article [Gro13] for an overview.
One purpose of this work is to propose a notion of fibered (multi)derivator which enhances the notion of a fibration of (monoidal) triangulated categories in the same way as the notion of usual derivator enhances the notion of triangulated category. We emphasize that this new context is very well suited to reformulate (and reprove the theorems of) the classical theory of cohomological descent and to establish a completely dual theory of homological descent which should be satisfied by the $f_!, f^!$-functors.

1.6.1. As we have seen, the questions of homological and cohomological descent cannot be treated in a satisfactory way by considering a "classical" six-functor-formalism $\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$ whose fibers are *derived categories* (of sheaves, D-modules, motives, etc.). This does not allow to consider operations between *coherent* diagrams in $\mathcal{D}$ over arbitrary diagrams in $\mathcal{S}$ or even over diagrams of correspondences i.e. diagrams in $\mathcal{S}^{\mathrm{cor}}$. Enhancing only the fibers as derivators, is obviously also not sufficient. There are two approaches to correct this:

1. To enlarge the *domain* of a derivator to "diagrams in $\mathcal{S}$" (or here even to "diagrams in $\mathcal{S}^{\mathrm{cor}}$"). This road has been taken for example in the work of Ayoub, Cisinski and Deglise [Ayo07a, Ayo07b, CD09] under the name *algebraic derivator*.

2. To consider fibered (multi)derivators, which are morphisms of pre-(multi)derivators, satisfying some axioms.

In view of the philosophy that it should be preferred to consider a compact structure like (op)fibered multicategories instead of a bunch of functors and compatibilities, we follow the second approach.

The first approach has also the disadvantage that one has to choose between two types of categories of diagrams in $\mathcal{S}$, the 2-categories $\mathrm{Dia}(\mathcal{S})$ and $\mathrm{Dia}^{\mathrm{op}}(\mathcal{S})$. A fibered derivator according to the second approach gives rise to pseudo-functors from both types, but the definition is itself completely self-dual (in the non-multi case at least). The two approaches will be (partly) conciliated later (cf. Section 1.7) observing that a (left) fibered multiderivator can be seen as a pseudo-functor $\mathrm{Dia}^{\mathrm{cor}}(\mathcal{S}) \to \mathcal{MCAT}$, i.e. as a kind of Wirthmüller context, and we have (contravariant) embeddings of both $\mathrm{Dia}(\mathcal{S})$ and $\mathrm{Dia}^{\mathrm{op}}(\mathcal{S})$ into $\mathrm{Dia}^{\mathrm{cor}}(\mathcal{S})$.

Let Dia be a category of diagrams (a full subcategory of the category of small categories satisfying some closure properties).

Definition 1.6.2 (cf. Definition 4.2.1)**.** *A* **pre-derivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-op} \to \mathcal{CAT}$$

into the 2-"category"[6] *of categories.*
A **pre-multiderivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-op} \to \mathcal{MCAT}$$

into the 2-"category" of multicategories. A morphism of pre-derivators is a natural transformation.
For a morphism $\alpha : I \to J$ in Dia *the corresponding functor $\mathbb{D}(\alpha)$*

$$\mathbb{D}(J) \to \mathbb{D}(I)$$

will be denoted by α^*.
We call a pre-multiderivator **symmetric** *(resp.* **braided***), if the images are symmetric (resp. braided), and the morphisms α^* are compatible with the actions of the symmetric (resp. braid groups).*

Definition 1.6.3 (cf. Definition 4.3.5)**.** *We consider the following axioms on a pre-(multi)derivator $\mathbb{D}$:*

(Der1) *For I, J in* Dia*, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence. Moreover, $\mathbb{D}(\varnothing)$ is not empty.*

(Der2) *For I in* Dia *the 'underlying diagram' functor*

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Fun}(I, \mathbb{D}(\cdot))$$

is conservative.

In addition, we consider the following axioms for a morphism of pre-(multi)derivators $p : \mathbb{D} \to \mathbb{S}$ (here only the left versions of the axioms are listed; they all have corresponding dual right versions):

[6] where we put "category" into quotation marks to indicate that is has classes replaced with 2-classes (or, if you prefer, is constructed w.r.t. a larger universe).

(FDer0 left) *For each I in* Dia *the morphism p specializes to an opfibered (multi)category and any functor $\alpha : I \to J$ induces a diagram*

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of opfibered (multi)categories, i.e. the top horizontal functor maps coCartesian morphisms to coCartesian morphisms.

(FDer3 left) *For each functor $\alpha : I \to J$ in* Dia *and $S \in \mathbb{S}(J)$ the functor α^* between fibers*

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!^S$.

(FDer4 left) *For each functor $\alpha : I \to J$ in* Dia, *and for any object $j \in J$ and the 2-cell*[7]

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $\alpha_{j!}(\mathbb{S}(\mu))_\bullet \iota^* \to j^* \alpha_!$ is an isomorphism w.r.t. all bases $S \in \mathbb{S}(J)$.

(FDer5 left) *(if $\mathbb{S}$ is strong, only needed for the multiderivator case). For any Grothendieck opfibration $\alpha : I \to J$ in* Dia, *and for any morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors*

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-, \ - \ , \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -, \ \alpha_!- \ , -, \cdots, -)$$

are isomorphisms.

In axiom (FDer4 left), which makes only sense in the presence of (FDer0 left) and (FDer3 left), $(\mathbb{S}(\mu))_\bullet$ is an arbitrary choice of push-forward along $\mathbb{S}(\mu)$. Similarly, in (FDer5 left) $\xi_\bullet$ is a functor defined by choosing a coCartesian arrow, which makes only sense in the presence of (FDer0 left).

Question 1.6.4. *It seems natural to allow also multicategories, in particular operads, as* domain *for a fibered multiderivator. The author however did not succeed yet in writing down a neat generalization of (FDer3–4) which would encompass (FDer5).*

[7]The diagram $I \times_{/J} j$ is the 2-pullback of the diagram $\left(\begin{tikzcd} & I \arrow[d] \\ \{j\} \arrow[r] & J \end{tikzcd}\right)$ and is also called the **slice** or **comma** category.

Definition 1.6.5 (cf. Definition 4.3.6). *A morphism of pre-(multi)derivators $p : \mathbb{D} \to \mathbb{S}$ is called a* **left fibered (multi)derivator**, *if axioms (Der1–2) hold for $\mathbb{D}$ and $\mathbb{S}$ and (FDer0–5 left) hold for p. Similarly it is called a* **right fibered (multi)derivator** *with domain* Dia, *if instead the corresponding dual axioms (FDer0–5 right) hold. It is called* **fibered** *if it is both left and right fibered.*

1.6.6. Let $S \in \mathbb{S}(\cdot)$ be an object, and consider a (left, resp. right) fibered (multi)derivator $p : \mathbb{D} \to \mathbb{S}$. The association

$$I \mapsto \mathbb{D}(I)_{p^*S},$$

where $p : I \to \cdot$ is the projection, defines a (left, resp. right) derivator in the usual sense which we call the **fiber of** p **over** S. In the multi-setting, the fiber is monoidal in the sense of [Gro12, Definition 2.4], if S extends to a section $\{\cdot\} \to \mathbb{S}(\cdot)$ of multicategories. The axioms (FDer6–7) below involve only these fibers.
More generally, if $S \in \mathbb{S}(J)$ we may consider the association

$$I \mapsto \mathbb{D}(I \times J)_{p^*S},$$

where $p : I \times J \to J$ is the projection. This defines again a (left, resp. right) derivator in the usual sense which we call the **fiber of** p **over** S.

Definition 1.6.7. *Let $p : \mathbb{D} \to \mathbb{S}$ be a (left and right) fibered derivator. We call $\mathbb{D}$* **pointed** *(relative to p) if the following axiom holds:*

(FDer6) For any $S \in \mathbb{S}(\cdot)$, the category $\mathbb{D}(\cdot)_S$ has a zero object.

Definition 1.6.8. *Let $p : \mathbb{D} \to \mathbb{S}$ be a (left and right) fibered derivator. We call $\mathbb{D}$* **stable** *(relative to p) if the fibers of p are strong*[8] *and the following axiom holds:*

(FDer7) For any $S \in \mathbb{S}(\cdot)$, an object in the category $\mathbb{D}(\square)_{p^*S}$ is homotopy Cartesian if and only if it is homotopy coCartesian.

1.6.9. Recall from [Gro13] that axiom (FDer7) implies that the fibers of a stable fibered derivator are triangulated categories in a natural way. Since the push-forward, and the (relative and absolute) tensor product commute with homotopy colimits (FDer5 left, cf. also 1.6.10 below) they induce, in particular, triangulated functors between the fibers.

1.6.10. Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator. Let $\alpha : I \to J$ be a functor, and let $f : S \to T$ be a morphism in $\mathbb{S}(J)$. Axiom (FDer0 left) implies that we have a canonical isomorphism

$$(\alpha^*(f))_\bullet \alpha^* \to \alpha^* f_\bullet$$

determined by the choice of push-forward functors. We get an associated exchange morphism

$$\alpha_!(\alpha^*(f))_\bullet \to f_\bullet \alpha_!. \tag{5}$$

[8] See [Gro13, Definition 1.13]

If $\mathbb{S}$ is strong, axiom (FDer4 left) implies that this is an isomorphism (cf. Proposition 4.3.26). In other words $f_\bullet$ commutes with homotopy left Kan extensions (in particular with homotopy colimits). This also follows from (FDer0 left) *and* (FDer0 right) because these imply that $f_\bullet$ is a left adjoint. In particular (FDer5 left) follows from (FDer4 left) if we are considering plain fibered derivators (not multiderivators). Also in the multi-case, (FDer5 left) follows from (FDer0 left) *and* (FDer0 right) together.

Analogously to (op)fibrations of multicategories we have the following (cf. Section 4.4):

Proposition 1.6.11 (cf. Proposition 4.4.1)**.** *Let*

$$\mathbb{E} \xrightarrow{p_1} \mathbb{D} \xrightarrow{p_2} \mathbb{S}$$

be two left (resp. right) fibered (multi)derivators. Then also the composition $p_2 \circ p_1 : \mathbb{E} \to \mathbb{S}$ *is a left (resp. right) fibered (multi)derivator.*

The canonical source for fibered multiderivators, at least if the base $\mathbb{S}$ is associated with a usual multicategory $\mathcal{S}$, is the following:

Theorem 1.6.12. *Suppose we are given a bifibered multicategory* $\mathcal{D} \to \mathcal{S}$ *and a collection of model category structures*

$$(\mathcal{D}_S, \mathrm{Fib}_S, \mathrm{Cof}_S, \mathcal{W}_S) \tag{6}$$

on each $\mathcal{D}_S$ *such that for every morphism* $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ *the "push-forward"*

$$X_1, \dots, X_n \mapsto f_\bullet(X_1, \dots, X_n)$$

is a left Quillen functor in n-variables.
Consider the disjoint union $\mathcal{W}$ *of the* $\mathcal{W}_S$ *and for each diagram I the class* $\mathcal{W}_I$ *of morphisms in* $\mathrm{Fun}(I, \mathcal{D})$ *which are point-wise in* $\mathcal{W}$*.*
Then the association[9]

$$\mathbb{D} : I \mapsto \mathrm{Fun}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

is a fibered multiderivator over $\mathbb{S}$ whose fibers are just the associated derivators (monoidal if a section $\{\cdot\} \to \mathcal{S}$ *of multicategories is given) of the model categories (6). It is pointed (resp. stable) if the model categories (6) are pointed (resp. stable).*

We prove this Theorem in Section 4.7, cf. especially Theorem 4.7.5, with the restriction to directed diagrams for the left case and with the restriction to inverse diagrams for the right case. There are methods due to Cisinski [Cis03, Théorème 6.11] to extend such a construction to Dia = Cat. This will be written down in a forthcoming article for the fibered case.
The fact that it is reasonable to just invert the union of the $\mathcal{W}_S$ to get the fibered category associated with the *derived* push-forwards and pull-backs the author learned from Deligne [SGA73, Exposé XVII, §2][10].

[9]The localization $\mathcal{D}[\mathcal{W}^{-1}]$ is defined for multicategories in a similar manner as for categories (only 1-ary morphisms become inverted).
[10]which starts with the words: "Le rédacteur insiste pour que le lecteur s'abstinne de lire ce §".

1.6.13. Every fibered derivator $\mathbb{D} \to \mathbb{S}$ does indeed give rise to two pseudo-functors

$$\begin{array}{rcl} \mathbb{D} : \mathrm{Dia}(\mathbb{S})^{1-op} & \to & \mathcal{CAT} \\ (I,F) & \mapsto & \mathbb{D}(I)_F \end{array} \tag{7}$$

and

$$\begin{array}{rcl} \mathbb{D} : \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-op} & \to & \mathcal{CAT} \\ (I,F) & \mapsto & \mathbb{D}(I)_F \end{array} \tag{8}$$

where $\mathrm{Dia}(\mathbb{S})$ is the 2-category of pairs (I,F), with $I \in \mathrm{Dia}$, and $F \in \mathbb{S}(I)$, and in which a morphism $(I,F) \to (J,G)$ is given by a functor $\alpha : I \to J$, and a morphism $f : F \to \alpha^* G$. The pullback $\mathbb{D}(\alpha, f)$, denoted also by $(\alpha, f)^*$, is given by $f^\bullet \alpha^*$. The 2-morphisms are defined in the obvious way. The 2-category $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ has the same objects but with 1-morphisms containing $f : \alpha^* G \to F$. The pullback $(\alpha, f)^*$ here is given by $f_\bullet \alpha^*$. We have $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}) = \mathrm{Dia}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}}$. Both functors (7) and (8) can be incorporated into a single one in a nice way, see Section 1.7.
All of the following has a corresponding dual version, which we will not state explicitly.

1.6.14. *For the rest of this section, we assume for simplicity that $\mathcal{S}$ is a category with fibered products and $\mathbb{S}$ is its associated pre-derivator.* Then there is a notion of comma object $D_1 \times_{/D_2} D_3$ in $\mathrm{Dia}(\mathbb{S})$ which satisfies a universal property in the 2-category $\mathrm{Dia}(\mathbb{S})$ similar to the one which the usual comma category satisfies in Dia.

Definition 1.6.15 (cf. Definition 4.5.3)**.** *Let $p : \mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right). We say that a morphism $f : U \to X$ in $\mathcal{S}$ is* $\mathbb{D}$**-local** *if*

(Dloc1 left) The morphism f satisfies **base change***: for any diagram $Q \in \mathbb{D}(\square)$ with underlying diagram*

$$\begin{tikzcd} A \arrow[r, "\widetilde{F}"] \arrow[d, "\widetilde{G}"'] & B \arrow[d, "\widetilde{g}"] \\ C \arrow[r, "\widetilde{f}"'] & D \end{tikzcd}$$

such that $p(Q)$ in $\mathbb{S}(\square)$ is Cartesian, the following holds true: If $\widetilde{F}$ and $\widetilde{f}$ are Cartesian, and $\widetilde{g}$ is coCartesian then also $\widetilde{G}$ is coCartesian.[11]

[11] In other words, if

$$\begin{tikzcd} U \times_X Y \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ U \arrow[r, "f"'] & X \end{tikzcd}$$

is the diagram $p(Q)$ then the exchange morphism

$$G_\bullet F^\bullet \to f^\bullet g_\bullet$$

is an isomorphism.

(Dloc2 left) *The morphism of derivators (cf. Lemma 4.3.14)*

$$f^\bullet : \mathbb{D}_X \to \mathbb{D}_U$$

commutes with homotopy colimits as well.

Similarly a morphism f in $\mathbb{S}(I)$ is called local if it is object-wise local.
The associated pseudo-functor (7) satisfies the following:

Proposition 1.6.16 (cf. Proposition 4.6.9)**.** *Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator with existing pull-backs (i.e. (FDer0 right) holds true as well). Then the associated pseudo-functor (7) satisfies the following properties:*

1. *For a morphism of diagrams $(\alpha, f) : D_1 \to D_2$ the corresponding pullback*

$$(\alpha, f)^* : \mathbb{D}(D_2) \to \mathbb{D}(D_1)$$

   has a left-adjoint $(\alpha, f)_!$.

2. *For a slice diagram*

$$\begin{tikzcd} D_1 \times_{/D_3} D_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\swArrow^{\alpha}"] & D_1 \arrow[d, "\beta_1"] \\ D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

   the corresponding exchange morphism

$$p_{2!} p_1^* \to \beta_2^* \beta_{1!}$$

   is an isomorphism in $\mathbb{D}(D_2)$ provided β_2 is $\mathbb{D}$-local (in particular, if the underlying morphism f_2 of β_2 is an isomorphism — in which case we say that β_2 is of pure diagram type).

Note that 1. follows immediately from (FDer4 left) and (FDer0 left/right): If $D_1 = (I_1, F_1)$ and $D_2 = (I_2, F_2)$ then for $\alpha : I_1 \to I_2$ and $f : F_1 \to \alpha^* F_2$ a left adjoint $(\alpha, f)_!$ is obviously given by $\alpha_! f_\bullet$.

1.7 Fibered multiderivators as (special) six-functor-formalisms

A fibered multiderivator, or more simply a monoidal derivator, specifies, in addition to the pull-back and Kan extension functors, a monoidal structure on the categories $\mathbb{D}(I)$ which satisfies some additional axioms as for example (part of FDer0 left)

$$\alpha^*(- \otimes -) = (\alpha^* -) \otimes (\alpha^* -)$$

and the projection formula (FDer5 left)

$$\alpha_!(- \otimes (\alpha^* -)) = ((\alpha_! -) \otimes -)$$

for *certain* functors α. Together with the base change formula (FDer4 left)

$$\beta^* \alpha_! = B_! A^*$$

for *certain* functors α, β, A, B forming a Cartesian square, this resembles a lot the datum of a six-functor-formalism in which $f^* = f^!$, i.e. a Wirthmüller context. By defining a 2-multicategory $\mathrm{Dia}^{\mathrm{cor}}$ of multicorrespondences of diagrams (cf. Definition 5.2.3) we make this analogy precise by showing the following general theorem. (Note that a monoidal derivator is the same as a left fibered multiderivator over $\{\cdot\}$.)

Theorem 1.7.1 (cf. Corollary 5.4.3). *Let $\mathbb{D}$ and $\mathbb{S}$ be pre-multiderivators satisfying (Der1) and (Der2). A strict morphism of pre-multiderivators $\mathbb{D} \to \mathbb{S}$ is a left (resp. right) fibered multiderivator if and only if* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *is a 1-opfibration (resp. 1-fibration) of 2-multicategories.*

Here $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ is defined for any pre-multiderivator as an extension of the 2-multicategory of correspondences of diagrams $\mathrm{Dia}^{\mathrm{cor}}$. We have $\mathrm{Dia}^{\mathrm{cor}} = \mathrm{Dia}^{\mathrm{cor}}(\{\cdot\})$. However it is not essential that a pre-multiderivator is given a priori. From *any* 1-(op)fibration and 2-fibration $\mathcal{D} \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ with 1-categorical fibers a (non-strict) pre-multiderivator can be reconstructed (cf. 6.2.6).
Using the correspondence (cf. 2.4.16) between 1-opfibrations (and 2-fibrations) of 2-multicategories over $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ with 1-categorical fibers and pseudo-functors $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathcal{CAT}$ into the 2-multicategory of all categories (where the multistructure is given by multivalued functors) this formulation unifies in a nice way the two pseudofunctors

$$\mathrm{Dia}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT} \quad \text{and} \quad \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$$

that are associated with a fibered multiderivator (cf. 1.6.13) because there are embeddings of $\mathrm{Dia}(\mathbb{S})^{1-\mathrm{op}}$ and $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}}$ into $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ (cf. 5.3.8–5.3.9).
For example, Ayoub has defined in [Ayo07a, Ayo07b] an *algebraic derivator* as a pseudo-functor $\mathrm{Dia}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$ satisfying certain axioms, mentioning that this involved a choice because $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ is an equally justified forming. This problem led the author to the definition of a fibered multiderivator instead of using Ayoub's notion of algebraic derivator. The new viewpoint has thus the advantage of not only of clarifying the difference of these two approaches but also of encoding most axioms of a fibered multiderivator in a more elegant way.
The formal equalization of six-functor-formalisms and monoidal derivators explains many of their similarities. This is worked out in Chapter 6. For example, in both cases there is an internal monoidal product $\otimes$ (with adjoint denoted $\mathcal{HOM}$) and an external monoidal product $\boxtimes$ (with adjoint denoted $\mathbf{HOM}$). The external monoidal product and Hom are compatible with the one on $\mathcal{S}^{\mathrm{cor}}$ given by $S \otimes T = S \times T$, and $\mathcal{HOM}(S,T) = S \times T$, and with the one on $\mathrm{Dia}^{\mathrm{cor}}$ given by $I \otimes J = I \times J$, and $\mathcal{HOM}(I,J) = I^{\mathrm{op}} \times J$. This is just a common feature of 1-/2- (op)fibrations of 2-multicategories: The notions are transitive. Hence if $\mathcal{D} \to \mathcal{S} \to \{\cdot\}$ is a sequence of 1- (op)fibrations of multicategories, where $\{\cdot\}$ is the final multicategory, also $\mathcal{D} \to \{\cdot\}$ is a 1- (op)fibration. While $\mathcal{D} \to \mathcal{S}$

being a 1-opfibration encodes the existence of the internal monoidal product, $\mathcal{D} \to \{\cdot\}$ being a 1-opfibration encodes the existence of the external monoidal product.
From the abstract properties of 1-/2- (op)fibrations of 2-multicategories we can derive

$$\mathcal{E} \otimes \mathcal{F} = \Delta_\bullet(\mathcal{E} \boxtimes \mathcal{F}), \tag{9}$$

$$\mathcal{HOM}(\mathcal{E}, \mathcal{F}) = (\Delta')^\bullet(\mathbf{HOM}(\mathcal{E}, \mathcal{F})), \tag{10}$$

and that the external product, resp. the external Hom, can also be reconstructed from the internal ones in an analogous way. For the meaning of Δ and Δ' see Section 6.1. Explicitly, formula (9) specializes to:

$$\mathcal{E} \otimes \mathcal{F} = \Delta^*(\mathcal{E} \boxtimes \mathcal{F})$$

for the diagonal map $\Delta : S \to S \times S$ in the six-functor-formalism case, resp. $\Delta : I \to I \times I$ in the monoidal derivator case. Formula (10) specializes to

$$\mathcal{HOM}(\mathcal{E}, \mathcal{F}) = \Delta^! \mathbf{HOM}(\mathcal{E}, \mathcal{F})$$

in the six-functor-formalism case and to

$$\mathcal{HOM}(\mathcal{E}, \mathcal{F}) = \mathrm{pr}_{2,*}\, \pi_* \pi^* \mathbf{HOM}(\mathcal{E}, \mathcal{F})$$

in the monoidal derivator case, with the following functors:

$$\mathrm{tw}(I) \xrightarrow{\pi} I^{\mathrm{op}} \times I \xrightarrow{\mathrm{pr}_2} I,$$

where $\mathrm{tw}(I)$ is the twisted arrow category. The slightly different behavior is due to the different definitions of $\mathcal{S}^{\mathrm{cor}}$ and $\mathrm{Dia}^{\mathrm{cor}}$. The definition of $\mathrm{Dia}^{\mathrm{cor}}$ takes the 2-categorical nature of Dia into account. For the same reason, it encodes the more complicated base change formula of derivators involving comma categories as opposed to the simpler base change formula of a six-functor-formalism. And for the same reason the duality on $\mathrm{Dia}^{\mathrm{cor}}$ is not given by the identity $S \mapsto S$ as for $\mathcal{S}^{\mathrm{cor}}$ but by $I \mapsto I^{\mathrm{op}}$.
The upshot is that the theory of 1-/2- (op)fibrations of 2-multicategories is sufficiently powerful to treat classical six-functor formalisms and monoidal (resp. fibered multi-) derivators alike. It is even sufficiently powerful to deal with *derivator six-functor-formalisms*, an amalgam of the two theories (cf. Section 1.9 of this introduction).

1.8 (Co)homological descent revisited

1.8.1. Using fibered derivators, there is a neat conceptual solution to the problem of (co)homological descent. Analogously to a derivator $\mathbb{D}$ which associates a (derived) category $\mathbb{D}(I)$ with each diagram shape I, we now have a (derived) category $\mathbb{D}(I)_F$ for each diagram $F : I \to \mathcal{S}$ (resp. $F : I \to \mathcal{S}^{\mathrm{op}}$), cf. also 1.6.13. Let a simplicial resolution $\pi : U_\bullet \to S$ as in Section 1.4 be given, considered as a morphism $p : (\Delta^{\mathrm{op}}, U_\bullet) \to (\cdot, S)$ of diagrams in $\mathcal{S}$, resp. as a morphism $i : (\Delta, (U_\bullet)^{\mathrm{op}}) \to (\cdot, S)$ of diagrams in $\mathcal{S}^{\mathrm{op}}$. In a fibered derivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ the corresponding pull-back i^* has a right adjoint i_*, and in a fibered derivator $\mathbb{D} \to \mathbb{S}$, the pull-back p^* does have a left adjoint $p_!$, respectively. Then the first question of (co)homological descent becomes simply:

Q1: Is the corresponding unit $\mathrm{id} \to i_* i^*$ (resp. counit $p_! p^* \to \mathrm{id}$) an isomorphism?

More generally, we may consider **Cartesian** (resp. **coCartesian**) objects (cf. Section 7.4) in the fiber over a diagram $(\Delta^{\mathrm{op}}, U_\bullet)$ (resp. over $(\Delta, (U_\bullet)^{\mathrm{op}})$), and denote the corresponding subcategories by $\mathbb{D}(\Delta^{\mathrm{op}})^{\mathrm{cart}}_{U_\bullet}$ (resp. $\mathbb{D}(\Delta)^{\mathrm{cocart}}_{U_\bullet^{\mathrm{op}}}$).
The second question of (co)homological descent becomes:

Q2: Do these categories depend only on $U_\bullet$ up to taking (finite) hypercovers w.r.t. the Grothendieck topology on $\mathcal{S}$? In particular, if an object S in $\mathcal{S}$ is presented by a Čech cover (or finite hypercover) $U_\bullet$, do we have

$$\mathbb{D}(\Delta^{\mathrm{op}})^{\mathrm{cart}}_{U_\bullet} \cong \mathbb{D}(\cdot)_S, \quad \text{resp.} \quad \mathbb{D}(\Delta)^{\mathrm{cocart}}_{U_\bullet^{\mathrm{op}}} \cong \mathbb{D}(\cdot)_S \ ?$$

The categories of coCartesian objects can also be seen as a generalization of the **equivariant derived categories** of Bernstein and Lunts (cf. 7.4.3).
We call a fibered derivator **(co)local** w.r.t. a given Grothendieck pre-topology on the base (cf. section 4.5) if a few simple axioms are satisfied, i.e. if for each covering $\{f_i : U_i \to S\}$ in the given Grothendieck pre-topology, the corresponding pull-backs f_i^* (resp. $f_i^!$)

1. are jointly conservative,
2. satisfy base-change,
3. and have adjoints on the other side.

In Section 7.5 we prove that these axioms imply that (co)homological descent as described in Q1 and Q2 for all finite hypercovers is satisfied. The stronger form Q2 only holds under the stronger technical hypothesis that the fibers are stable (hence triangulated) and compactly generated. In this case the additional adjoints of f^* (resp. $f^!$) do exist automatically provided f^* commutes with infinite products as well (resp. $f^!$ commutes with infinite coproducts as well). This follows from Brown representability, cf. Chapter 8.
As a special case of *cohomological* descent we recover the theory of Grothendieck and Deligne developed in SGA IV. The present theory, however, is more general in that it is not restricted to diagrams of simplicial shape and is completely **self-dual**, leading to a theory of homological descent as well.

1.8.2. We will explain our theory of (co)homological descent in fibered derivators now in detail. We concentrate on the case of homological descent. The cohomological case is formally dual.
For this we assume that $\mathcal{S}$ is a category equipped with a Grothendieck pre-topology and $\mathbb{S}$ is its associated pre-derivator.

Definition 1.8.3 (cf. Definition 4.5.4)**.** *A left fibered derivator $p : \mathbb{D} \to \mathbb{S}$ is called* **local** *w.r.t. the pre-topology on $\mathcal{S}$, if the following conditions hold:*

1. *Every morphism $U_i \to S$ which is part of a cover is $\mathbb{D}$-local (see 1.6.15).*

2. *For a cover $\{f_i : U_i \to S\}$ the family*

$$f_i^\bullet : \mathbb{D}(S) \to \mathbb{D}(U_i)$$

is jointly conservative.

Definition 1.8.4 (cf. Definition 7.2.3). *A subclass $\mathcal{W}$ of morphisms in* $\mathrm{Dia}(\mathcal{S})$ *is called an* **absolute localizer** *(or just* **localizer***) if the following properties are satisfied:*

(L1) $\mathcal{W}$ is weakly saturated.

(L2 left) If $D = (I, F) \in \mathrm{Dia}(\mathcal{S})$, and I has a final object e, then the projection $D \to (e, F(e))$ is in $\mathcal{W}$.

(L3 left) For any commutative diagram in $\mathrm{Dia}(\mathcal{S})$

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 = (E, F) & \end{tikzcd}$$

and chosen covering $\{U_{i,e} \to F_3(e)\}$ for all $e \in E$, the following implication holds true:

$$\forall e \in E \ \forall i \quad w \times_{/D_3} (e, U_i) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

(L4 left) For any morphism $w : D_1 \to D_2 = (E, F)$ of pure diagram type, the following implication holds true:

$$\forall e \in E \quad (e, F(e)) \times_{/D_2} D_1 \to (e, F(e)) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

The corresponding dual version with axioms (L1), (L2–L4 right) is called a **colocalizer**. The notion of localizer over $\mathcal{S} = \{\cdot\}$ is due to Grothendieck. In this case (L2–L3 left) and (L2–L3 right) are equivalent and (L4 left/right) are implied by the other axioms. The class of localizers is obviously closed under intersection, hence there is a smallest localizer $\mathcal{W}^{\min}$. In the case $\mathcal{S} = \{\cdot\}$, Cisinski [Cis04] showed that $\mathcal{W}^{\min} = \mathcal{W}_\infty$, the class of functors α such that $N(\alpha)$ is a weak equivalence. There should be a connection between this more general notion of (co)localizer and the homotopy theory of simplicial sheaves on $\mathcal{S}$. In particular

Theorem 1.8.5 (cf. Theorem 7.3.15). *Let $\mathcal{W}$ be a localizer in* $\mathrm{Dia}(\mathcal{S})$. *If $F : (\Delta^{\mathrm{op}}, S_\bullet) \to (\Delta^{\mathrm{op}}, T_\bullet)$ is a morphism of simplicial diagrams in $\mathcal{S}$ which is a finite hypercover, then F is in $\mathcal{W}$.*

or more simply:

Example 1.8.6 (Mayer-Vietoris)**.** *The easiest example of an interesting morphism in $\mathcal{W}$ arises from a cover $\{U_1 \to S, U_2 \to S\}$ consisting of two monomorphisms. Then the projection*

$$p : \left(\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_1 \\ U_2 & \end{tikzcd} \right) \to S$$

is in $\mathcal{W}$, as is easily derived from the axioms (L1–L4).

Question Q1 (cf. 1.8.1) is answered by

Theorem 1.8.7 (cf. Main Theorem 7.5.5, 1)**.** *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator which is local w.r.t. the pre-topology on $\mathcal{S}$. Then the classes $\mathcal{W}_{\mathbb{D},S}$, where $S \in \mathcal{S}$ runs through all objects, of those morphisms $f : D_1 \to D_2$ in* $\mathrm{Dia}(\mathcal{S})/(\cdot, S)$ *such that the induced morphism*

$$p_{1!}p_1^* \to p_{2!}p_2^*$$

is an isomorphism, form a system of relative localizers.

The notion of system of relative localizers (cf. Definition 7.2.2) is a relative version of the notion of (absolute) localizer given above.

Definition 1.8.8 (cf. Definition 7.4.1)**.** *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator. Let $I, E \in$* Dia *be diagrams and let $\alpha : I \to E$ be a functor in* Dia*. We say that an object*

$$X \in \mathbb{D}(I)$$

*is E-***(co)Cartesian***, if for any morphism $\mu : i \to j$ in I mapping to an identity in E, the corresponding morphism $\mathbb{D}(\mu) : i^*X \to j^*X$ is (co)Cartesian.*
If E is the trivial category, we omit it from the notation, and talk about (co)Cartesian objects.

Question Q2 (cf. 1.8.1) is answered by

Theorem 1.8.9 (cf. Main Theorem 7.5.5, 2)**.** *Let $\mathbb{D} \to \mathbb{S}$ be a stable fibered derivator which is well-generated (cf. Chapter 8) and local w.r.t. the pre-topology on $\mathcal{S}$. Under some additional conditions, the set $\mathcal{W}_{\mathbb{D}}$ of those $f : D_1 \to D_2$ in* $\mathrm{Dia}(\mathcal{S})$ *such that $f^* : \mathbb{D}(D_2) \to \mathbb{D}(D_1)$ induces an equivalence*

$$f^* : \mathbb{D}(D_2)^{\mathrm{cart}} \to \mathbb{D}(D_1)^{\mathrm{cart}}$$

form a localizer.

Note that hence, in particular, f^* induces an equivalence between categories of Cartesian objects, if f is a finite hypercover $(\Delta^{\mathrm{op}}, S_\bullet) \to (\Delta^{\mathrm{op}}, T_\bullet)$. There is a variant including arbitrary hypercovers which, however, requires more axioms.

1.8.10. If $f : D_1 \to D_2$ is given by (p, φ) where D_2 is of the form $(\cdot, S)$, and $p : I \to \cdot$ is the projection, then the inverse of the equivalence f^* of Theorem 1.8.9 is just given by f_* which is the push-forward $\varphi_\bullet$ along φ followed by a homotopy colimit (see Theorem 1.6.16, 1.). In this case obviously $\mathbb{D}(D_2)^{\mathrm{cart}} = \mathbb{D}(D_2) = \mathbb{D}(S)$. In general, the inverse is given by $\varphi_\bullet$ followed by a *Cartesian projection* (cf. 7.4.4).

1.8.11. Resuming Example 1.8.6: If $\mathbb{D} \to \mathbb{S}$ is local and stable, and $A \in \mathbb{D}(\cdot)_S$ this yields

$$A \cong p_! p^* A,$$

i.e. the homotopy colimit of

$$\begin{tikzcd} i_{1,2,\bullet} i_{1,2}^\bullet A \arrow[r] \arrow[d] & i_{1,\bullet} i_1^\bullet A \\ i_{2,\bullet} i_2^\bullet A & \end{tikzcd}$$

is isomorphic to A which translates to the usual Mayer-Vietoris distinguished triangle

$$i_{1,2,\bullet} i_{1,2}^\bullet A \longrightarrow i_{1,\bullet} i_1^\bullet A \oplus i_{2,\bullet} i_2^\bullet A \longrightarrow A \longrightarrow i_{1,2,\bullet} i_{1,2}^\bullet A[1]$$

in the language of triangulated categories.

1.8.12. If $\mathbb{D}$ is a stable derivator (not fibered) which is well-generated then we get by the above result and Cisinski's Theorem that the categories

$$\mathbb{D}(I)^{\mathrm{cart}}$$

depend (up to equivalence) only on the homotopy type of $N(I)$. Note that in the non-fibered setting "cart" and "cocart" are synonymous. The results presented in this section lead to the expectation that a similar statement holds for a local fibered derivator over $\mathcal{S}$ and homotopy types replaced by homotopy classes of simplicial sheaves over $\mathcal{S}$.

1.9 Derivator six-functor-formalisms

In most of their occurrences in nature the values of a six-functor-formalism, i.e. the fibers of the fibration $\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$, are *derived categories*. It is therefore natural to seek to enhance the situation to a *fibered multiderivator*.
As we have seen, enhancements of this sort are essential to deal with questions of (co)homological descent (cf. Section 1.8). The notion of a fibered multiderivator given in 4.3.6, however, is not sufficient because $\mathcal{S}^{\mathrm{cor}}$ is a 2-multicategory (as opposed to a usual multicategory). Although the 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ gives rise to a usual (not-represented) pre-multiderivator, by identifying 2-isomorphic 1-morphisms in the diagram categories $\mathcal{S}^{\mathrm{cor}}(I)$, a fibered multiderivator over *that* pre-multiderivator would not encode what we

want[12]. It turns out that the theory of fibered multiderivators over pre-multiderivators has a straightforward extension to 2-pre-multiderivators in which the knowledge of the 2-morphisms of the base is preserved. In Section 5.4, we therefore develop the theory of fibered multiderivators over 2-pre-multiderivators. This allows to consider the symmetric 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$ represented by the symmetric 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ and to give the following:

Definition 1.9.1 (cf. Definition 9.1.1)**.**
A **(symmetric) derivator six-functor-formalism** *is a left and right fibered (symmetric) multiderivator*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}.$$

It becomes important to have notions of (symmetric) *(op)lax* fibered multiderivators as well. Those are useful to enhance to derivators the definition of a proper or etale six-functor-formalism which arises, for instance, whenever for some class of morphisms one has isomorphisms $f^! \cong f^*$ or $f_! \cong f_*$ which are part of the formalism. If this is the case for all morphisms, one speaks of a Wirthmüller, or Grothendieck context, respectively. Section 9.2 is devoted to the *construction* of derivator six-functor-formalisms. There, we concentrate on the case in which $f_! = f_*$ for all morphisms f in $\mathcal{S}$, i.e. to Grothendieck contexts. In the classical case this construction is almost tautological:

1. One starts with a four-functor-formalism $(f_*, f^*, \otimes, \mathcal{HOM})$ encoded by a bifibration of usual (symmetric) multicategories $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ (where $\mathcal{S}^{\mathrm{op}}$ becomes a multicategory via the product, cf. Example 1.5.11), or equivalently by a pseudo-functor of (2-)multicategories $\mathcal{S}^{\mathrm{op}} \to \mathcal{CAT}$. Then one simply defines a pseudo-functor $\mathcal{S}^{\mathrm{cor}} \to \mathcal{CAT}$ by mapping a multicorrespondence (3) to the functor

   $$f_*((g_1^*-) \otimes \cdots \otimes (g_n^*-)).$$

   It is straightforward (but slightly tedious) to check that this defines a pseudo-functor if and only if base-change and projection formula hold (cf. Proposition 3.3.3).

2. In the derived world, using theorems on Brown representability, one gets formally that the f_* functors have right adjoints $f^!$ (provided that f_* commutes with infinite coproducts as well) hence the 1-opfibration (and 2-bifibration) with 1-categorical fibers $\mathcal{E} \to \mathcal{S}^{\mathrm{cor}}$, which corresponds to the pseudo-functor in 1., is also a 1-fibration.

It is surprising, however, that constructions 1. and 2. are also possible in the world of fibered multiderivators, although they become more involved. The first is however still completely formal. We will describe the results in detail in Section 1.10. It turns out that one can relax the condition that $\mathcal{S}^{\mathrm{op}}$ is a multicategory coming from a usual category $\mathcal{S}$

[12]E.g. the push-forward along a correspondence of the form $\{\cdot\} \leftarrow X \to \{\cdot\}$ should yield something like the cohomology with compact support of X with constant coefficients. Identifying 2-isomorphic 1-morphisms in $\mathcal{S}^{\mathrm{cor}}$ would force this to become the invariant part (in a derived sense) under automorphisms of X.

via the categorical product — one can start with any opmulticategory $\mathcal{S}$. The definition of $\mathcal{S}^{\mathrm{cor}}$ generalizes readily to this situation.
The ultimate goal is to construct derivator six-functor-formalisms in interesting situations (e.g. etale constructible sheaves over schemes, or (pro-)quasi-coherent sheaves over schemes). This case, in which $f_! \neq f_*$, is much more involved. Imitating the classical constructions of $f_!$ using compactifications and $f^!$ as its right adjoint using Brown representability, we will achieve this in forthcoming work [Hör17a] which is not part of this habilitation thesis. The degree of generality (in particular which category Dia can be taken) is not yet completely clear. The input is, in any case, a bifibration of monoidal model categories like in 1.6.12 over $\mathcal{S}^{\mathrm{op}}$ satisfying a bunch of additional axioms, including the existence of compactifications.

1.10 Construction of derivator six-functor-formalisms

Let $\mathcal{S}$ be an opmulticategory with multipullbacks. It may also be equipped with the structure of symmetric or braided opmulticategory. As before, $\mathcal{S}$ will mostly be a usual category with the structure of opmulticategory given by the product, i.e.

$$\mathrm{Hom}_{\mathcal{S}}(Y; X_1, \dots, X_n) = \mathrm{Hom}_{\mathcal{S}}(Y, X_1) \times \cdots \times \mathrm{Hom}_{\mathcal{S}}(Y, X_n). \tag{11}$$

This structure is canonically symmetric. However, $\mathcal{S}$ may be arbitrary (it also does not need to be representable). If $\mathcal{S}$ is symmetric, or braided, all other multicategories and 2-multicategories occurring, e.g. $\mathcal{S}^{\mathrm{cor}}$, will also be symmetric, resp. braided, and all functors have to be compatible with the corresponding actions.

Definition 1.10.1. *Let $\mathcal{S}$ be an opmulticategory with multipullbacks. Let $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be a bifibration of usual multicategories. We say that it satisfies* **multi-base-change***, if for every multipullback in $\mathcal{S}$*

$$\begin{tikzcd} X_1, \dots, X_i, \dots, X_n & Z \arrow[l, "g"'] \\ X_1, \dots, X_i', \dots, X_n \arrow[u, "{(\mathrm{id},\dots,f,\dots,\mathrm{id})}"] & Z' \arrow[l, "G"] \arrow[u, "F"'] \end{tikzcd}$$

the natural transformation

$$g_\bullet(-, \dots, \underbrace{f^\bullet -}_{at\ i}, \dots, -) \longrightarrow F^\bullet G_\bullet(-, \dots, -)$$

is an isomorphism.

If $\mathcal{S}$ is a usual category equipped with the opmulticategory structure (11) this encodes *projection formula* and *base change*.
In this definition, $f^\bullet$ denotes the pull-back along f^{op} in $\mathcal{S}^{\mathrm{op}}$, that is, the usual push-forward f_* along f in $\mathcal{S}$. The reason for this notation is that we stick to the convention that $f^\bullet$ is always right adjoint to $f_\bullet$ and, at the same time, we want to avoid the notation $f_*, f^*, f^!, f_!$ because of the possible confusion with the left and right Kan extension functors which are be denoted by $\alpha_!$, and α_*, respectively.

Theorem 1.10.2 (cf. Main Theorem 9.2.1)**.** *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}^{\mathrm{op}}$ be the (symmetric) pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (symmetric) left and right fibered multiderivator satisfying the following conditions:*

1. *The pullback along 1-ary morphisms (i.e. the pushforward along 1-ary morphisms in $\mathcal{S}$) commutes also with homotopy* colimits *(of shape in* Dia*).*
2. *In the underlying bifibration $\mathbb{D}(\cdot) \to \mathbb{S}(\cdot)$ multi-base-change holds in the sense of Definition 1.10.1.*

Then there exists a (symmetric) oplax left fibered multiderivator

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$$

with the following properties:

a) The corresponding (symmetric) 1-opfibration, and 2-opfibration, of 2-multicategories with 1-categorical fibers

$$\mathbb{E}(\cdot) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(\cdot) = \mathcal{S}^{\mathrm{cor},G}$$

is just (up to equivalence) obtained from $\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$ by the procedure described in Definition 3.3.2.

b) For every $S \in \mathcal{S}$ there is a canonical equivalence between the fibers (which are usual left and right derivators):

$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Using standard theorems on Brown representability etc. (cf. Chapter 8) we can refine this:

Theorem 1.10.3 (cf. Main Theorem 9.2.2)**.** *Let* Dia *be an infinite diagram category (cf. 4.1.1) which contains all finite posets. Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}$ be the corresponding represented (symmetric) pre-multiderivator. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be an* infinite *(symmetric) left and right fibered multiderivator satisfying conditions 1. and 2. of Theorem 9.2.1,* with stable, perfectly generated fibers *(cf. Definition 4.3.19 and Definition 8.1.7).*
Then the restriction of the left fibered multiderivator $\mathbb{E}$ from Main Theorem 9.2.1 is a (symmetric) left and right *fibered multiderivator*

$$\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}} \to \mathbb{S}^{\mathrm{cor}}$$

and has an extension as a (symmetric) lax *right fibered multiderivator*

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}.$$

We call $\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}}$ together with the extensions to $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$, and to $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$, respectively, a **derivator Grothendieck context**, cf. Definition 9.1.1.
The construction in Definition 3.3.2 recalled above is quite tautological. Why is Theorem 9.2.1 not similarly tautological? To understand this point, let us look at the following simple example: A fibered derivator $\mathbb{D}$ over Δ_1, the (represented pre-derivator of the) usual category with one arrow, encodes an enhancement of an adjunction between derivators (the two fibers of $\mathbb{D}$). Think about the case in which this is the derived adjunction coming from an adjunction of underived functors f_*, f^*. This includes, for instance, as fiber of $\mathbb{D}(\Delta_1)$ over the identity in $\mathrm{Fun}(\Delta_1, \Delta_1)$, the category of *coherent* diagrams of the form $X \to f_*Y$, or equivalently $f^*X \to Y$, up to quasi-isomorphisms between such diagrams. Here f_*, f^* are the *underived* functors. The extension $\mathbb{E}$ in Theorem 9.2.1 allows to consider *coherent* diagrams of the form $Rf_*X \to Y$, or equivalently $X \to f^!Y$, if Rf_* has a right adjoint $f^!$, the point being that, however, $f^!$ may not exist before passing to the derived categories. Nevertheless, we are now allowed to speak about "coherent diagrams of the form $X \to f^!Y$" although this does not make literally sense. In particular, Theorem 9.2.1 yields a *coherent enhancement* in $\mathbb{D}(\Delta_1)_{p^*e_i}$ for $i = 0$, and $i = 1$, respectively, of the unit and counit

$$Rf_*f^!\mathcal{E} \to \mathcal{E} \qquad \mathcal{E} \to f^!Rf_*\mathcal{E}.$$

In this particular case, the *construction* boils down to the following. The fiber of $\mathbb{E}(\Delta_1)$ over the correspondence

$$\begin{tikzcd} & e_0 \arrow[dl] \arrow[dr, equal] & \\ e_1 & & e_0 \end{tikzcd}$$

will consist of coherent diagrams of the form $f_*X \leftarrow Z \to Y$ *in the original fibered derivator* $\mathbb{D}$ with the property that the induced morphism $Rf_*X \leftarrow Z$ is a quasi-isomorphism, i.e. the morphism $X \leftarrow Z$ in $\mathbb{D}(\Delta_1)$ becomes *coCartesian*, when considered as a morphism in $\mathbb{D}(\cdot)$. The purpose of Sections 9.2–9.3 is thus to make this construction work in the full generality of Theorem 9.2.1. Although the idea is still very simple, the construction of $\mathbb{E}$ (cf. Definition 9.2.17), and the proof that it really is a (left) fibered multiderivator over $\mathbb{S}^{\mathrm{cor}}$, becomes quite technical.

1.11 Localization triangles and n-angles

As an application of the general definition of derivator six-functor-formalisms, in Section 9.4 we explain that the appearance of distinguished triangles like

$$j_!j^!\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \overline{j}_*\overline{j}^*\mathcal{E} \xrightarrow{[1]}$$

for an "open immersion" j and its complementary "closed immersion" $\overline{j}$ can be treated elegantly. Actually there are four flavours of these sequences, two for proper derivator six-functor-formalisms, and two for etale derivator six-functor-formalisms. More generally, a sequence of "open embeddings"

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$$

leads immediately to so called $(n+1)$-angles in the sense of [GS14, §13] in the fiber over X_n which is a usual stable derivator.

1.12 The six functors for stacks

The results described in this section will appear in a forthcoming article which is not part of this habilitation thesis.

Consider a derivator six-functor-formalism with stable, compactly generated fibers $\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}$ as in Section 1.9. We neglect the multi- (i.e. monoidal) aspect in this section.

1.12.1. Consider again the case in which $\mathcal{S}$ is equipped with a Grothendieck pretopology.

With a simplicial object $X_\bullet$, i.e. an object in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$, the derivator six-functor-formalism associates canonically two categories:

$$\mathbb{D}(X_\bullet)^{\mathrm{cart}} \qquad \mathbb{D}((X_\bullet)^{\mathrm{op}})^{\mathrm{cocart}} \tag{12}$$

If the restriction $\mathbb{D} \to \mathbb{S}$ is local (cf. 4.5.4) then by *homological* decent the first category does only depend on $X_\bullet$ up to taking finite hypercovers. If the restriction $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is colocal (cf. 4.5.6) then by *cohomological* decent the second category does only depend on $X_\bullet$ up to taking finite hypercovers. In particular if both properties are satisfied, and if $X_\bullet$ presents an algebraic stack, then both categories do only depend on the equivalence class of the stack.

1.12.2. Let $\alpha : X_\bullet \to Y_\bullet$ be a morphism of simplicial objects in $\mathcal{S}$, i.e. a morphism in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$. It induces the following pairs of adjoint functors:

$$\begin{tikzcd} \mathbb{D}(X_\bullet)^{\mathrm{cart}} \arrow[d, shift right, "\alpha_!"'] & \mathbb{D}((X_\bullet)^{\mathrm{op}})^{\mathrm{cocart}} \arrow[d, shift left, "\alpha_*"] \\ \mathbb{D}(Y_\bullet)^{\mathrm{cart}} \arrow[u, shift right, "\alpha^!"'] & \mathbb{D}((Y_\bullet)^{\mathrm{op}})^{\mathrm{cocart}} \arrow[u, shift left, "\alpha^*"] \end{tikzcd}$$

Here (cf. also 1.8.10) α_* is the usual pull-back $(\alpha^{\mathrm{op}})^\bullet$ in the fibered derivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ followed by a right coCartesian projection, and $\alpha_!$ is the usual push-forward $\alpha_\bullet$ in the fibered derivator $\mathbb{D} \to \mathbb{S}$ followed by a left Cartesian projection. To extend all four functors (and eventually all six) to stacks, we have to be able to identify the two categories (12) in a canonical way. This is achieved by the following

Theorem 1.12.3. *If $X_\bullet$ presents a 1-stack, and if the derivator six-functor-formalism has certain additional locality properties w.r.t. the Grothendieck pretopology, then we have a canonical equivalence*

$$\mathbb{D}(X_\bullet)^{\mathrm{cart}} \cong \mathbb{D}((X_\bullet)^{\mathrm{op}})^{\mathrm{cocart}}.$$

Therefore we dispose of the 4 functors in either version. These satisfy base-change w.r.t. the fiber product of 1-stacks.

Idea of proof. Here the utility of the general derivator version of a four- (or six-) functor-formalism is revealed: We construct a diagram of correspondences $X^{\mathrm{cor}}_{\bullet,\bullet} \in \mathrm{Fun}(\Delta \times \Delta^{\mathrm{op}}, \mathcal{S}^{\mathrm{cor}})$ (for this it is crucial that $X_\bullet$ presents a 1-stack) with morphisms of diagrams in $\mathcal{S}^{\mathrm{cor}}$

$$\begin{tikzcd} & X^{\mathrm{cor}}_{\bullet,\bullet} \arrow[dl] \arrow[dr] & \\ X_\bullet & & X_\bullet^{\mathrm{op}} \end{tikzcd}$$

and show that these induce equivalences

$$\mathbb{D}(X_\bullet)^{\mathrm{cart}} \cong \mathbb{D}(X^{\mathrm{cor}}_{\bullet,\bullet})^{\Delta^{\mathrm{op}}-\mathrm{cart},\Delta-\mathrm{cocart}} \cong \mathbb{D}((X_\bullet)^{\mathrm{op}})^{\mathrm{cocart}}.$$

The base change formula can be proven using the symmetric diagrams $X^{\mathrm{cor}}_{\bullet,\bullet}$. The proof is completely formal. □

Unfortunately, so far, the author has not been able to make this construction coherent, that is, to construct something like a fibered derivator over correspondences of stacks. This would probably be necessary to extend the result to higher stacks.

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Notation

We denote by $\mathcal{CAT}$ the 2-“category”[13] of categories, by $(\mathcal{S})\mathcal{MCAT}$ the 2-“category” of (symmetric) multicategories, and by Cat the 2-category of small categories. We consider a partially ordered set (poset) X as a small category by considering the relation $x \le y$ to be equivalent to the existence of a unique morphism $x \to y$. We denote the positive integers (resp. the non-negative integers) by $\mathbb{N}$ (resp. by $\mathbb{N}_0$). The ordered set $\{0, \dots, n\} \subset \mathbb{N}_0$ considered as a small category is denoted by Δ_n. We denote by $\mathrm{Mor}(\mathcal{D})$ (resp. $\mathrm{Iso}(\mathcal{D})$) the class of morphisms (resp. isomorphisms) in a category $\mathcal{D}$. The final category (which consists of only one object and its identity) is denoted by $\cdot$ or Δ_0. The same notation is also used for the final multicategory, which has one object and precisely one n-ary morphism for any $n \in \mathbb{N}_0$.

[13] where we put “category” into quotation marks to indicate that is has classes replaced with 2-classes (or, if you prefer, is constructed w.r.t. a larger universe).

2 Categorical generalities

2.1 Classical (op)fibrations

For the convenience of the reader we recall the theory of classical (op)fibrations of categories in this section. The theory will be generalized to multicategories and finally to 2-multicategories in Section 2.4.

2.1.1 (right)**.** Let $p : \mathcal{D} \to \mathcal{S}$ be a functor, and let $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E}' \to \mathcal{E}$ over f is called **Cartesian** if the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{F}, \mathcal{E}') \cong \mathrm{Hom}_{f \circ g}(\mathcal{F}, \mathcal{E})$$

for any morphism $g : R \to S$ in $\mathcal{S}$ and for every $\mathcal{F} \in \mathcal{D}_R$.
The functor p is called a **(Grothendieck) fibration** if for any $f : S \to T$ and for every object $\mathcal{E}$ in $\mathcal{D}_T$ (i.e. such that $p(\mathcal{E}) = T$) there exists a Cartesian morphism $\mathcal{E}' \to \mathcal{E}$.

2.1.2 (left)**.** Let $p : \mathcal{D} \to \mathcal{S}$ be a functor, and let $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E} \to \mathcal{E}'$ over f is called **coCartesian** if the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{E}', \mathcal{F}) \cong \mathrm{Hom}_{g \circ f}(\mathcal{E}, \mathcal{F})$$

for any morphism $g : T \to U$ in $\mathcal{S}$ and for every $\mathcal{F} \in \mathcal{D}_U$.
The functor p is called **a (Grothendieck) opfibration** if for any $f : S \to T$ and for every object $\mathcal{E}$ in $\mathcal{D}_S$ there exists a coCartesian morphism $\mathcal{E} \to \mathcal{E}'$.

2.1.3. The functor p is an opfibration if and only if $p^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}$ is a fibration. We say that p is a **bifibration** if is a fibration and an opfibration at the same time. If $p : \mathcal{D} \to \mathcal{S}$ is a fibration we may choose an associated pseudo-functor, i.e. to each $S \in \mathcal{S}$ we associate the category $\mathcal{D}_S$, and to each $f : S \to T$ we associate a push-forward functor

$$f_\bullet : \mathcal{D}_S \to \mathcal{D}_T$$

such that for each $\mathcal{E}$ in $\mathcal{D}_S$ there is a Cartesian morphism $\mathcal{E} \to f_\bullet \mathcal{E}$. The same holds similarly for an opfibration with the pull-back $f^\bullet$ instead of the push-forward. If the functor p is a bifibration, $f_\bullet$ is left adjoint to $f^\bullet$. Situations where this is the opposite can be modeled by considering bifibrations $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$.

2.2 2-multicategories

The notion of 2-multicategory is a straight-forward generalization of the notion of 2-category. For lack of reference and because we want to stick to the case of (strict) 2-categories as opposed to bicategories, we list the relevant definitions here:

Definition 2.2.1. *A* **2-multicategory** $\mathcal{D}$ *consists of*

- *a class of objects* $\mathrm{Ob}(\mathcal{D})$*;*

- *for all $n \in \mathbb{Z}_{\geq 0}$, and for all objects $X_1, \dots, X_n, Y$ in $\mathrm{Ob}(\mathcal{D})$ a category*

$$\mathrm{Hom}(X_1, \dots, X_n; Y);$$

- *a composition, i.e. for all objects $X_1, \dots, X_n$, $Y_1, \dots, Y_m$, Z in $\mathrm{Ob}(\mathcal{D})$ and for all $i \in \{1, \dots, m\}$ a functor:*

$$\begin{array}{rcl} \mathrm{Hom}(X_1, \dots, X_n; Y_i) \times \mathrm{Hom}(Y_1, \dots, Y_m; Z) & \to & \mathrm{Hom}(Y_1, \dots, \underbrace{X_1, \dots, X_n}_{at\ i}, \dots, Y_m; Z) \\ f, g & \mapsto & g \circ_i f; \end{array}$$

- *for all $X \in \mathrm{Ob}(\mathcal{D})$ an identity object id_X in the category $\mathrm{Hom}(X; X)$;*

satisfying strict associativity and identity laws. The composition w.r.t. independent slots is commutative, i.e. for $1 \leq i < j \leq m$ if $f \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ and $f' \in \mathrm{Hom}(X'_1, \dots, X'_k; Y_j)$ and $g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ we have

$$(g \circ_i f) \circ_{j+n-1} f' = (g \circ_j f') \circ_i f. \tag{13}$$

A symmetric (braided) 2-multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms of categories

$$\alpha : \mathrm{Hom}(X_1, \dots, X_n; Y) \to \mathrm{Hom}(X_{\alpha(1)}, \dots, X_{\alpha(n)}; Y)$$

for $\alpha \in S_n$ (resp. $\alpha \in B_n$) forming an action which is strictly compatible with composition in the obvious way (in the braided case: substitution of strings).

The 1-composition of 2-morphisms is (as for usual 2-categories) determined by the following whiskering operations: Let $f, g \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ and $h \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ be 1-morphisms and let $\mu : f \Rightarrow g$ be a 2-morphism, i.e. a morphism in the category $\mathrm{Hom}(X_1, \dots, X_n; Y_i)$. Then we define

$$h * \mu := \mathrm{id}_h \cdot \mu$$

where the right hand side is the image of the 2-morphism $\mathrm{id}_h \times \mu$ under the composition functor. Similarly we define $\mu * h$ for $\mu : f \Rightarrow g$ with $f, g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ and $h \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$.

2.2.2. In the same way, we define a **2-opmulticategory** having categories of 1-morphisms of the form

$$\mathrm{Hom}(X; Y_1, \dots, Y_n).$$

For each 2-multicategory $\mathcal{D}$ there is a natural 2-opmulticategory $\mathcal{D}^{1-\mathrm{op}}$, and vice versa, where the direction of the 1-morphisms is flipped.

Definition 2.2.3. *A* **pseudo-functor** *$F : \mathcal{C} \to \mathcal{D}$ between 2-multicategories is given by the following data:*

- *for* $X \in \mathrm{Ob}(\mathcal{C})$ *an object* $F(X) \in \mathrm{Ob}(\mathcal{C})$*;*
- *for* $X_1, \dots, X_n; Y \in \mathrm{Ob}(\mathcal{C})$*, a functor*

$$\mathrm{Hom}(X_1, \dots, X_n; Y) \to \mathrm{Hom}(F(X_1), \dots, F(X_n); F(Y));$$

- *for* $X \in \mathrm{Ob}(\mathcal{C})$ *a 2-isomorphism*

$$F_X : F(\mathrm{id}_X) \Rightarrow \mathrm{id}_{F(X)};$$

- *for* $X_1, \dots, X_n; Y_1, \dots, Y_m; Z \in \mathrm{Ob}(\mathcal{C})$ *and* $i \in \{1, \dots, m\}$ *a natural isomorphism*

$$F_{-,-} : F(-) \circ_i F(-) \Rightarrow F(- \circ_i -)$$

of functors

$$\begin{aligned} &\mathrm{Hom}(X_1, \dots, X_n; Y_i) \times \mathrm{Hom}(Y_1, \dots, Y_m; Z) \\ \to &\mathrm{Hom}(F(Y_1), \dots, F(Y_{i-1}), F(X_1), \dots, F(X_n), F(Y_{i+1}), \dots, F(Y_m); F(Z)); \end{aligned}$$

satisfying

$$F_{\mathrm{id}_Y, f} = F_Y * F(f) \qquad F_{g, \mathrm{id}_{Y_i}} = F(g) * F_{Y_i}$$

for $f \in \mathrm{Hom}(X_1, \dots, X_n; Y)$*, and* $g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$*, respectively, and for composable* f, g, h *that*

$$\begin{tikzcd} F(h) \circ_j F(g) \circ_i F(f) \arrow[r, "F(h) * F_{g,f}"] \arrow[d, "F_{h,g} * F(f)"'] & F(h) \circ_j F(g \circ_i f) \arrow[d, "F_{h,gf}"] \\ F(h \circ_j g) \circ_i F(f) \arrow[r, "F_{hg,f}"] & F(h \circ_j g \circ_i f) \end{tikzcd}$$

commutes. A pseudo-functor is called a **strict functor** *if all* $F_{g,f}$ *and* F_X *are identities.*

Definition 2.2.4. *A* **pseudo-natural transformation** $\alpha : F_1, \dots, F_m \Rightarrow G$ *between pseudo-functors* $F_1, \dots, F_m; G : \mathcal{C} \to \mathcal{D}$ *is given by:*

- *for* $X \in \mathrm{Ob}(\mathcal{C})$ *a 1-morphism* $\alpha(X) \in \mathrm{Hom}(F_1(X), \dots, F_m(X); G(X))$*;*
- *for each 1-morphism* f *in* $\mathrm{Hom}(X_1, \dots, X_n; Y)$ *a 2-isomorphism*

$$\alpha_f : \alpha(Y) \circ (F_1(f), \dots, F_m(f)) \Rightarrow G(f) \circ (\alpha(X_1), \dots, \alpha(X_n));$$

such that all the following diagrams commute:

- *for* $f \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ *and* $g \in \mathrm{Hom}(Y_1, \dots, Y_k; Z)$*:*

$$\begin{tikzcd} \alpha(Z)(F_1(g)F_1(f), \dots, F_m(g)F_m(f)) \arrow[r, "(G(g) * \alpha_f)(\alpha_g * F(f))"] \arrow[d, "\alpha(Z) * ((F_1)_{g,f}, \dots, (F_m)_{g,f})"'] & G(g)G(f)(\alpha(Y_1), \dots, \alpha(X_1), \dots, \alpha(X_n), \dots, \alpha(Y_k)) \arrow[d, "G_{g,f} * (\dots)"] \\ \alpha(Z)(F_1(gf), \dots, F_m(gf)) \arrow[r, "\alpha_{gf}"] & G(gf)(\alpha(Y_1), \dots, \alpha(X_1), \dots, \alpha(X_n), \dots, \alpha(Y_k)) \end{tikzcd}$$

- *for $X \in \mathrm{Ob}(\mathcal{C})$:*

$$\begin{tikzcd}
\alpha(X)(F_1(\mathrm{id}_X),\dots,F_n(\mathrm{id}_X)) \arrow[r, "\alpha_{\mathrm{id}_X}"] \arrow[d, "\alpha(X)*((F_1)_X,\dots,(F_n)_X)"] & G(\mathrm{id}_X)\alpha(X) \arrow[d, "G_X*\alpha(X)"] \\
\alpha(X)(\mathrm{id}_{F_1(X)},\dots,\mathrm{id}_{F_n(X)}) \arrow[r, equal] & \mathrm{id}_{G(X)}\,\alpha(X)
\end{tikzcd}$$

- *for each 2-morphism $f \Rightarrow g$ in $\mathrm{Hom}(X_1,\dots,X_n;Y)$:*

$$\begin{tikzcd}
\alpha(Y)\cdot(F_1(f),\dots,F_m(f)) \arrow[r] \arrow[d] & G(f)\cdot(\alpha(X_1),\dots,\alpha(X_n)) \arrow[d] \\
\alpha(Y)\cdot(F_1(g),\dots,F_m(g)) \arrow[r] & G(g)\cdot(\alpha(X_1),\dots,\alpha(X_n))
\end{tikzcd}$$

Similarly we define an **oplax natural transformation**, *if the morphism α_f is no longer required to be a 2-isomorphism but can be any 2-morphism. We define a* **lax natural transformation** *requiring that the morphism α_f goes in the other direction, with the diagrams above changed suitably.*

Definition 2.2.5. *A* **modification** *$\mu : \alpha \Rrightarrow \beta$ between $\alpha, \beta : F_1,\dots,F_m \Rightarrow G$ (pseudo-, lax-, or oplax-) natural transformations is given by the following data:*

- *For $X \in \mathrm{Ob}(\mathcal{C})$ a 2-morphism*

$$\mu_X : \alpha(X) \Rightarrow \beta(X)$$

such that for each 1-morphism $f \in \mathrm{Hom}(X_1,\dots,X_n;Y)$ the following diagram commutes:

$$\begin{tikzcd}
\alpha(Y)\circ(F_1(f),\dots,F_m(f)) \arrow[r] \arrow[d] & G(f)\circ(\alpha(X_1),\dots,\alpha(X_n)) \arrow[d] \\
\beta(Y)\circ(F_1(f),\dots,F_m(f)) \arrow[r] & F(f)\circ(\beta(X_1),\dots,\beta(X_n))
\end{tikzcd}$$

resp. (in the lax case) the analogous diagram with the horizontal arrows reversed.

Lemma 2.2.6. *Let $\mathcal{C}$, $\mathcal{D}$ be 2-multicategories. Then the collection*

$$\mathrm{Fun}(\mathcal{C},\mathcal{D})$$

of pseudo-functors, pseudo-natural transformations and modifications forms a 2-multicategory. Similarly the collections

$$\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{D}) \qquad \mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C},\mathcal{D})$$

of pseudo-functors, (op)lax natural transformations and modifications form 2-multicategories.

Proof. We leave the proof to the reader, but will explicitly spell out how pseudo-natural transformations are composed:
Let $\alpha : F_1, \dots, F_m \Rightarrow G_i$ and $\beta : G_1, \dots, G_n \Rightarrow H$ be pseudo-natural transformations. Then the pseudo-natural transformation

$$\beta \circ_i \alpha : G_1, \dots, G_{i-1}, F_1, \dots, F_m, G_{i+1}, \dots, G_n \Rightarrow H$$

is given as follows. $(\beta \circ_i \alpha)(X)$ is just the composition of $\beta(X) \circ \alpha(X)$ and the 2-morphism

$$\begin{aligned}(\beta \circ_i \alpha)_f : \beta(X) \circ_i \alpha(X) \circ (G_1(f), \dots, F_1(f), \dots, F_m(f), \dots, G_n(f)) \\ \Rightarrow H(f) \circ (\beta(X_1)\alpha(X_1), \dots, \beta(X_n)\alpha(X_n))\end{aligned}$$

is given by the composition

$$\beta(X) \circ_i \alpha(X) \circ (G_1(f), \dots, F_1(f), \dots, F_m(f), \dots, G_n(f)) \Rightarrow$$

$$\beta(X) \circ_i (G_1(f), \dots, G_i(f), \dots, G_n(f)) \circ (\alpha(X_1), \dots, \alpha(X_n)) \Rightarrow$$

$$H(f) \circ (\beta(X_1) \circ_i \alpha(X_1), \dots, \beta(X_n) \circ_i \alpha(X_n)).$$

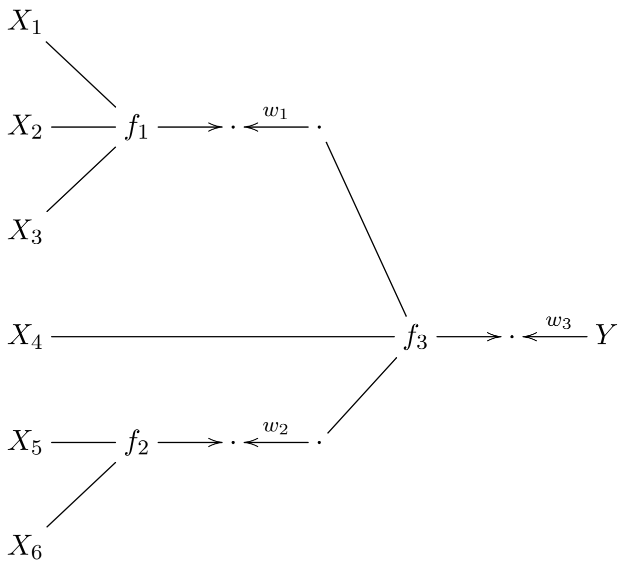
□

2.3 Localization of multicategories

The following proposition deals with usual multicategories (not 2-multicategories).

Proposition 2.3.1. *Let $\mathcal{D}$ be a (symmetric, braided) multicategory and let $\mathcal{W}$ be a subclass of 1-ary morphisms. Then there exists a (symmetric, braided) multicategory $\mathcal{D}[\mathcal{W}^{-1}]$, which is not necessarily locally small if $\mathcal{D}$ is, together with a functor $\iota : \mathcal{D} \to \mathcal{D}[\mathcal{W}^{-1}]$ of (symmetric, braided) multicategories with the property that $\iota(w)$ is an isomorphism for all $w \in \mathcal{W}$ and which is universal w.r.t. this property.*

Proof. This construction is completely analogous to the construction for usual categories. Morphisms $\mathrm{Hom}(X_1, \dots, X_n; Y)$ are formal compositions of i-ary morphisms in $\mathcal{D}$ and formal inverses of morphisms in $\mathcal{W}$, for example:



More precisely: Morphisms are defined to be the class of lists of n_i-ary morphisms $f_i \in \mathrm{Hom}(X_{i,1}, \dots, X_{i,n_i}; Y_i)$, morphisms $w_i : Y_i' \to Y_i$ in $\mathcal{W}$ and integers k_i as follows

$$(f_1, w_1), k_1, (f_2, w_2), k_2, \dots, k_{n-1}, (f_n, w_n)$$

such that $Y_i' = X_{i+1,k_i}$, modulo relations coming from commutative squares, the relations (13), and relations forcing the (id, w_i) to become the left and right inverse of (w_i, id). $\square$

2.4 (Op)fibrations of 2-multicategories

For (op)fibrations of (usual) multicategories the reader may consult [Her00, Her04], and for (op)fibrations of 2-categories [Bak09, Buc14, Her99]. The definitions in this section however are slightly different from those in any of these sources.

2.4.1. Let

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow^{\mu}"] & \mathcal{B} \arrow[d, "\beta"] \\ \mathcal{C} \arrow[r, "\gamma"'] & \mathcal{D} \end{tikzcd}$$

be a 2-commutative diagram of (usual) categories, where μ is a natural isomorphism. Then we say that the diagram is **2-Cartesian** if it induces an equivalence

$$\mathcal{A} \cong \mathcal{B} \times^{\sim}_{/\mathcal{D}} \mathcal{C},$$

where $\mathcal{B} \times^{\sim}_{/\mathcal{D}} \mathcal{C}$ is the full subcategory of the comma category $\mathcal{B} \times_{/\mathcal{D}} \mathcal{C}$ consisting of those objects $(b, c, \nu : \beta(b) \to \gamma(c))$, with $b \in \mathcal{B}$, $c \in \mathcal{C}$ in which the morphism ν is an isomorphism.

If μ is an identity then the diagram is said to be **Cartesian**, if it induces an equivalence of categories

$$\mathcal{A} \cong \mathcal{B} \times_{\mathcal{D}} \mathcal{C}.$$

Lemma 2.4.2. *If*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\delta"] \arrow[d, "\alpha"'] & \mathcal{B} \arrow[d, "\beta"] \\ \mathcal{C} \arrow[r, "\gamma"'] & \mathcal{D} \end{tikzcd} \tag{14}$$

is a strictly commutative diagram of categories then:

1. *If β is an iso-fibration (i.e. the corresponding functor between the groupoids of isomorphisms is a fibration or, equivalently, an opfibration) then for (14) the two notions*

   2-Cartesian and Cartesian

   are equivalent.

2. *If α is an iso-fibration then (14) is Cartesian if and only if*

   $$\mathcal{A} \to \mathcal{B} \times_{\mathcal{D}} \mathcal{C} \tag{15}$$

   is fully-faithful and for any $b \in \mathcal{B}$ and $c \in \mathcal{C}$ with $\beta(b) = \gamma(c)$ there exists an $a \in \mathcal{A}$ with $\alpha(a) = c$ and an isomorphism $\kappa : \delta(a) \to b$ with $\beta(\kappa) = \mathrm{id}_{\beta(b)}$.

3. *If α and β are fibrations (resp. opfibrations) and δ maps Cartesian (resp. coCartesian) morphisms to Cartesian (resp. coCartesian) morphisms then (15) is fully-faithful if and only if δ induces an isomorphism*

   $$\mathrm{Hom}_{\mathcal{A},\mathrm{id}_c}(a, a') \cong \mathrm{Hom}_{\mathcal{B},\mathrm{id}_{\gamma(c)}}(\delta(a), \delta(a'))$$

   for all $c \in \mathcal{C}$ and $a, a' \in \mathcal{A}$ with $\alpha(a) = \alpha(a') = c$. In particular (14) is Cartesian, or equivalently 2-Cartesian, if and only if δ induces an equivalence of categories between the fibers

   $$\mathcal{A}_c \cong \mathcal{B}_{\gamma(c)}$$

   for all objects $c \in \mathcal{C}$.

Proof. 1. Indeed, if β is an iso-fibration, the obvious functor

$$\mathcal{B} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{B} \times_{/\mathcal{D}}^{\sim} \mathcal{C}$$

has a quasi-inverse functor which maps an object $(b, c, \nu : \beta(b) \to \gamma(c))$ to (b', c) for any choice of coCartesian morphism $b \to b'$ (necessarily an isomorphism as well) over ν.
2. Obviously if the condition is satisfied then the functor (15) is essentially surjective. If it is in turn essentially surjective, for any $b \in \mathcal{B}$ and $c \in \mathcal{C}$ with $\beta(b) = \gamma(c)$ there exists an $a' \in \mathcal{A}$, an isomorphism $\tau : \alpha(a') \to c$, and an isomorphism $\kappa' : \delta(a') \to b$ with

$\beta(\kappa') = \gamma(\tau)$. Now choose a coCartesian morphism $\xi : a' \to a$ in $\mathcal{A}$ lying over τ which exists by assumption. It is necessarily an isomorphism. Then we have $\alpha(a) = c$ and an isomorphism $\kappa := \kappa' \circ \delta(\xi^{-1})$ with $\beta(\kappa) = \mathrm{id}_{\beta(b)}$. Hence the statement of 2. is satisfied.
3. The only if part is clear. For the if part, let $f : c \to c'$ be a morphism in $\mathcal{C}$. We have to show that

$$\mathrm{Hom}_{\mathcal{A},f}(a, a'') \cong \mathrm{Hom}_{\mathcal{B},\gamma(f)}(\delta(a), \delta(a'')).$$

for any $a, a'' \in \mathcal{A}$ with $\alpha(a) = c, \alpha(a'') = c'$. Choose a Cartesian morphism $g : a' \to a''$ over f. Since δ maps g to a Cartesian morphism we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A},\mathrm{id}_c}(a, a') \arrow[r, "\delta"] \arrow[d, "g\circ"'] & \mathrm{Hom}_{\mathcal{A},\mathrm{id}_{\gamma(c)}}(\delta(a), \delta(a'))) \arrow[d, "\delta(g)\circ"] \\
\mathrm{Hom}_{\mathcal{A},f}(a, a'') \arrow[r, "\delta"'] & \mathrm{Hom}_{\mathcal{B},\gamma(f)}(\delta(a), \delta(a''))
\end{tikzcd}$$

in which the vertical maps are isomorphisms. Hence it suffices to see the assertion of 3. to show fully-faithfulness. If α, β are opfibrations one proceeds analogously choosing a coCartesian morphism. □

Definition 2.4.3. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories. A 1-morphism*

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

in $\mathcal{D}$ over a 1-morphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is called **coCartesian** *w.r.t. p, if for all i and objects $\mathcal{F}_1, \dots, \mathcal{F}_m, \mathcal{G} \in \mathcal{D}$ with $\mathcal{F}_i = \mathcal{F}$, lying over $T_1, \dots, T_m, U \in \mathcal{S}$ the diagram of categories*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) \arrow[r, "\circ_i \xi"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_m; U) \arrow[r, "\circ_i f"] & \mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_{i-1}, S_1, \dots, S_n, T_{i+1}, \dots, T_m; U)
\end{tikzcd}$$

is 2-Cartesian.
A 1-morphism

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

is called **weakly coCartesian** *w.r.t. p, if*

$$\mathrm{Hom}_{\mathrm{id}_T}(\mathcal{F}; \mathcal{G}) \xrightarrow{\circ\xi} \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{G})$$

is an equivalence of categories for all $\mathcal{G} \in \mathcal{D}$ with $p(\mathcal{G}) = T$.

If $p : \mathcal{D} \to \mathcal{S}$ is a 2-isofibration (cf. Definition 2.4.5) then a coCartesian 1-morphism is weakly coCartesian by the proof of Proposition 2.4.6 below.

Definition 2.4.4. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories. A 1-morphism*

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

in $\mathcal{D}$ over $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is called **Cartesian** *w.r.t. p and w.r.t. the i-th slot, if for all $\mathcal{G}_1, \dots, \mathcal{G}_m \in \mathcal{D}$ lying over $U_1, \dots, U_m \in \mathcal{S}$ the diagram of categories*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \dots, \mathcal{G}_m; \mathcal{E}_i) \arrow[r, "\xi \circ_i"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, \mathcal{G}_1, \dots, \mathcal{G}_m, \mathcal{E}_{i+1}, \dots, \mathcal{E}_n; \mathcal{F}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(U_1, \dots, U_m; S_i) \arrow[r, "f \circ_i"] & \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_{i-1}, U_1, \dots, U_m, S_{i+1}, \dots, S_n; T)
\end{tikzcd}$$

is 2-Cartesian.
A 1-morphism

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

is called **weakly Cartesian** *w.r.t. p and the w.r.t. i-th slot, if*

$$\mathrm{Hom}_{\mathrm{id}_{S_i}}(\mathcal{G}; \mathcal{E}_i) \xrightarrow{\xi \circ_i} \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{G}, \dots, \mathcal{E}_n; \mathcal{F})$$

is an equivalence of categories for all $\mathcal{G} \in \mathcal{D}$ with $p(\mathcal{G}) = S_i$.

Definition 2.4.5. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories.*

- *p is called a* **1-opfibration of 2-multicategories** *if for all 1-morphisms $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$ and all objects $\mathcal{E}_1, \dots, \mathcal{E}_n \in \mathcal{D}$ lying over $S_1, \dots, S_n \in \mathcal{S}$ there is an object $\mathcal{F} \in \mathcal{D}$ with $p(\mathcal{F}) = T$ and a coCartesian 1-morphism in $\mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$.*

- *p is called a* **2-opfibration of 2-multicategories** *if for $\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F} \in \mathcal{D}$ lying over $S_1, \dots, S_n; T \in \mathcal{S}$ the functors*

  $$\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$$

  are opfibrations, and the composition functors in $\mathcal{D}$ are morphisms of opfibrations, i.e. they map pairs of coCartesian 2-morphisms to coCartesian 2-morphisms.

- *p is called a* **1-fibration of 2-multicategories** *if for all 1-morphisms*

  $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$, for all $i = 1, \dots, n$ and for all objects $\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; \mathcal{F} \in \mathcal{D}$ lying over $S_1, \overset{\widehat{i}}{\dots}, S_n; T \in \mathcal{S}$ there is an object $\mathcal{E}_i \in \mathcal{D}$ with $p(\mathcal{E}_i) = S_i$ and a Cartesian 1-morphism w.r.t. the i-th slot in $\mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$.

- *p is called a* **2-fibration of 2-multicategories** *if for $\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F} \in \mathcal{D}$ lying over $S_1, \dots, S_n; T \in \mathcal{S}$ the functors*

  $$\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; Y)$$

  are fibrations, and the composition functors in $\mathcal{D}$ are morphisms of fibrations, i.e. they map pairs of Cartesian 2-morphisms to Cartesian 2-morphisms.

- *Similarly we define the notions of* **1-bifibration** *and* **2-bifibration**.
- *Let S be an object in $\mathcal{S}$. The 2-category consisting of those objects, (1-ary) 1-morphisms, and 2-morphisms which p maps to S, id_S and $\mathrm{id}_{\mathrm{id}_S}$ respectively is called the* **fiber** *$\mathcal{D}_S$ of p above S.*
- *We say that p has* **1-categorical fibers**, *if all fibers $\mathcal{D}_S$ are equivalent to 1-categories (this is also equivalent to all 2-morphism sets in the fibers being either empty or consisting of exactly one isomorphism).*
- *We say that p has* **discrete fibers**, *if all fibers $\mathcal{D}_S$ are equivalent to sets (this is also equivalent to all morphism categories in the fibers being either empty or equivalent to the terminal category).*
- *p is called a* **2-isofibration** *if p induces a 2-fibration (or equivalently a 2-opfibration) when restricted to the strict 2-functor*

  $$\mathcal{D}^{2-\sim} \to \mathcal{S}^{2-\sim}$$

  where the 2-morphisms sets are the subsets of 2-isomorphisms in $\mathcal{D}$ and $\mathcal{S}$, respectively.

  Obviously every 2-fibration (or 2-opfibration) is a 2-isofibration.

Note that p is a **2-isofibration** precisely if the restriction $\mathcal{D}^{2-\sim} \to \mathcal{S}^{2-\sim}$ is full on 2-morphisms, i.e. if 2-isomorphisms have a preimage under p.
For 2-isofibrations, by Lemma 2.4.2, we could have defined (co)Cartesian 1-morphisms equivalently using the notion of Cartesian diagram instead of 2-Cartesian diagram.

Proposition 2.4.6. *A 2-fibration or 2-opfibration of 2-multicategories $p : \mathcal{D} \to \mathcal{S}$ is a 1-fibration if and only if the following two conditions hold:*

1. *For all 1-morphisms $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$ and all $i = 1, \dots, n$ and all objects $\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; \mathcal{F} \in \mathcal{D}$ with $p(\mathcal{E}_k) = S_k$ and $p(\mathcal{F}) = T$ there is an object $\mathcal{E}_i$ with $p(\mathcal{E}_i) = S_i$ and a* weakly *Cartesian 1-morphism w.r.t. the i-th slot in $\mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$;*
2. *The composition of weakly Cartesian 1-morphisms is weakly Cartesian.*

A similar statement holds for 1-opfibrations where it is important that the Cartesian morphisms are composed w.r.t. the correct slot (otherwise see 2.4.7).

Proof. Let $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T_i)$, and let $\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_i)$ be a weakly coCartesian morphism with $p(\xi) = f$. We have to show that ξ is coCartesian.
By Lemma 2.4.2, 3., to prove that p is a 1-fibration, it suffices to show that

$$\mathrm{Hom}_g(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) \to \mathrm{Hom}_{g \circ_i f}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G})$$

is an equivalence of categories for all $g \in \mathrm{Hom}(T_1, \dots, T_m; U)$. Now choose another weakly coCartesian 1-morphism

$$\xi' \in \mathcal{F}_1, \dots, \mathcal{F}_m \to \mathcal{G}'$$

over g. We get the following sequence of functors

$$\mathrm{Hom}_{\mathrm{id}_U}(\mathcal{G}'; \mathcal{G}) \xrightarrow{\circ \xi'} \mathrm{Hom}_g(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G})$$

$$\xrightarrow{\circ \xi} \mathrm{Hom}_{g \circ f}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G}).$$

Since the composition $\xi' \circ \xi$ is also weakly coCartesian the left functor *and* the composition are equivalences of categories. Hence also the right functor is an equivalence.
To show the converse, we show that coCartesian morphisms are weakly coCartesian. The following Lemma states that, in general, coCartesian morphisms are stable under composition. Let $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$, and let $\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ be a coCartesian morphism with $p(\xi) = f$. In particular, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}; \mathcal{G}) & \xrightarrow{\circ_i \xi} & \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{S}}(U; U) & \xrightarrow{\circ_i f} & \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; U) \end{array}$$

is 2-Cartesian and hence (this uses that we have a 2-isofibration) satisfies the statement of Lemma 2.4.2, 2. which implies that

$$\mathrm{Hom}_{\mathrm{id}_U}(\mathcal{F}; \mathcal{G}) \to \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{G})$$

is an equivalence. □

Lemma 2.4.7. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor between 2-multicategories. Then the composition of (co)Cartesian 1-morphisms (resp. 2-morphisms) is (co)Cartesian. For* Cartesian *1-morphisms this holds true only if the slot used for the composition agrees with the slot at which the second morphism is Cartesian. Otherwise we have the following statement: If $\xi \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_i)$ is a* coCartesian *1-morphism and $\xi' \in \mathrm{Hom}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G})$ is a* Cartesian *1-morphism w.r.t. the j-th slot ($i \neq j$) then the composition*

$$\xi' \circ_j \xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G})$$

is Cartesian *w.r.t. the j-th slot if $j < i$ and w.r.t. the $j+n-1$-th slot if $j > i$. (This holds true in particular also in the case $n = 0$).*

Proof. The 1-categorical statement is well-known, hence the composition of (co)Cartesian 2-morphisms is (co)Cartesian. We now show that the composition of coCartesian 1-morphisms is coCartesian. Let $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n, T_i)$ and $f' \in \mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_m, U_j)$ be arbitrary 1-morphisms in $\mathcal{S}$, and let

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_i)$$

and

$$\xi' \in \mathrm{Hom}_{f'}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}_j)$$

be coCartesian morphisms. We want to show that their composition w.r.t. the i-th-slot

$$\xi' \circ_i \xi \in \mathrm{Hom}_{f' \circ_i f}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G}_j)$$

is Cartesian.
Let $\mathcal{G}_1, \overset{\widehat{j}}{\dots}, \mathcal{G}_k \in \mathcal{D}$ be objects lying over $U_1, \overset{\widehat{j}}{\dots}, U_k \in \mathcal{S}$, and let $\mathcal{H} \in \mathcal{D}$ an object over $V \in \mathcal{S}$ (all arbitrary). Consider the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}\begin{pmatrix} \mathcal{G}_1 & \\ \dots & ;\mathcal{H} \\ \mathcal{G}_k & \end{pmatrix} \arrow[r, "\circ_j \xi'"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}\begin{pmatrix} \mathcal{G}_1 & \\ \dots & \\ \mathcal{G}_{j-1} & \\ \mathcal{F}_1 & \\ \dots & ;\mathcal{H} \\ \mathcal{F}_m & \\ \mathcal{G}_{j+1} & \\ \dots & \\ \mathcal{G}_k & \end{pmatrix} \arrow[r, "\circ_i \xi"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}\begin{pmatrix} \mathcal{G}_1 & \\ \dots & \\ \mathcal{G}_{j-1} & \\ \mathcal{F}_1 & \\ \dots & \\ \mathcal{F}_{i-1} & \\ \mathcal{E}_1 & \\ \dots & ;\mathcal{H} \\ \mathcal{E}_n & \\ \mathcal{F}_{i+1} & \\ \dots & \\ \mathcal{F}_m & \\ \mathcal{G}_{j+1} & \\ \dots & \\ \mathcal{G}_k & \end{pmatrix} \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}\begin{pmatrix} U_1 & \\ \dots & ;V \\ U_k & \end{pmatrix} \arrow[r, "\circ_j f'"] & \mathrm{Hom}_{\mathcal{S}}\begin{pmatrix} U_1 & \\ \dots & \\ U_{j-1} & \\ T_1 & \\ \dots & ;V \\ T_m & \\ U_{j+1} & \\ \dots & \\ U_k & \end{pmatrix} \arrow[r, "\circ_i f"] & \mathrm{Hom}_{\mathcal{S}}\begin{pmatrix} U_1 & \\ \dots & \\ U_{j-1} & \\ T_1 & \\ \dots & \\ T_{i-1} & \\ S_1 & \\ \dots & ;V \\ S_n & \\ T_{i+1} & \\ \dots & \\ T_m & \\ U_{j+1} & \\ \dots & \\ U_k & \end{pmatrix}
\end{tikzcd}
$$

The right hand square is 2-Cartesian because ξ is coCartesian, and the left square is 2-Cartesian because ξ' is coCartesian. Hence also the composed square is 2-Cartesian, i.e. $\xi' \circ \xi$ is coCartesian as well.
The assertion about the composition of 1-Cartesian morphisms is proven in the same way.
For the additional statement, let $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n, T_i)$ and $f' \in \mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_m, U)$ be arbitrary 1-morphisms in $\mathcal{S}$, and let

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_i)$$

be coCartesian (here $n = 0$ is possible) and

$$\xi' \in \mathrm{Hom}_{f'}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G})$$

be Cartesian w.r.t. to the slot $j \neq i$. To fix notation assume $i < j$.
We want to show that their composition w.r.t. the i-th-slot

$$\xi' \circ_i \xi \in \mathrm{Hom}_{f' \circ_i f}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G})$$

is Cartesian w.r.t. to the slot $j + n - 1$.
Let $\mathcal{E}'_1, \dots, \mathcal{E}'_k \in \mathcal{D}$ be objects lying over $S'_1, \dots, S'_k \in \mathcal{S}$ (all arbitrary). Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}\left(\begin{matrix} \mathcal{E}'_1 \\ \dots \\ \mathcal{E}'_k \end{matrix} ; \mathcal{F}_j\right) \arrow[r, "\xi' \circ_j"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}\left(\begin{matrix} \mathcal{F}_1 \\ \dots \\ \mathcal{F}_{j-1} \\ \mathcal{E}'_1 \\ \dots \\ \mathcal{E}'_k \\ \mathcal{F}_{j+1} \\ \dots \\ \mathcal{F}_m \end{matrix} ; \mathcal{G}\right) \arrow[r, "\circ_i \xi"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}\left(\begin{matrix} \mathcal{F}_1 \\ \dots \\ \mathcal{F}_{i-1} \\ \mathcal{E}_1 \\ \dots \\ \mathcal{E}_n \\ \mathcal{F}_{i+1} \\ \dots \\ \mathcal{F}_{j-1} \\ \mathcal{E}'_1 \\ \dots \\ \mathcal{E}'_k \\ \mathcal{F}_{j+1} \\ \dots \\ \mathcal{F}_m \end{matrix} ; \mathcal{G}\right) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}\left(\begin{matrix} S'_1 \\ \dots \\ S'_k \end{matrix} ; T_j\right) \arrow[r, "f' \circ_j"] & \mathrm{Hom}_{\mathcal{S}}\left(\begin{matrix} T_1 \\ \dots \\ T_{j-1} \\ S'_1 \\ \dots \\ S'_k \\ T_{j+1} \\ \dots \\ T_m \end{matrix} ; U\right) \arrow[r, "\circ_i f"] & \mathrm{Hom}_{\mathcal{S}}\left(\begin{matrix} T_1 \\ \dots \\ T_{i-1} \\ S_1 \\ \dots \\ S_n \\ T_{i+1} \\ \dots \\ T_{j-1} \\ S'_1 \\ \dots \\ S'_k \\ T_{j+1} \\ \dots \\ T_m \end{matrix} ; U\right)
\end{tikzcd}$$

Now note that the composed functor

$$\rho \mapsto (\xi' \circ_j \rho) \circ_i \xi$$

is the same as

$$\rho \mapsto (\xi' \circ_i \xi) \circ_{j+n-1} \rho$$

because of the independence of slots (analogously for the bottom line functors). The right hand square is 2-Cartesian because ξ is coCartesian, and the left square is 2-Cartesian because ξ' is Cartesian w.r.t. the i-th slot. Hence also the composed square is 2-Cartesian, i.e. $\xi' \circ_j \xi$ is Cartesian w.r.t. the slot $i+n-1$ as well. □

2.4.8. Recall the definition of pseudo-functor between strict 2-categories, pseudo-natural transformations, and modifications (Definitions 2.2.3–2.2.5). Let F, G be pseudo-functors from a 2-category $\mathcal{D}$ to a 2-category $\mathcal{D}'$. A pseudo-natural transformation $\xi : F \to G$ is called an **equivalence** if there are a pseudo-natural transformation $\eta : G \to F$, and modifications (isomorphisms) $\xi \circ \eta \cong \mathrm{id}_G$, and $\eta \circ \xi \cong \mathrm{id}_F$.

Lemma 2.4.9. *A pseudo-natural transformation $\xi : F \to G$ is an equivalence if and only if for all $\mathcal{E} \in \mathcal{D}$*

$$\xi_{\mathcal{E}} : F(\mathcal{E}) \to G(\mathcal{E})$$

is an equivalence in the target-2-category $\mathcal{D}'$. In other words, choosing a point-wise inverse sets up automatically a pseudo-natural transformation as well, and the point-wise natural transformations between the compositions constitute the required modifications.

Proof. The "only if" implication is clear. For the "if" part choose a quasi-inverse $\xi'(\mathcal{E}) : G(\mathcal{E}) \to F(\mathcal{E})$ to $\xi(\mathcal{E}) : F(\mathcal{E}) \to G(\mathcal{E})$ for all objects $\mathcal{E} \in \mathcal{D}$. Hence, for all $\mathcal{E} \in \mathcal{D}$, we can find isomorphisms $\mathrm{id}_{G(\mathcal{E})} \Rightarrow \xi(\mathcal{E}) \circ \xi'(\mathcal{E})$ and $\xi'(\mathcal{E}) \circ \xi(\mathcal{E}) \Rightarrow \mathrm{id}_{F(\mathcal{E})}$ satisfying the unit-counit equations. Let $f : \mathcal{E} \to \mathcal{F}$ be a 1-morphism in $\mathcal{D}$. Define ξ'_f to be the following composition:

$$\xi'(\mathcal{F}) \circ G(f) \Rightarrow \xi'(\mathcal{F}) \circ G(f) \circ \xi(\mathcal{E}) \circ \xi'(\mathcal{E}) \Leftarrow \xi'(\mathcal{F}) \circ \xi(\mathcal{F}) \circ F(f) \circ \xi'(\mathcal{E}) \Rightarrow F(f) \circ \xi'(\mathcal{E}).$$

We leave to reader to check that this defines indeed a pseudo-natural transformation. □

Definition 2.4.10. *Recall that an object $\mathcal{E}$ in a strict 2-category defines a strict 2-functor*

$$\begin{array}{rcl} \mathrm{Hom}(\mathcal{E}, -) : \mathcal{D} & \to & \mathcal{CAT} \\ \mathcal{F} & \mapsto & \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \end{array}$$

A pseudo-functor from a 2-category $\mathcal{D}$

$$F : \mathcal{D} \to \mathcal{CAT}$$

is called **representable** *if there is an object $\mathcal{E}$ and a pseudo-natural transformation*

$$\nu : F \to \mathrm{Hom}(\mathcal{E}, -)$$

which is an equivalence, cf. 2.4.8.

Lemma 2.4.11. *An object $\mathcal{E}$ which represents a functor F is determined up to equivalence.*

Proof. We have to show that every pseudo-natural transformation

$$\xi : \mathrm{Hom}(\mathcal{E}, -) \to \mathrm{Hom}(\mathcal{E}', -)$$

which has an inverse up to modification, induces an equivalence $\mathcal{E} \to \mathcal{E}'$. Let η be the quasi-inverse of ξ. We have a 2-commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{E}, \mathcal{E}') \arrow[r, "\xi_{\mathcal{E}'}"] \arrow[d, "\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})\circ -"'] \arrow[dr, phantom, "\Downarrow\sim"] & \mathrm{Hom}(\mathcal{E}', \mathcal{E}') \arrow[d, "\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})\circ -"] \\
\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \arrow[r, "\xi_{\mathcal{E}}"'] & \mathrm{Hom}(\mathcal{E}', \mathcal{E})
\end{tikzcd}$$

by the definition of pseudo-natural transformation. Hence also a 2-commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{E}, \mathcal{E}') \arrow[d, "\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})\circ -"'] \arrow[dr, phantom, "\Downarrow\sim"] & \mathrm{Hom}(\mathcal{E}', \mathcal{E}') \arrow[l, "\eta_{\mathcal{E}'}"'] \arrow[d, "\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})\circ -"] \\
\mathrm{Hom}(\mathcal{E}, \mathcal{E}) & \mathrm{Hom}(\mathcal{E}', \mathcal{E}) \arrow[l, "\eta_{\mathcal{E}}"]
\end{tikzcd}$$

In particular, we get 2-isomorphisms

$$\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}}) \circ \eta_{\mathcal{E}'}(\mathrm{id}_{\mathcal{E}'}) \Rightarrow \eta_{\mathcal{E}}(\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})) \Rightarrow \mathrm{id}_{\mathcal{E}}$$

where the second one comes from the fact that η and ξ are inverse to each other up to 2-isomorphism. Similarly, there is a 2-isomorphism

$$\eta_{\mathcal{E}'}(\mathrm{id}_{\mathcal{E}'}) \circ \xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}}) \Rightarrow \mathrm{id}_{\mathcal{E}'}.$$

Hence we get the required equivalence

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "\eta_{\mathcal{E}'}(\mathrm{id}_{\mathcal{E}'})"] & \mathcal{E}' \arrow[l, shift left, "\xi_{\mathcal{E}}(\mathrm{id}_{\mathcal{E}})"]
\end{tikzcd}$$

□

The previous lemma shows that the following definition makes sense:

Definition 2.4.12. *1. Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories which is a 1-opfibration and 2-isofibration. The target object $\mathcal{F}$ of a coCartesian 1-morphism (cf. Definition 2.4.3) starting from $\mathcal{E}_1, \dots, \mathcal{E}_n$ and lying over a 1-multimorphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathcal{S}$ is denoted by $f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)$.*

2. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories which is a 1-fibration and 2-isofibration. The i-th source object $\mathcal{F}$ of a Cartesian 1-morphism w.r.t. to the i-th slot (cf. Definition 2.4.4) starting from $\mathcal{E}_1, \overset{\widehat{i}}{\ldots}, \mathcal{E}_n$ with target $\mathcal{F}$ and lying over a 1-multimorphism $f \in \mathrm{Hom}(S_1, \ldots, S_n; T)$ in $\mathcal{S}$ is denoted by $f^{\bullet,i}(\mathcal{E}_1, \overset{\widehat{i}}{\ldots}, \mathcal{E}_n; \mathcal{F})$.*

In both cases the objects are uniquely determined up to equivalence in $\mathcal{D}_T$.

Note that for two different objects $f_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ and $f_\circ(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ each representing the 2-functor

$$\mathcal{F} \mapsto \mathrm{Hom}_f(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F})$$

on the 2-category $\mathcal{D}_T$, we get an equivalence $f_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) \leftrightarrow f_\circ(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ by Lemma 2.4.11.

2.4.13. The 2-category $\mathcal{CAT}$ has a natural structure of a symmetric 2-multicategory setting

$$\mathrm{Hom}(\mathcal{C}_1, \ldots, \mathcal{C}_n; \mathcal{D}) := \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D}).$$

$\mathcal{CAT}$ is obviously opfibered over $\{\cdot\}$ with the monoidal product given by the product of categories and with the final category as neutral element.

Definition 2.4.14 (2-categorical Grothendieck construction)**.** *For a pseudo-functor of 2-multicategories*

$$\Xi : \mathcal{S} \to \mathcal{CAT}$$

where $\mathcal{CAT}$ is equipped with the structure of 2-multicategory of 2.4.13, we get a 2-multicategory $\int \Xi$ and a strict functor

$$\int \Xi \to \mathcal{S}$$

which is 1-opfibered and 2-fibered and whose fiber over $S \in \mathcal{S}$ is isomorphic to $\Xi(S)$ (hence it is a 1-category), as follows: The objects of $\int \Xi$ are pairs

$$(\mathcal{E}, S)$$

where S is an object of $\mathcal{S}$, and $\mathcal{E}$ is an object of $\Xi(S)$. The 1-morphisms in

$$\mathrm{Hom}_{\int \Xi}((\mathcal{E}_1, S_1), \ldots, (\mathcal{E}_n, S_n); (\mathcal{F}, T))$$

are pairs (α, f) where $f \in \mathrm{Hom}(S_1, \ldots, S_n; T)$ is a 1-morphism in $\mathcal{S}$ and $\alpha : \Xi(f)(\mathcal{E}_1, \ldots, \mathcal{E}_n) \to \mathcal{F}$ is a morphism in $\Xi(T)$. The 2-morphisms

$$\nu : (\alpha, f) \Rightarrow (\beta, g)$$

are those 2-morphisms $\nu : f \Rightarrow g$ such that $\beta \circ \Xi(\nu) = \alpha$.

Similarly there is a Grothendieck construction which starts from a pseudo-functor of 2-multicategories

$$\Xi : \mathcal{S}^{2-\mathrm{op}} \to \mathcal{CAT}$$

and produces a 1-opfibration and 2-opfibration.

2.4.15. There is also a Grothendieck construction which starts from a pseudo-functor of 2-categories (not 2-multicategories)

$$\Xi : \mathcal{S}^{1-\mathrm{op}} \to \mathcal{CAT}$$

and produces a 1-fibration and 2-opfibration $\nabla\Xi \to \mathcal{S}$, or from a pseudo-functor

$$\Xi : \mathcal{S}^{1-\mathrm{op},2-\mathrm{op}} \to \mathcal{CAT}$$

respectively, and produces a 1-fibration and 2-fibration $\nabla\Xi \to \mathcal{S}$. A 1-fibration of (2-)*multi*categories cannot be so easily described by a pseudo-functor because one gets several pullback functors depending on the slot (e.g. $\mathcal{HOM}_l, \mathcal{HOM}_r$).

Proposition 2.4.16. *For a strict functor between 2-multicategories $p : \mathcal{D} \to \mathcal{S}$ which is 1-opfibered and 2-fibered* with 1-categorical fibers*, we get an associated pseudo-functor of 2-multicategores:*

$$\begin{array}{rcl} \Xi_{\mathcal{D}} : \mathcal{S} & \to & \mathcal{CAT} \\ S & \mapsto & \mathcal{D}_S \end{array}$$

The construction is inverse (up to isomorphism of pseudo-functors, resp. 1-opfibrations/2-fibrations) to the one given in the previous definition.

An analogous proposition is true for 1-(op)fibrations and 2-(op)fibrations, with the restriction that for 1-fibrations the multi-aspect has to be neglected.

Proof (Sketch). The pseudo-functor $\Xi_{\mathcal{D}}$ maps a 1-morphism $f : S_1, \dots, S_n \to T$ to the functor (cf. Definition 2.4.12)

$$f_\bullet(-, \dots, -) : \mathcal{D}_{S_1} \times \cdots \times \mathcal{D}_{S_n} \to \mathcal{D}_T.$$

A 2-morphism $\nu : f \Rightarrow g$ is mapped to the following natural transformation between $f_\bullet(-, \dots, -)$ and $g_\bullet(-, \dots, -)$. With the definition (or characterization) of $f_\bullet(-, \dots, -)$ there comes a natural equivalence of *discrete* categories

$$\mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}_{\mathcal{D}_T}(f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n); \mathcal{F}). \tag{16}$$

Because p is 2-fibered and any 2-isomorphism is Cartesian, ν induces a well-defined isomorphism

$$\mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \cong \mathrm{Hom}_g(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}).$$

Since this is true for any $\mathcal{F}$, using the natural equivalences (16) for f and g, we get a morphism in $\mathcal{D}_T$

$$f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \to g_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n).$$

One checks that this defines a natural transformation and that the whole construction Ξ is indeed a pseudo-functor of 2-multicategories. □

Corollary 2.4.17. *The concept of functor between 1-multicategories $p : \mathcal{D} \to \{\cdot\}$ which are (1-)opfibered is equivalent to the concept of a monoidal category. The functor is, in addition, (1-)fibered if the corresponding monoidal category is closed.*

2.4.18. For a strict functor between 2-multicategories $p : \mathcal{D} \to \mathcal{S}$ which is 1-opfibered and 2-fibered but *with arbitrary 2-categorical fibers*, and every $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$ and $\mathcal{E}_1, \dots, \mathcal{E}_n$ we get still an object

$$f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)$$

which is well-defined up to equivalence. This defines a certain kind of pseudo-3-functor

$$\mathcal{S} \to 2 - \mathcal{CAT}.$$

Since this becomes confusing and we will not need it, we will not go into any details of this. For example, if $\mathcal{S} = \{\cdot\}$ then a 2-multicategory $\mathcal{D}$ which is 1-opfibered[14] over $\{\cdot\}$ is the same as a **monoidal 2-category** in the sense of [KV94, Shu10, Gur06, GPS95]. The (symmetric) prototype here is $\mathcal{CAT}$ with the structure of 2-multicategory considered above.

Example 2.4.19. *Let $\mathcal{S}$ be a usual category. Then $\mathcal{S}$ may be turned into a symmetric multicategory by setting*

$$\mathrm{Hom}(S_1, \dots, S_n; T) \coloneqq \mathrm{Hom}(S_1; T) \times \cdots \times \mathrm{Hom}(S_n; T).$$

If $\mathcal{S}$ has coproducts, then $\mathcal{S}$ (with this multicategory structure) is opfibered over $\{\cdot\}$. Let $p : \mathcal{D} \to \mathcal{S}$ be an opfibered (usual) category. Then a multicategory structure on $\mathcal{D}$ which turns p into an opfibration w.r.t. this multicategory structure on $\mathcal{S}$, is equivalent to a monoidal structure on the fibers of p such that the push-forwards $f_\bullet$ are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a covariant monoidal pseudo-functor *in [LH09, (3.6.7)].*

Example 2.4.20. *Let $\mathcal{S}$ be a usual category. Then $\mathcal{S}^{\mathrm{op}}$ may be turned into a symmetric multicategory (or equivalently $\mathcal{S}$ into a symmetric opmulticategory) by setting*

$$\mathrm{Hom}(S_1, \dots, S_n; T) \coloneqq \mathrm{Hom}(T; S_1) \times \cdots \times \mathrm{Hom}(T; S_n).$$

If $\mathcal{S}$ has products then $\mathcal{S}^{\mathrm{op}}$ (with this multicategory structure) is opfibered over $\{\cdot\}$. Let $p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be an opfibered (usual) category. Then a multicategory structure on $\mathcal{D}$ which turns p into an opfibration w.r.t. this multicategory structure on $\mathcal{S}^{\mathrm{op}}$, is equivalent to a monoidal structure on the fibers of p such that the pull-backs f^* (along morphisms in $\mathcal{S}$) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a contravariant monoidal pseudo-functor *in [LH09, (3.6.7)].*

[14] Note that everything is trivially 2-(op)fibered over $\{\cdot\}$

Lemma 2.4.21. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories. Any equivalence in $\mathcal{D}$ is a Cartesian and coCartesian 1-morphism.*

Proof. An equivalence $\mathcal{F} \to \mathcal{F}'$ has the property that the composition

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}')$$

is an equivalence of categories for all objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ of $\mathcal{D}$. We hence get a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}') \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T) \arrow[r] & \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T')
\end{tikzcd}$$

where the two horizontal morphisms are equivalences. It is automatically 2-Cartesian. □

Lemma 2.4.22. *Let $p : \mathcal{D} \to \mathcal{S}$ be a functor of 2-multicategories. If $\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ is a (co)Cartesian 1-morphism and $\alpha : \xi \Rightarrow \xi'$ is a 2-isomorphism in $\mathcal{D}$, then ξ' is (co)Cartesian as well.*

Proof. The 2-isomorphism α induces a natural isomorphism between the functor 'composition with ξ' and the functor 'composition with ξ''. And $p(\alpha)$ induces a natural isomorphism between the functor 'composition with $p(\xi)$' and the functor 'composition with $p(\xi')$'. Therefore the diagram expressing the coCartesianity of ξ is 2-Cartesian if and only if the corresponding diagram for ξ' is 2-Cartesian. □

2.4.23. Consider 2-multicategories $\mathcal{D}$, $\mathcal{S}$, $\mathcal{S}'$ and a diagram

$$\begin{tikzcd}
 & \mathcal{D} \arrow[d, "p"] \\
\mathcal{S}' \arrow[r, "F"'] & \mathcal{S}
\end{tikzcd}$$

where p is a strict 2-functor and F is a pseudo-functor. We define the **pull-back** of p along F as the following 2-multicategory $F^*\mathcal{D}$:

1. The objects of $F^*\mathcal{D}$ are pairs of objects $\mathcal{F} \in \mathcal{D}$ and $S \in \mathcal{S}'$ such that $p(\mathcal{F}) = F(S)$.
2. The 1-morphisms $(S_1, \mathcal{F}_1), \dots, (S_n, \mathcal{F}_n) \to (T, \mathcal{G})$ are pairs consisting of a 1-morphism $\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_n; \mathcal{G})$ and a 1-morphism $\beta \in \mathrm{Hom}_{\mathcal{S}'}(S_1, \dots, S_n; T)$ and a 2-isomorphism

$$\begin{tikzcd}
(p(\mathcal{F}_1), \dots, p(\mathcal{F}_n)) \arrow[r, "p(\alpha)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & p(\mathcal{G}) \arrow[d, equal] \\
(F(S_1), \dots, F(S_n)) \arrow[r, "F(\beta)"'] & F(T)
\end{tikzcd}$$

3. The 2-morphisms $(\alpha,\beta,\gamma) \Rightarrow (\alpha',\beta',\gamma')$ are 2-morphisms $\mu : \alpha \Rightarrow \alpha'$ and $\nu : \beta \Rightarrow \beta'$ such that $\gamma' p(\mu) = F(\nu)\gamma$.

4. Composition for the γ's is given by the following pasting (here depicted for 1-ary morphisms):

$$\begin{tikzcd} p(\mathcal{F}) \arrow[r, "p(\alpha_1)"] \arrow[rr, bend left, "p(\alpha_2\alpha_1)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma_1"] & p(\mathcal{F}') \arrow[r, "p(\alpha_2)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma_2"] & p(\mathcal{F}'') \arrow[d, equal] \\ F(S) \arrow[r, "F(\beta_1)"] \arrow[rr, bend right, "F(\beta_2\beta_1)"'] \arrow[rr, phantom, bend right=20, "\Downarrow F_{\beta_2,\beta_1}"] & F(S') \arrow[r, "F(\beta_2)"] & F(S'') \end{tikzcd}$$

Here F_{β_2,β_1} is the 2-isomorphism given by the pseudo-functoriality of F (cf. Definition 2.2.3). Associativity follows from the axioms of a pseudo-functor.

We get a commutative diagram of 2-multicategories in which the vertical 2-functors are strict:

$$\begin{tikzcd} F^*\mathcal{D} \arrow[r] \arrow[d, "F^*p"'] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{S}' \arrow[r, "F"'] & \mathcal{S} \end{tikzcd}$$

Proposition 2.4.24. *If p is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) then F^*p is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration).*

Proof. We show the proposition for 1-opfibrations and 2-opfibrations. The other assertions are shown similarly. Consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_{F^*\mathcal{D}}((S_1,\mathcal{F}_1),\dots,(S_m,\mathcal{F}_m);(T,\mathcal{G})) \arrow[r] \arrow[d, "F^*p"'] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1,\dots,\mathcal{F}_m;\mathcal{G}) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathcal{S}'}(S_1,\dots,S_m;T) \arrow[r, "F"] & \mathrm{Hom}_{\mathcal{S}}(F(S_1),\dots,F(S_m);F(T)) \end{tikzcd}$$

where $S_1,\dots,S_m,T$ are objects of $\mathcal{S}'$ and $\mathcal{F}_1,\dots,\mathcal{F}_m,\mathcal{G}$ are objects of $\mathcal{D}$ such that $F(S_i) = p(\mathcal{F}_i)$ and $F(T) = p(\mathcal{G})$. By definition of pull-back this diagram is 2-Cartesian.
Hence if p is an opfibration then so is F^*p. Furthermore a 2-morphism in $F^*\mathcal{D}$, i.e. a morphism in the category $\mathrm{Hom}_{F^*\mathcal{D}}((T_1,\mathcal{F}_1),\dots,(T_m,\mathcal{F}_m);(U,\mathcal{G}))$ is coCartesian for F^*p if and only if its image in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1,\dots,\mathcal{F}_m;\mathcal{G})$ is coCartesian for p.
Let $f \in \mathrm{Hom}(S_1,\dots,S_n;T_i)$ be a 1-morphism in $\mathcal{S}'$ and $\mathcal{E}_1,\dots,\mathcal{E}_n$ be objects of $\mathcal{D}$ such that $F(S_i) = p(\mathcal{E}_i)$. Choose a coCartesian 1-morphism $\xi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}_i)$ over $F(f)$ and consider the corresponding morphism

$$(\xi, f) \in \mathrm{Hom}_{F^*\mathcal{D}}((S_1,\mathcal{E}_1),\dots,(S_m,\mathcal{E}_m);(T_i,\mathcal{F}_i))$$

over f. We will show that the 1-morphism (ξ, f) is coCartesian for $F^*\mathcal{D} \to \mathcal{S}'$. Consider the following 2-commutative diagram of categories (we omitted the natural isomorphisms which occur in the left, right, bottom and top faces):

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{D}}\left(\begin{smallmatrix} \mathcal{F}_1 \\ \dots \\ \mathcal{F}_m \end{smallmatrix} ; \mathcal{G}\right) \arrow[rr, "\circ_i \xi"] \arrow[dd] & & \mathrm{Hom}_{\mathcal{D}}\left(\begin{smallmatrix} \mathcal{F}_1 \\ \dots \\ \mathcal{F}_{i-1} \\ \mathcal{E}_1 \\ \dots \\ \mathcal{E}_n \\ \mathcal{F}_{i+1} \\ \dots \\ \mathcal{F}_m \end{smallmatrix} ; \mathcal{G}\right) \arrow[dd] \\
\mathrm{Hom}_{F^*\mathcal{D}}\left(\begin{smallmatrix} (T_1, \mathcal{F}_1) \\ \dots \\ (T_m, \mathcal{F}_m) \end{smallmatrix} ; (U, \mathcal{G})\right) \arrow[ur] \arrow[rr, "\circ_i (\xi, f)"] \arrow[dd] & & \mathrm{Hom}_{F^*\mathcal{D}}\left(\begin{smallmatrix} (T_1, \mathcal{F}_1) \\ \dots \\ (T_{i-1}, \mathcal{F}_{i-1}) \\ (S_1, \mathcal{E}_1) \\ \dots \\ (S_n, \mathcal{E}_n) \\ (T_{i+1}, \mathcal{F}_{i+1}) \\ \dots \\ (T_m, \mathcal{F}_m) \end{smallmatrix} ; (U, \mathcal{G})\right) \arrow[ur] \arrow[dd] & \\
& \mathrm{Hom}_{\mathcal{S}}\left(\begin{smallmatrix} F(T_1) \\ \dots \\ F(T_m) \end{smallmatrix} ; F(U)\right) \arrow[rr, "\circ_i F(f)"] & & \mathrm{Hom}_{\mathcal{S}}\left(\begin{smallmatrix} F(T_1) \\ \dots \\ F(T_{i-1}) \\ F(S_1) \\ \dots \\ F(S_n) \\ F(T_{i+1}) \\ \dots \\ F(T_m) \end{smallmatrix} ; F(U)\right) \\
\mathrm{Hom}_{\mathcal{S}'}\left(\begin{smallmatrix} T_1 \\ \dots \\ T_m \end{smallmatrix} ; U\right) \arrow[ur] \arrow[rr, "\circ_i f"] & & \mathrm{Hom}_{\mathcal{S}'}\left(\begin{smallmatrix} T_1 \\ \dots \\ T_{i-1} \\ S_1 \\ \dots \\ S_n \\ T_{i+1} \\ \dots \\ T_m \end{smallmatrix} ; U\right) \arrow[ur] &
\end{tikzcd}$$

The back face of the cube is 2-Cartesian by the definition of coCartesian for ξ. The left and right face of the cube are 2-Cartesian by the definition of pull-back. Therefore also the front face is 2-Cartesian, and hence (ξ, f) is a Cartesian 1-morphism.
Furthermore, for the composition with any (not necessarily coCartesian) 1-morphism we may draw a similar diagram and have to show that if the top horizonal functor in the back face is a morphism of opfibrations then the front face is a morphism of opfibrations. This follows from the characterization of coCartesian 2-morphisms given in the beginning of the proof. □

Proposition 2.4.25. *If $p_1 : \mathcal{E} \to \mathcal{D}$ and $p_2 : \mathcal{D} \to \mathcal{S}$ are 1-fibrations (resp. 1-opfibrations, resp. 2-fibrations, resp. 2-opfibrations) of 2-multicategories then the composition $p_2 \circ p_1 : \mathcal{E} \to \mathcal{S}$ is a 1-fibration, (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) of 2-multicategories. An i-morphism ξ is (co)Cartesian w.r.t. $p_2 \circ p_1$ if and only if it is i-(co)Cartesian w.r.t. p_1 and $p_1(\xi)$ is i-(co)Cartesian w.r.t. p_2.*

Proof. Let $\xi \in \mathrm{Hom}_{\mathcal{E}}(\Sigma_1, \dots, \Sigma_n; \Xi_i)$ be a 1-morphism which is coCartesian for p_1 and such that $p_1(\xi)$ is coCartesian for p_2. Then we have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(\Xi_1, \dots, \Xi_m; \Pi) \arrow[r, "\circ_i \xi"] \arrow[d] & \mathrm{Hom}_{\mathcal{E}}(\Xi_1, \dots, \Xi_{i-1}, \Sigma_1, \dots, \Sigma_n, \Xi_{i+1}, \dots, \Xi_m; \Pi) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) \arrow[r, "\circ_i p_1(\xi)"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_{i+1}, \dots, \mathcal{F}_m; \mathcal{G}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_m; U) \arrow[r, "\circ_i p_2(p_1(\xi))"] & \mathrm{Hom}_{\mathcal{S}}(T_1, \dots, T_{i-1}, S_1, \dots, S_n, T_{i+1}, \dots, T_m; U)
\end{tikzcd} \tag{17}$$

in which both small squares commute and are 2-Cartesian. Hence also the composite square is 2-Cartesian, that is, ξ is coCartesian for $p_2 \circ p_1$.
Let $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$ be a 1-morphism and $\Sigma_1, \dots, \Sigma_n$ be objects of $\mathcal{E}$ over $S_1, \dots, S_n$. Choose a coCartesian 1-morphism $\mu \in \mathrm{Hom}_{\mathcal{D}}(p_1(\Sigma_1), \dots, p_1(\Sigma_n); \mathcal{F}_i)$ in $\mathcal{D}$ over f. Choose a coCartesian 1-morphism (for p_1) $\xi \in \mathrm{Hom}_{\mathcal{E}}(\Sigma_1, \dots, \Sigma_n; \Xi_i)$ over μ. We have seen before that ξ is coCartesian for $p_2 \circ p_1$ as well.
Let $\xi' \in \mathrm{Hom}_{\mathcal{E}}(\Sigma_1, \dots, \Sigma_n; \Xi_i')$ be a different coCartesian 1-morphism for $p_2 \circ p_1$ over f. We still have to prove the implication that ξ' is coCartesian for p_1 and that $p_1(\xi')$ is coCartesian for p_2.
By Lemma 2.4.11 there is an equivalence $\alpha : \Xi_i' \to \Xi_i$ such that ξ' is isomorphic to $\alpha \circ \xi$. Then $p_1(\xi')$ is isomorphic to $p_1(\alpha) \circ \mu$. The 1-morphism $\alpha \circ \xi$ is coCartesian for p_1, being a composition of coCartesian 1-morphisms for p_1 (cf. Lemma 2.4.7 and Lemma 2.4.21). Therefore, by Lemma 2.4.22, also ξ' is coCartesian for p_1, and hence $p_1(\alpha) \circ \mu$ is a composition of coCartesian morphisms for p_2. Therefore, by Lemma 2.4.22, also $p_1(\xi')$ is coCartesian for p_2. □

There is a certain converse to Proposition 2.4.25:

Proposition 2.4.26. *Let $p_1 : \mathcal{E} \to \mathcal{D}$ and $p_2 : \mathcal{D} \to \mathcal{S}$ be 2-isofibrations of 2-multicategories. Then $p_1 : \mathcal{E} \to \mathcal{D}$ is a 1-fibration (resp. 1-opfibration), if the following conditions hold:*

1. *$p_2 \circ p_1$ is a 1-fibration (resp. 1-opfibration);*
2. *p_1 maps (co)Cartesian 1-morphisms w.r.t. $p_2 \circ p_1$ to (co)Cartesian 1-morphisms w.r.t. p_2;*
3. *p_1 induces a 1-fibration (resp. 1-opfibration) between fibers[15] $\mathcal{E}_S \to \mathcal{D}_S$ for any $S \in \mathcal{S}$ and (co)Cartesianity of 1-morphisms in the fibers of p_1 is stable under pull-back (resp. push-forward) w.r.t. $p_2 \circ p_1$.*

   More precisely (here for the opfibered case, the other case is similar): For a morphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$, for objects $\mathcal{E}_i$ over S_i, and morphisms $\tau_i : \mathcal{E}_i \to \mathcal{F}_i$

[15] Note that these fibers are usual 2-categories, not 2-multicategories.

over id_{S_i}*, consider a diagram in* $\mathcal{D}$

$$\begin{tikzcd} \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[r, "(\tau_1, \dots, \tau_n)"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[d, "\xi'"] \\ \mathcal{G} \arrow[r] & \mathcal{H} \end{tikzcd}$$

over the diagram in $\mathcal{S}$

$$\begin{tikzcd} S_1, \dots, S_n \arrow[r, equal] \arrow[d, "f"'] & S_1, \dots, S_n \arrow[d, "f"] \\ T \arrow[r, equal] & T \end{tikzcd}$$

where ξ *and* ξ' *are coCartesian 1-morphisms (in particular the 1-morphism* $\mathcal{G} \to \mathcal{H}$ *is uniquely determined up to 2-isomorphism). Given a diagram in* $\mathcal{E}$

$$\begin{tikzcd} \Xi_1, \dots, \Xi_n \arrow[r, "(\mu_1, \dots, \mu_n)"] \arrow[d, "\kappa"'] \arrow[dr, phantom, "\swarrow\sim"] & \Phi_1, \dots, \Phi_n \arrow[d, "\kappa'"] \\ \Pi \arrow[r, "\nu"] & \Sigma \end{tikzcd}$$

over the other two, the following holds true: If κ *and* κ' *are coCartesian 1-morphisms w.r.t.* $p_2 \circ p_1$ *and if* $\mu_1, \dots, \mu_n$ *are coCartesian 1-morphisms w.r.t.* p_1 *(restriction to the respective fiber) then also* ν *is a coCartesian 1-morphism w.r.t.* p_1 *(restriction to the fiber over* T*).*

Proof. We have to show that coCartesian 1-morphisms w.r.t. p_1 exist. To ease notation we will neglect the multi-aspect.
Let $\tau : \mathcal{E} \to \mathcal{F}$ be a 1-morphism over $f : S \to T$ and let Ξ be an object over $\mathcal{E}$. Choose a coCartesian 1-morphism $\xi : \Xi \to \Xi'$ over f w.r.t. $p_2 \circ p_1$ which exists by property 1. By property 2. we have that $p_1(\xi) : \mathcal{E} \to \mathcal{E}'$ is a coCartesian 1-morphism over f w.r.t. p_2. We therefore have an induced 1-morphism $\widetilde{\tau} : \mathcal{E}' \to \mathcal{F}$ over id_T and a 2-isomorphism

$$\eta : \widetilde{\tau} \circ p_1(\xi) \Rightarrow \tau.$$

Now choose a coCartesian 1-morphism $\xi' : \Xi' \to \Xi''$ w.r.t. $p_{1,T} : \mathcal{E}_T \to \mathcal{D}_T$ over $\widetilde{\tau}$. We claim that

$$\eta_*(\xi' \circ \xi) : \Xi \to \Xi''$$

is a coCartesian 1-morphism over τ. Using Lemma 2.4.22 this is equivalent to $\xi' \circ \xi$ being a coCartesian 1-morphism over $\widetilde{\tau} \circ p_1(\xi)$. Using diagram (17) from the proof of the previous proposition we see that ξ is a coCartesian 1-morphism for p_1 as well. Since the composition of coCartesian 1-morphisms is coCartesian we are left to show that ξ' is coCartesian for p_1. Let $f : T \to U$ be a morphism in $\mathcal{S}$ and Σ an object over $\mathcal{G}$ over

U. We have to show that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E},f}(\Xi'',\Sigma) \arrow[r, "\circ\xi'"] \arrow[d] & \mathrm{Hom}_{\mathcal{E},f}(\Xi',\Sigma) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D},f}(\mathcal{F},\mathcal{G}) \arrow[r, "\circ\widetilde{\tau}"] & \mathrm{Hom}_{\mathcal{D},f}(\mathcal{E}',\mathcal{G})
\end{tikzcd} \tag{18}$$

is 2-Cartesian (or Cartesian, which amounts to the same). We can form a 2-commutative diagram

$$\begin{tikzcd}
\Xi' \arrow[r, "\xi'"] \arrow[d] \arrow[dr, phantom, "\swarrow\sim"] & \Xi'' \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=3ex, "\swarrow\sim"] & \\
\widetilde{\Xi}' \arrow[r, "\widetilde{\xi}'"] & \widetilde{\Xi}'' \arrow[r] & \Sigma
\end{tikzcd}$$

in which the vertical morphisms are coCartesian 1-morphisms w.r.t. $p_2 \circ p_1$ over f. The diagram (18) is point-wise equivalent to the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E},\mathrm{id}_U}(\widetilde{\Xi}'',\Sigma) \arrow[r, "\widetilde{\xi}'"] \arrow[d] & \mathrm{Hom}_{\mathcal{E},\mathrm{id}_U}(\widetilde{\Xi}',\Sigma) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D},\mathrm{id}_U}(p_1(\widetilde{\Xi}''),\mathcal{G}) \arrow[r, "\circ p_1(\widetilde{\xi}')"] & \mathrm{Hom}_{\mathcal{D},\mathrm{id}_U}(p(\widetilde{\Xi}'),\mathcal{G})
\end{tikzcd}$$

which is 2-Cartesian because $\widetilde{\xi}'$ is coCartesian w.r.t. $p_{1,U}$ by property 3. $\square$

3 Correspondences in a category and abstract six-functor-formalisms

3.1 Categories of multicorrespondences

Let $\mathcal{S}$ be a usual 1-category with fiber products and final object and assume that strictly associative fiber products have been chosen in $\mathcal{S}$.

Definition 3.1.1. *We define the* **2-multicategory $\mathcal{S}^{\mathrm{cor}}$ of correspondences in $\mathcal{S}$** *to be the following 2-multicategory.*

1. *The objects are the objects of $\mathcal{S}$.*

2. *The 1-morphisms* $\mathrm{Hom}(S_1, \dots, S_n; T)$ *are the (multi-)correspondences*[16]

$$\begin{tikzcd} & & & U \arrow[dlll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\ S_1 & \cdots & S_n & ; & T. \end{tikzcd}$$

3. *The 2-morphisms* $(U, \alpha_1, \dots, \alpha_n, \beta) \Rightarrow (U', \alpha_1', \dots, \alpha_n', \beta')$ *are the* iso*morphisms* $\gamma : U \to U'$ *such that in*

$$\begin{tikzcd} & & & U \arrow[dlll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] \arrow[dd, "\gamma"] & \\ S_1 & \cdots & S_n & & T \\ & & & U' \arrow[ulll, "\alpha_1'"] \arrow[ul, "\alpha_n'"'] \arrow[ur, "\beta'"'] & \end{tikzcd} \tag{19}$$

all triangles are commutative.

4. *The composition is given by the fiber product in the following way: the correspondence*

$$\begin{tikzcd} & & & & & U \times_{T_i} V \arrow[dll] \arrow[drr] & & & & & \\ & & & U \arrow[dlll] \arrow[dl] \arrow[drrr, "\beta_U"] & & & & V \arrow[dlll] \arrow[dl, "\alpha_{V,i}"'] \arrow[dr] \arrow[drrr] & & & \\ S_1 & \cdots & S_n & ; & T_1 & \cdots & T_i & \cdots & T_m & ; & W \end{tikzcd}$$

in $\mathrm{Hom}(T_1, \dots, T_{i-1}, S_1, \dots, S_n, T_{i+1}, \dots, T_m; W)$ *is the composition w.r.t. the i-th slot of the left correspondence in* $\mathrm{Hom}(S_1, \dots, S_n; T_i)$ *and the right correspondence in* $\mathrm{Hom}(T_1, \dots, T_m; W)$*.*

[16] as usual, $n = 0$ is allowed.

The 2-multicategory $\mathcal{S}^{\text{cor}}$ is symmetric, representable (i.e. opfibered over $\{\cdot\}$), closed (i.e. fibered over $\{\cdot\}$) and self-dual, with tensor product and internal hom *both* given by the product $\times$ in $\mathcal{S}$ and having as unit the final object of $\mathcal{S}$.

Definition 3.1.2. *We define also the larger category $\mathcal{S}^{\text{cor},G}$ where in addition every morphism $\gamma : U \to U'$, such that in (19) all triangles commute, is a 2-morphism (i.e. γ does not necessarily have to be an isomorphism).*

3.1.3. The previous definition can be generalized to the case of a general opmulticategory (2.2.2) $\mathcal{S}$ which has multipullbacks: Given a multimorphism $T \to S_1, \dots, S_n$ and a morphism $S_i' \to S_i$ for some $1 \le i \le n$, a **multipullback** is a universal square of the form

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & S_1, \dots, S_i', \dots, S_n \arrow[d] \\ T \arrow[r] & S_1, \dots, S_n. \end{tikzcd}$$

A usual category $\mathcal{S}$ becomes an opmulticategory setting

$$\operatorname{Hom}(T; S_1, \dots, S_n) := \operatorname{Hom}(T, S_1) \times \cdots \times \operatorname{Hom}(T, S_n). \tag{20}$$

In case that a usual category $\mathcal{S}$ has pullbacks it automatically has multipullbacks w.r.t. opmulticategory structure given by (20). Those are given by Cartesian squares

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & S_i' \arrow[d] \\ T \arrow[r] & S_i. \end{tikzcd}$$

For any opmulticategory $\mathcal{S}$ with multipullbacks we define $\mathcal{S}^{\text{cor}}$ to be the 2-category whose objects are the objects of $\mathcal{S}$, whose 1-morphisms are the multicorrespondences of the form

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ S_1, \dots, S_n & & T \end{tikzcd}$$

and whose 2-morphisms are commutative diagrams of multimorphisms

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] \arrow[dd] & \\ S_1, \dots, S_n & & T. \\ & U' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The composition is given by forming the multipullback. The reader may check that if the opmulticategory structure on $\mathcal{S}$ is given by (20) we reobtain the 2-multicategory $\mathcal{S}^{\text{cor}}$ defined in 3.1.1.

Definition 3.1.4. *Let $\mathcal{S}$ be a opmulticategory with multipullbacks. A* **(symmetric) six-functor-formalism** *on $\mathcal{S}$ is a 1-bifibered and 2-bifibered (symmetric) 2-multicategory with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}.$$

A **(symmetric) Grothendieck context** *on $\mathcal{S}$ is a 1-bifibered and 2-opfibered (symmetric) 2-multicategory with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor},G}.$$

A **(symmetric) Wirthmüller context** *on $\mathcal{S}$ is a 1-bifibered and 2-fibered (symmetric) 2-multicategory with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor},G}.$$

For an explanation of the terminology "Grothendieck" and "Wirthmüller" cf. Section 6.2.

3.1.5. If we are given a class of "proper" (resp. "etale") 1-ary morphisms $\mathcal{S}_0$ in $\mathcal{S}$, it is convenient to define $\mathcal{S}^{\mathrm{cor},0}$ to be the category where the morphisms $\gamma : U \to U'$ entering the definition of 2-morphism are the morphisms in $\mathcal{S}_0$. Then we would consider a 1-bifibration

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor},0}$$

which is a 2-opfibration in the proper case and a 2-fibration in the etale case. We call this respectively a **(symmetric) proper six-functor-formalism** and a **(symmetric) etale six-functor-formalism**.

3.2 Multicorrespondences and the six functors

3.2.1. We have a morphism of opfibered (over $\{\cdot\}$) symmetric multicategories $\mathcal{S}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{cor}}$. However, if $\mathcal{S}$ has the opmulticategory structure (20), i.e. if $\mathcal{S}^{\mathrm{cor}}$ is as defined in 3.1.1, there is no reasonable morphism of opfibered multicategories $\mathcal{S} \to \mathcal{S}^{\mathrm{cor}}$ where $\mathcal{S}$ is equipped with the symmetric multicategory structure as in 2.4.19[17]. This reflects the fact that, in the classical formulation of the six functors, there is no compatibility involving only '$\otimes$' and '!'. From a six-functor-formalism over $\mathcal{S}$ equipped with the opmulticategory structure (20) we get operations g_*, g^* as the pull-back and the push-forward along the correspondence

$$\begin{tikzcd} & S \arrow[dl, "g"'] \arrow[dr, equal] & \\ T & ; & S. \end{tikzcd}$$

[17]There is though a morphism of multicategories $\mathcal{S} \to \mathcal{S}^{\mathrm{cor}}$, where $\mathcal{S}$ is equipped with the multicategory structure $\mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T) := \mathrm{Hom}(S_1 \times \cdots \times S_n; T)$.

We get $f^!$ and $f_!$ as the pull-back and the push-forward along the correspondence

$$\begin{tikzcd}[column sep=small] & S \arrow[dl, equal] \arrow[dr, "f"] & \\ S & ; & T. \end{tikzcd}$$

We get the monoidal product $\mathcal{E} \otimes \mathcal{F}$ for objects $\mathcal{E}, \mathcal{F}$ above S as the target of any coCartesian morphism $\otimes$ over the correspondence

$$\xi_S = \left(\begin{tikzcd}[column sep=small] & & S \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ S & S & ; & S \end{tikzcd} \right).$$

Alternatively, we have

$$\mathcal{E} \otimes \mathcal{F} = \Delta^*(\mathcal{E} \boxtimes \mathcal{F})$$

where Δ^* is the push-forward along the correspondence

$$\left(\begin{tikzcd}[column sep=small] & S \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ S \times S & ; & S \end{tikzcd} \right)$$

induced by the canonical 1-morphism $\xi_S \in \mathrm{Hom}(S, S; S)$, and where $\boxtimes$ is the absolute monoidal product which exists because by Proposition 2.4.25 the composition $\mathcal{D} \to \{\cdot\}$ is opfibered as well, i.e. $\mathcal{D}$ is monoidal.

3.2.2. It is easy to derive from the definition of bifibered multicategory over $\mathcal{S}^{\mathrm{cor}}$ that the absolute monoidal product $\mathcal{E} \boxtimes \mathcal{F}$ can be reconstructed from the fiber-wise product as $\mathrm{pr}_1^* \mathcal{E} \otimes \mathrm{pr}_2^* \mathcal{F}$ on $S \times T$, whereas the absolute $\mathbf{HOM}(\mathcal{E}, \mathcal{F})$ is given by $\mathcal{HOM}(\mathrm{pr}_1^* \mathcal{E}, \mathrm{pr}_2^! \mathcal{F})$ on $S \times T$. In particular, for an object $\mathcal{E}$ of $\mathcal{D}$ lying over an object S in $\mathcal{S}$, we can define the absolute duality by $D\mathcal{E} := \mathbf{HOM}(\mathcal{E}, 1)$. It is then equal to $\mathcal{HOM}(\mathcal{E}, \pi^! 1)$ for $\pi : S \to \cdot$ being the final morphism. Here 1 is the unit object w.r.t. to the monodal structure on $\mathcal{D}_\bullet$, i.e. an object representing $\mathrm{Hom}_{\mathcal{D}_\bullet}(; -)$. The unit object 1 seen as an object in $\mathcal{D}$ is also the unit w.r.t. the *absolute* monoidal structure. We will discuss this more thoroughly in Section 6.1.

Proposition 3.2.3. *Given a six-functor-formalism on $\mathcal{S}$*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$$

where $\mathcal{S}$ is a usual category equipped with the opmulticategory structure (20) for the six functors as extracted in 3.2.1 there exist naturally the following compatibility isomophisms:

	isomorphisms between left adjoints	*isomorphisms between right adjoints*
$(*,*)$	$(fg)^* \xrightarrow{\sim} g^*f^*$	$(fg)_* \xrightarrow{\sim} f_*g_*$
$(!,!)$	$(fg)_! \xrightarrow{\sim} f_!g_!$	$(fg)^! \xrightarrow{\sim} g^!f^!$
$(!,*)$	$g^*f_! \xrightarrow{\sim} F_!G^*$	$G_*F^! \xrightarrow{\sim} f^!g_*$
$(\otimes,*)$	$f^*(-\otimes-) \xrightarrow{\sim} f^*-\otimes f^*-$	$f_*\mathcal{HOM}(f^*-,-) \xrightarrow{\sim} \mathcal{HOM}(-,f_*-)$
$(\otimes,!)$	$f_!(-\otimes f^*-) \xrightarrow{\sim} (f_!-)\otimes -$	$f_*\mathcal{HOM}(-,f^!-) \xrightarrow{\sim} \mathcal{HOM}(f_!-,-)$
		$f^!\mathcal{HOM}(-,-) \xrightarrow{\sim} \mathcal{HOM}(f^*-,f^!-)$
$(\otimes,\otimes)$	$(-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-)$	$\mathcal{HOM}(-\otimes-,-) \xrightarrow{\sim} \mathcal{HOM}(-,\mathcal{HOM}(-,-))$

Here f,g,F,G are morphisms in $\mathcal{S}$ which, in the $(!,*)$-row, are related by a Cartesian *diagram*

$$\begin{tikzcd} \cdot \arrow[r, "G"] \arrow[d, "F"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

Remark 3.2.4. *In the right column the corresponding adjoint natural transformations are listed. In each case the left hand side natural isomorphism determines the right hand side one and conversely. (In the $(\otimes,!)$-case there are 2 versions of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two.) The $(!,*)$-isomorphism (between left adjoints) is called* **base change***, the $(\otimes,!)$-isomorphism is called the* **projection formula***, and the $(*,\otimes)$-isomorphism is usually part of the definition of a* **monoidal functor***. The $(\otimes,\otimes)$-isomorphism is the associativity of the tensor product and usually part of the definition of a monoidal category. The $(*,*)$-isomorphism, and the $(!,!)$-isomorphism express that the corresponding functors arrange as a pseudo-functor with values in categories.*

Proof. The existence of all isomorphisms is a consequences of the fact that the composition of coCartesian morphisms is coCartesian. For example, the projection formula $(\otimes,!)$ is derived from the following composition in $\mathcal{S}^{\mathrm{cor}}$:

$$\left(\begin{tikzcd}[column sep=small] & & Y \arrow[dll, equal] \arrow[dl, equal] \arrow[drr, equal] & & \\ Y & Y & & ; & Y \end{tikzcd}\right) \circ_1 \left(\begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & ; & Y \end{tikzcd}\right) \cong \left(\begin{tikzcd}[column sep=small] & & X \arrow[dll, equal] \arrow[dl, "f"'] \arrow[drr, "f"] & & \\ X & Y & & ; & Y \end{tikzcd}\right),$$

where $\circ_1$ means that we compose w.r.t. the first slot.
The "monoidality of f^*" $(*,\otimes)$ is derived from the following composition in $\mathcal{S}^{\mathrm{cor}}$:

$$\left(\begin{tikzcd}[column sep=small] & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & ; & X \end{tikzcd}\right) \circ \left(\begin{tikzcd}[column sep=small] & & Y \arrow[dll, equal] \arrow[dl, equal] \arrow[drr, equal] & & \\ Y & Y & & ; & Y \end{tikzcd}\right) \cong \left(\begin{tikzcd}[column sep=small] & & X \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[drr, equal] & & \\ Y & Y & & ; & X \end{tikzcd}\right).$$

Base change $(!,*)$ is derived from:

$$\left(\begin{tikzcd}[column sep=small] & X \arrow[dl, "g"'] \arrow[dr, equal] & \\ A & ; & X \end{tikzcd}\right) \circ \left(\begin{tikzcd}[column sep=small] & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ Y & ; & A \end{tikzcd}\right) \cong \left(\begin{tikzcd}[column sep=small] & Y \times_A X \arrow[dl, "F"'] \arrow[dr, "G"] & \\ Y & ; & X \end{tikzcd}\right).$$

$\square$

Remark 3.2.5. *This raises the question about to what extent a converse of Proposition 3.2.3 holds true. In the literature a six-functor-formalism is often introduced merely as a collection of functors such that the isomorphisms of Proposition 3.2.3 exist, without specifying explicitly their compatibilities. In view of the theory developed in this section the question becomes: how can the 1- and 2-morphisms in the 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ be presented by generators and relations? We will not try to answer this question because all compatibilities, if needed, can be easily derived from the definition of $\mathcal{S}^{\mathrm{cor}}$. As an illustration, we prove that the diagram of isomorphisms*

$$\begin{tikzcd}
G_! F^*(A \otimes g^* B) \arrow[d, "(\otimes,*)"'] & & f^* g_!(A \otimes g^* B) \arrow[ll, "(!,*)"'] \arrow[d, "(\otimes,!)"] \\
G_!((F^* A) \otimes F^* g^* B) & & f^*(g_! A \otimes B) \arrow[ddd, "(\otimes,*)"] \\
G_!((F^* A) \otimes (gF)^*) \arrow[u, "(*,*)"] \arrow[d, "(*,*)"'] & & \\
G_!((F^* A) \otimes G^* f^* B) \arrow[dr, "(\otimes,!)"'] & & \\
 & (G_! F^* A) \otimes f^* B & (f^* g_! A) \otimes f^* B \arrow[l, "(*,!)"]
\end{tikzcd} \tag{21}$$

commutes. For this we only have to check that the two chains of obvious 2-isomorphisms in $\mathcal{S}^{\mathrm{cor}}$ given in Figure 5 and Figure 6 are equal.
To see this, observe that the multicorrespondences in the lines are all 2-isomorphic to the multicorrespondence

$$\begin{tikzcd} & & X \arrow[dll, "gF"'] \arrow[dl, "F"] \arrow[dr, "G"] & \\ W & Z & ; & Y. \end{tikzcd}$$

and that all the 2-isomorphisms in the chains (which induce the isomorphisms in Lemma 3.2.3 used in the diagram (21)) respect these 2-isomorphisms.
See 6.2.4 for a similar calculation involving also an (iso-)morphism $f_! \to f_*$, i.e. involving a proper six-functor-formalism.

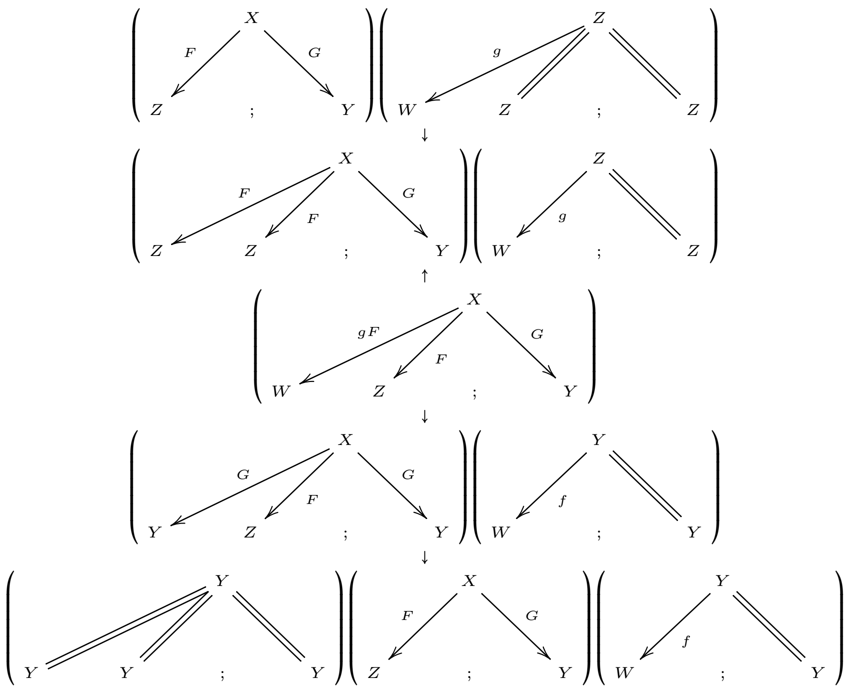


Figure 5: The first composition

3.3 Canonical Grothendieck contexts

3.3.1. Let $\mathcal{S}$ be a 1-opmulticategory with multipullbacks and let $p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be an ordinary bifibration of 1-multicategories. Let $\mathcal{S}_0$ be a subcategory of "proper" morphisms for which projection formula and base change formula hold true. This means that for every multipullback with $f_i \in \mathcal{S}_0$

$$\begin{tikzcd} T' \arrow[r, "G"] \arrow[d, "F_i"'] & S_1, \dots, S_i', \dots, S_n \arrow[d, "(\mathrm{id}_{S_1}, \dots, f_i, \dots, \mathrm{id}_{S_n})"] \\ T \arrow[r, "g"'] & S_1, \dots, S_i, \dots, S_n \end{tikzcd}$$

the canonical exchange natural transformation

$$g_\bullet \circ_i f_i^\bullet \to F_i^\bullet \circ G_\bullet \tag{22}$$

is an isomorphism. Note that the morphisms are morphisms in $\mathcal{S}$ (and not in $\mathcal{S}^{\mathrm{op}}$), e.g. $F_i^\bullet : \mathcal{D}_{T'} \to \mathcal{D}_T$ denotes a right-adjoint *push-forward* along the corresponding morphism in $\mathcal{S}$.

Assume that $\mathcal{S}_0$ is stable under multipullback, i.e. for any multipullback diagram as above, F_i is in $\mathcal{S}_0$ as well.

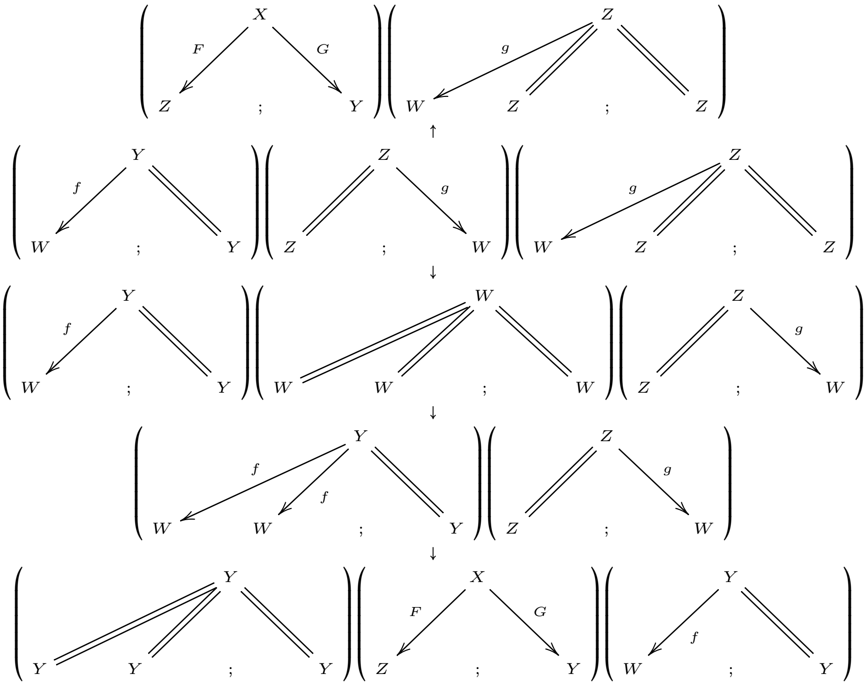


Figure 6: The second composition

Definition 3.3.2. *Define a category $\widetilde{\mathcal{D}}^{\mathrm{proper}}$ which has the same objects as $\mathcal{D}$ and whose 1-morphisms* $\mathrm{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F})$*, where $p(\mathcal{E}_i) = S_i$ and $p(\mathcal{F}) = T$, are the 1-morphisms*

$$\begin{tikzcd} & U \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (S_1, \ldots, S_n) & & T \end{tikzcd}$$

in $\mathcal{S}^{\mathrm{cor},0}$ (cf. 3.1.5) such that $f \in \mathcal{S}_0$*, together with a 1-morphism*

$$\rho \in \mathrm{Hom}_T(f^\bullet g_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n); \mathcal{F})$$

in $\mathcal{D}_T$. A 2-morphism $(U, g, f, \rho) \Rightarrow (U', g', f', \rho')$ is a morphism $h : U \to U'$ in $\mathcal{S}_0$ making the obvious diagrams commute and such that the diagram

$$\begin{tikzcd} (f')^\bullet g_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) \arrow[r, "\rho'"] \arrow[d, "\mathrm{unit}_h"'] & \mathcal{F} \\ (f')^\bullet h^\bullet h_\bullet g'_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) \arrow[r, "\sim"] & f^\bullet g_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) \arrow[u, "\rho"'] \end{tikzcd}$$

also commutes.

Proposition 3.3.3. *Definition 3.3.2 is reasonable, i.e. the composition induced by projection and base change formula, i.e. by the natural isomorphism (22), is associative. The obvious projection*

$$\widetilde{p} : \widetilde{\mathcal{D}}^{\mathrm{proper}} \to \mathcal{S}^{\mathrm{cor,proper},0}$$

where $\mathcal{S}^{\mathrm{cor,proper},0}$ is the subcategory of $\mathcal{S}^{\mathrm{cor},0}$ in whose multicorrespondences the morphism f is in $\mathcal{S}^0$, is a 1-opfibration and 2-opfibration of 2-multicategories with 1-categorical fibers.

Proof. This is a straight-forward check that we leave to the reader. For the second assertion note that the category $\widetilde{\mathcal{D}}^{\mathrm{proper}}$ is obviously 2-opfibered over $\mathcal{S}^{\mathrm{cor,proper},0}$, the 2-push-forward given by $\rho \mapsto \mathrm{unit}_h \circ \rho$. □

In particular, if (22) holds true for *all* multipullbacks in $\mathcal{S}$, and all $f^\bullet$ have right adjoints, we obtain the **canonical Grothendieck context** associated with $p : \mathcal{D} \to \mathcal{S}$:

$$\widetilde{p} : \widetilde{\mathcal{D}} \to \mathcal{S}^{\mathrm{cor},G}.$$

If (22) holds true only for a proper subclass of morphisms, it is possible under additional hypothesis to extend the so constructed partial six-functor-formalism to a 1-opfibration (which is still 2-opfibered with 1-categorical fibers) over the whole $\mathcal{S}^{\mathrm{cor},0}$:

$$\begin{tikzcd} \widetilde{\mathcal{D}}^{\mathrm{proper}} \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{D}} \arrow[d] \\ \mathcal{S}^{\mathrm{cor,proper},0} \arrow[r, hook] & \mathcal{S}^{\mathrm{cor},0} \end{tikzcd}$$

That is, if right adjoints exist, even to a six-functor-formalism. The right hand side 1-opfibration and 2-opfibration encodes also morphisms $f_! \to f_*$ for the corresponding operations and all their compatibilities (cf. 6.2.3). Its construction will be explained more generally in the derivator context in forthcoming work [Hör17a] and parallels the classical construction using compactifications.

4 Fibered multiderivators

4.1 Categories of diagrams

Definition 4.1.1. *A* **diagram category** *is a full sub-2-category* Dia ⊂ Cat*, satisfying the following axioms:*

(Dia1) *The empty category* $\varnothing$*, the final category* $\cdot$ *(or* Δ_0*), and* Δ_1 *are objects of* Dia.

(Dia2) Dia *is stable under taking finite coproducts and fibered products.*

(Dia3) *For each functor* $\alpha : I \to J$ *in* Dia *and object* $j \in J$ *the slice categories* $I \times_{/J} j$ *and* $j \times_{/J} I$ *are in* Dia.

A diagram category Dia *is called* **self-dual***, if it satisfies in addition:*

(Dia4) *If* $I \in$ Dia *then* $I^{\mathrm{op}} \in$ Dia.

A diagram category Dia *is called* **infinite***, if it satisfies in addition:*

(Dia5) Dia *is stable under taking arbitrary coproducts.*

In the following we mean by a **diagram** a small category.

Example 4.1.2. *We have the following diagram categories:*

Cat *the category of all* **diagrams***. It is self-dual.*

Inv *the category of* **inverse** *diagrams* C*, i.e. small categories* C *such that there exists a functor* $C \to \mathbb{N}_0$ *with the property that the preimage of an identity consists of identities*[18]*. An example is the injective simplex category* Δ°*:*

$$\cdots \;\begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix}\; \cdot \;\begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix}\; \cdot \;\begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix}\; \cdot$$

Dir *the category of* **directed** *diagrams* D*, i.e. small categories such that* D^{op} *is inverse. An example is the opposite of the injective simplex category* $(\Delta^{\circ})^{\mathrm{op}}$*:*

$$\cdots \;\begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix}\; \cdot \;\begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix}\; \cdot \;\begin{smallmatrix}\rightarrow\\ \rightarrow\end{smallmatrix}\; \cdot$$

Catf*,* Dirf*, and* Invf *are defined as before but consisting of* **finite diagrams***. Those are self-dual and* Dirf = Invf*.*

Catlf*,* Dirlf*, and* Invlf *are defined as before but consisting of* **locally finite diagrams***, i.e. those which have the property that a morphism* γ *factors as* $\gamma = \alpha \circ \beta$ *only in a finite number of ways.*

Pos*,* Posf*,* Dirpos*, and* Invpos*: the categories of* **posets, finite posets, directed posets,** *and* **inverse posets***.*

[18] In many sources $\mathbb{N}_0$ is replaced by any ordinal.

4.2 Pre-(multi)derivators

Definition 4.2.1. *A* **pre-derivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-\mathrm{op}} \to \mathcal{CAT}.$$

A **pre-multiderivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-\mathrm{op}} \to \mathcal{MCAT}$$

into the 2-"category" of multicategories. A morphism of pre-derivators is a natural transformation.
For a morphism $\alpha : I \to J$ in Dia *the corresponding functor*

$$\mathbb{D}(\alpha) : \mathbb{D}(J) \to \mathbb{D}(I)$$

will be denoted by α^*.
We call a pre-multiderivator **symmetric** *(resp.* **braided***), if its images are symmetric (resp. braided), and the morphisms α^* are compatible with the actions of the symmetric (resp. braid) groups.*

4.2.2. The pre-derivator associated with a (multi)category: Let $\mathcal{S}$ be a (multi)category. We associate with it the pre-derivator

$$\mathbb{S} : I \mapsto \mathrm{Fun}(I, \mathcal{S}).$$

The pull-back α^* is defined as composition with α. A 2-morphism $\kappa : \alpha \to \beta$ induces a natural 2-morphism $\mathbb{S}(\kappa) : \alpha^* \to \beta^*$.

4.2.3. The pre-derivator associated with a simplicial class (in particular with an ∞-category): Let $\mathcal{S}$ be a simplicial class, i.e. a functor

$$\mathcal{S} : \Delta \to \mathcal{CLASS}$$

into the "category" of classes. We associate with it the pre-derivator

$$\mathbb{S} : I \mapsto \mathrm{Ho}(\mathrm{Hom}(N(I), \mathcal{S})),$$

where $N(I)$ is the nerve of I and Ho is the left adjoint of N. In detail this means the objects of the category $\mathbb{S}(I)$ are morphisms $\alpha : N(I) \to \mathcal{S}$, the class of morphisms in $\mathbb{S}(I)$ is freely generated by morphisms $\mu : N(I \times \Delta_1) \to \mathcal{S}$ considered to be a morphism from its restriction to $N(I \times \{0\})$ to its restriction to $N(I \times \{1\})$ modulo the relations given by morphisms $\nu : N(I \times \Delta_2) \to \mathcal{S}$, i.e. if ν_1, ν_2 and ν_3 are the restrictions of ν to the 3 faces of Δ_2 then we have $\mu_3 = \mu_2 \circ \mu_1$. The pull-back α^* is defined as composition with the morphism $N(\alpha) : N(I) \to N(J)$. A 2-morphism $\kappa : \alpha \to \beta$ can be given as a functor $I \times \Delta_1 \to J$ which yields (applying N and composing) a natural transformation which we call $\mathbb{S}(\kappa)$.

4.2.4. The following will not be needed in the sequel. More generally, consider the full subcategory $m\Delta \subset \mathcal{MCAT}$ of all *finite connected* multicategories M that are freely generated by a finite set of multimorphisms $f_1, \dots, f_n$ such that each object of M occurs at most once as a source and at most once as the target of one of the f_i. Similarly consider the full subcategory $T \subset \mathcal{SMCAT}$ which is obtained from $m\Delta$ adding images under the operations of the symmetric groups. This category is usually called the symmetric tree category. With a functor

$$\mathcal{S} : m\Delta \to \mathcal{CLASS} \quad \text{resp.} \quad \mathcal{S} : T \to \mathcal{CLASS}$$

we associate the pre-multiderivator (resp. symmetric pre-multiderivator):

$$\mathbb{S} : I \mapsto \mathrm{Ho}(\mathrm{Hom}(N(I), \mathcal{S})),$$

where $N : \mathcal{MCAT} \to \mathcal{CLASS}^{m\Delta}$ (resp. $N : \mathcal{SMCAT} \to \mathcal{CLASS}^{T}$) is the nerve, I is considered to be a multicategory without any n-ary morphisms for $n \geq 2$, and Ho is the left adjoint of N. Objects in $\mathcal{SET}^T$ are called dendroidal sets in [MW07].

4.3 Fibered (multi)derivators

4.3.1. Let $p : \mathbb{D} \to \mathbb{S}$ be a strict morphism of pre-derivators with domain Dia, and let $\alpha : I \to J$ be a functor in Dia. Consider an object $S \in \mathbb{S}(J)$. The functor α^* induces a morphism between fibers (denoted the same way)

$$\alpha^* : \mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}.$$

We are interested in the case that the latter has a left adjoint $\alpha_!^S$, resp. a right adjoint α_*^S. These will be called ***relative*** **left/right homotopy Kan extension** functors with **base** S. For better readability we often omit the base from the notation. Though the base is not determined by the argument of $\alpha_!$, it will often be understood from the context, cf. also (4.3.28).

4.3.2. We are interested in the case in which all morphisms

$$p(I) : \mathbb{D}(I) \to \mathbb{S}(I)$$

are fibrations, resp. opfibrations (2.1) or, more generally, (op)fibrations of multicategories (cf. Definition 2.4.5 or Definition 1.5.8). We will later assume (cf. Axioms (FDer0 left/right)) that the functors $\alpha^* := \mathbb{D}(\alpha)$ map coCartesian morphisms to coCartesian morphisms but map Cartesian morphisms to Cartesian morphisms (for arity $n \geq 2$) only if α itself is a opfibration.
Then we will choose an associated pseudo-functor, i.e. for each $f : S \to T$ in $\mathbb{S}(I)$ a pair of adjoints functors

$$f_\bullet : \mathbb{D}(I)_S \to \mathbb{D}(I)_T,$$

resp.

$$f^\bullet : \mathbb{D}(I)_T \to \mathbb{D}(I)_S,$$

characterized by functorial isomorphisms:

$$\mathrm{Hom}_f(\mathcal{E},\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{id}_T}(\mathcal{E}, f^\bullet \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{id}_S}(f_\bullet \mathcal{E}, \mathcal{F}).$$

More generally, in the multicategorical setting, if f is a multimorphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ for some $n \geq 1$, we get an *adjunction of n variables*

$$f_\bullet : \mathbb{D}(I)_{S_1} \times \cdots \times \mathbb{D}(I)_{S_n} \to \mathbb{D}(I)_T,$$

and

$$f^{i,\bullet} : \mathbb{D}(I)_{S_1}^{\mathrm{op}} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(I)_{S_n}^{\mathrm{op}} \times \mathbb{D}(I)_T \to \mathbb{D}(I)_{S_i}.$$

4.3.3. For a diagram of categories

$$\begin{tikzcd} & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd}$$

the **slice category** $K \times_{/J} I$ is the category of triples (k, i, μ), where $k \in K$, $i \in I$ and $\mu : \beta(k) \to \alpha(i)$. It sits in a corresponding 2-commutative square:

$$\begin{tikzcd} K \times_{/J} I \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd}$$

which is universal w.r.t. such squares. This construction is associative, but of course not commutative unless J is a groupoid. The projection $K \times_{/J} I \to K$ is a fibration and the projection $K \times_{/J} I \to I$ is an opfibration (see 2.1). There is an adjunction

$$\begin{tikzcd} I \times_{/J} J \arrow[r, shift left] & I. \arrow[l, shift left] \end{tikzcd}$$

4.3.4. Consider an arbitrary 2-commutative square

$$\begin{tikzcd} L \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd} \tag{23}$$

and let $S \in \mathbb{S}(J)$ be an object and $\mathcal{E}$ a preimage in $\mathbb{D}(J)$ w.r.t. p. The 2-morphism (natural transformation) μ induces a functorial morphism

$$\mathbb{S}(\mu) : A^* \beta^* S \to B^* \alpha^* S$$

and therefore a functorial morphism

$$\mathbb{D}(\mu) : A^*\beta^*\mathcal{E} \to B^*\alpha^*\mathcal{E}$$

over $\mathbb{S}(\mu)$, or — if we are in the (op)fibered situation — equivalently

$$A^*\beta^*\mathcal{E} \to (\mathbb{S}(\mu))^\bullet B^*\alpha^*\mathcal{E}$$

respectively

$$(\mathbb{S}(\mu))_\bullet A^*\beta^*\mathcal{E} \to B^*\alpha^*\mathcal{E}$$

in the fiber above $A^*\beta^*S$, resp. $B^*\alpha^*S$,
Let now $\mathcal{F}$ be an object over α^*S. If relative right homotopy Kan extensions exist, we may form the following composition which will be called the (right) **base-change morphism**:

$$\beta^*\alpha_*\mathcal{F} \to A_*A^*\beta^*\alpha_*\mathcal{F} \to A_*(\mathbb{S}(\mu))^\bullet B^*\alpha^*\alpha_*\mathcal{F} \to A_*(\mathbb{S}(\mu))^\bullet B^*\mathcal{F}. \tag{24}$$

(We again omit the base S from the notation for better readability — it is always determined by the argument.)
Let now $\mathcal{F}$ be an object over β^*S. If relative left homotopy Kan extensions exist, we may form the composition, the (left) **base-change morphism**:

$$B_!(\mathbb{S}(\mu))_\bullet A^*\mathcal{F} \to B_!(\mathbb{S}(\mu))_\bullet A^*\beta^*\beta_!\mathcal{F} \to B_!B^*\alpha^*\beta_!\mathcal{F} \to \alpha^*\beta_!\mathcal{F}. \tag{25}$$

We will later say that the square (23) is **homotopy exact** if (24) is an isomorphism for all right fibered derivators (see Definition 4.3.6 below) and (25) is an isomorphism for all left fibered derivators. It is obvious a priori that for a left *and* right fibered derivator (24) is an isomorphism if and only if (25) is, one being the adjoint of the other (see [Gro13, §1.2] for analogous reasoning in the case of usual derivators).

Definition 4.3.5. *We consider the following axioms*[19] *on a pre-(multi)derivator $\mathbb{D}$:*

(Der1) *For I, J in* Dia*, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence of (multi-)categories. Moreover, $\mathbb{D}(\varnothing)$ is not empty.*

(Der2) *For I in* Dia *the 'underlying diagram' functor*

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Fun}(I, \mathbb{D}(\cdot))$$

is conservative.

In addition, we consider the following axioms for a strict morphism of pre-(multi)derivators

$$p : \mathbb{D} \to \mathbb{S} :$$

[19]The numbering is compatible with that of [Gro13] in the case of non-fibered derivators.

(FDer0 left) *For each I in* Dia *the morphism p specializes to an opfibered (multi)category and any functor $\alpha : I \to J$ in* Dia *induces a diagram*

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of opfibered (multi)categories, i.e. the top horizontal functor maps coCartesian morphisms to coCartesian morphisms.

(FDer3 left) *For each functor $\alpha : I \to J$ in* Dia *and $S \in \mathbb{S}(J)$ the functor α^* between fibers*

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!^S$.

(FDer4 left) *For each functor $\alpha : I \to J$ in* Dia, *and for any object $j \in J$, and the 2-cell*

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $\alpha_{j!}(\mathbb{S}(\mu))_\bullet \iota^* \to j^* \alpha_!$ is an isomorphism w.r.t. all bases $S \in \mathbb{S}(J)$.

(FDer5 left) *(if $\mathbb{S}$ is strong, only needed for the multiderivator case). For any opfibration $\alpha : I \to J$ in* Dia, *and for any morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors*

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-,\ -\ , \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -,\ \alpha_!-\ , -, \cdots, -)$$

are isomorphisms.

and their dual variants:

(FDer0 right) *For each I in* Dia *the morphism p specializes to a fibered (multi)category and any* opfibration *$\alpha : I \to J$ in* Dia *induces a diagram*

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of fibered (multi)categories, i.e. the top horizontal functor maps Cartesian morphisms w.r.t. the i-th slot to Cartesian morphisms w.r.t. the i-th slot.

(FDer3 right) *For each functor $\alpha : I \to J$ in* Dia *and $S \in \mathbb{S}(J)$ the functor α^* between fibers*

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a right-adjoint α_*^S.

(FDer4 right) *For each morphism $\alpha : I \to J$ in* Dia*, and for any object $j \in J$, and the 2-cell*

$$\begin{tikzcd} j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] \arrow[ur, phantom, "\nearrow\mu"] & J \end{tikzcd}$$

the induced natural transformation of functors $j^*\alpha_* \to \alpha_{j\,*}(\mathbb{S}(\mu))^\bullet\iota^*$ is an isomorphism w.r.t. all bases $S \in \mathbb{S}(J)$.

(FDer5 right) *(if $\mathbb{S}$ is strong, only needed for the multiderivator case). For any functor $\alpha : I \to J$ in* Dia*, and for any morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors*

$$\alpha_*(\alpha^*\xi)^{\bullet,i}(\alpha^*-, \cdots, \alpha^*- \;;\; -) \cong \xi^{\bullet,i}(-, \cdots, - \;;\; \alpha_*-)$$

are isomorphisms for all $1 \leq i \leq n$.

Definition 4.3.6. *A strict morphism of pre-(multi)derivators $p : \mathbb{D} \to \mathbb{S}$ with domain* Dia *is called a* **left fibered (multi)derivator** *with domain* Dia*, if axioms (Der1–2) hold for $\mathbb{D}$ and $\mathbb{S}$ and (FDer0–5 left) hold for p. Similarly it is called a* **right fibered (multi)derivator** *with domain* Dia*, if instead the corresponding dual axioms (FDer0–5 right) hold. It is called just* **fibered** *if it is both left and right fibered.*

The squares in axioms (FDer4 left/right) are in fact homotopy exact and it follows from the axioms (FDer4 left/right) that many more are (see 4.3.23).
There is some reduncancy in the axioms, cf. 4.3.8 and 4.3.27.

Question 4.3.7. *It seems natural to allow also (symmetric) multicategories, in particular operads, as* domain *for a fibered (symmetric) multiderivator. The author however did not succeed yet in writing down a neat generalization of (FDer3–4) which would encompass (FDer5).*

Lemma 4.3.8. *For a strict morphism of pre-derivators $\mathbb{D} \to \mathbb{S}$ such that both satisfy (Der1) and (Der2) and such that it induces a bifibration of multicategories $\mathbb{D}(I) \to \mathbb{S}(I)$ for all $I \in$ Dia we have the following implications:*

$$\text{(FDer0 left) for } n\text{-ary morphisms, } n \geq 1 \;\Leftrightarrow\; \text{(FDer5 right)} \tag{26}$$

$$\text{(FDer0 right)} \;\Leftrightarrow\; \text{(FDer5 left)} \tag{27}$$

Proof. We will only show the implication (26), the other being similar. Choosing pull-back functors $f^\bullet$, the remaining part of (FDer0) says that the natural 2-morphism

$$\begin{tikzcd} \mathbb{D}(J)_{S_1} \times \cdots \times \mathbb{D}(J)_{S_n} \arrow[rr, "f_\bullet"] \arrow[d, "\alpha^*"] & \arrow[d, phantom, "\swArrow"] & \mathbb{D}(J)_T \arrow[d, "\alpha^*"] \\ \mathbb{D}(I)_{\alpha^* S_1} \times \cdots \times \mathbb{D}(J)_{\alpha^* S_n} \arrow[rr, "(\alpha^* f)_\bullet"] & & \mathbb{D}(I)_{\alpha^* T} \end{tikzcd}$$

is an isomorphism. Taking the adjoint of this diagram (of $f_\bullet$ w.r.t. the i-th slot) we get the diagram

$$\begin{tikzcd} \mathbb{D}(I)^{\mathrm{op}}_{\alpha^* S_1} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(I)^{\mathrm{op}}_{\alpha^* S_n} & \times & \mathbb{D}(I)_{\alpha^* T} \arrow[rr, "(\alpha^* f)^{\bullet,i}"] \arrow[d, "\alpha_*"] & \arrow[d, phantom, "\swArrow"] & \mathbb{D}(J)_{\alpha^* S_i} \arrow[d, "\alpha_*"] \\ \mathbb{D}(I)^{\mathrm{op}}_{S_1} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(J)^{\mathrm{op}}_{S_n} \arrow[u, "(\alpha^*)^{\mathrm{op}}"] & \times & \mathbb{D}(J)_T \arrow[rr, "f^{\bullet,i}"] & & \mathbb{D}(J)_{S_i} \end{tikzcd}$$

That its 2-morphism is an isomorphism is the content of (FDer5 left). Hence (FDer0 left) and (FDer5 right) are equivalent in this situation.
For (27) note that for both (FDer0 right) and (FDer5 left), the functor α in question is restricted to the class of opfibrations. $\square$

Remark 4.3.9. *The axioms (FDer0) and (FDer3–5) are similar to the axioms of a six-functor-formalism (cf. Section 1.7). It is actually possible to make this analogy precise and* define *a fibered multiderivator as a bifibration of 2-multicategories* $[\mathbb{D}] \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *where* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *is a certain category of multicorrespondences of diagrams in* $\mathbb{S}$*, similar to our definition of a usual six-functor-formalism (cf. Definition 3.1.4). This also clarifies the existence and comparison of the internal and external monoidal structure, resp. duality, in a closed monoidal derivator (i.e. fibered multiderivator over* $\{\cdot\}$*) or more generally for any fibered multiderivator. We will explain this in detail in section 5.2.*

4.3.10. The pre-derivator associated with an ∞-category $\mathcal{S}$ is actually a left and right derivator (in the usual sense, i.e. fibered over $\{\cdot\}$) if $\mathcal{S}$ is complete and co-complete [GPS13]. This includes the case of pre-derivators associated with categories, which is, of course, classical — axiom (FDer4) expressing nothing else than Kan's formulas.

4.3.11. Let $S \in \mathbb{S}(\cdot)$ be an object and $p : \mathbb{D} \to \mathbb{S}$ be a (left, resp. right) fibered multiderivator. The association

$$I \mapsto \mathbb{D}(I)_{\pi^* S},$$

where $\pi : I \to \cdot$ is the projection, defines a (left, resp. right) derivator in the usual sense which we call its fiber $\mathbb{D}_S$ over S. The axioms (FDer6–7) stated below involve only these fibers.

Definition 4.3.12. *More generally, if $S \in \mathbb{S}(J)$ we may consider the association*

$$I \mapsto \mathbb{D}(I \times J)_{\mathrm{pr}_2^* S},$$

where $\mathrm{pr}_2 : I \times J \to J$ is the second projection. This defines again a (left, resp. right) derivator in the usual sense which we call its **fiber** *$\mathbb{D}_S$ over S.*

Lemma 4.3.13 (left). *Let $\mathbb{D} \to \mathbb{S}$ be a left fibered multiderivator and let $I \in \mathrm{Dia}$ be a diagram and $f \in \mathrm{Hom}_{\mathbb{S}(J)}(S_1, \dots, S_n; T)$ for some $n \geq 1$ be a morphism. Then the collection of functors for each $J \in \mathrm{Dia}$*

$$\begin{array}{rcl} f_\bullet : \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_1} \times \cdots \times \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_n} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} \\ \mathcal{E}_1, \dots, \mathcal{E}_n & \mapsto & (\mathrm{pr}_2^* f)_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \end{array}$$

defines a morphism of derivators $\mathbb{D}_{S_1} \times \cdots \times \mathbb{D}_{S_n} \to \mathbb{D}_T$. Furthermore, for a collection $\mathcal{E}_k \in \mathbb{D}(I), k \neq i$ the morphism of derivators:

$$\begin{array}{rcl} \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_i} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} \\ \mathcal{E}_i & \mapsto & (\mathrm{pr}_2^* f)_\bullet(\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \end{array}$$

is left continuous (i.e. commutes with left Kan extensions).

Proof. The only point which might not be clear is the left continuity of the bottom morphism of pre-derivators. Consider the following 2-commutative square, where $I, J, J' \in \mathrm{Dia}$, $\alpha : J \to J'$ is a functor, and $j' \in J'$

$$\begin{tikzcd} I \times (j' \times_{/J'} J) \arrow[r, "{(\mathrm{id},\iota)}"] \arrow[d, "{(\mathrm{id},p)}"'] \arrow[dr, phantom, "\Nearrow"] & I \times J \arrow[d, "{(\mathrm{id},\alpha)}"] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is homotopy exact by 4.3.23, 4. Therefore we have (using FDer3–5 left):

$$\begin{array}{rl} & (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_! (\mathrm{pr}_1^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_1^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{id}, \iota)^* (\mathrm{pr}_1^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_1^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{pr}_1^* f)_\bullet ((\mathrm{id}, \iota)^* \mathrm{pr}_1^* \mathcal{E}_1, \dots, (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, (\mathrm{id}, \iota)^* \mathrm{pr}_1^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{pr}_1^* f)_\bullet ((\mathrm{id}, p)^* \mathcal{E}_1, \dots, (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, (\mathrm{id}, p)^* \mathcal{E}_n) \\ \cong & f_\bullet (\mathcal{E}_1, \dots, (\mathrm{id}, p)_! (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, \mathcal{E}_n) \\ \cong & f_\bullet (\mathcal{E}_1, \dots, (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathcal{E}_n) \\ \cong & (\mathrm{id}, j')^* (\mathrm{pr}_1^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathrm{pr}_1^* \mathcal{E}_n) \end{array}$$

(Note that (id, p) is trivially an opfibration). A tedious check shows that the composition of these isomorphisms is $(\mathrm{id}, j')^*$ applied to the exchange morphism

$$(\mathrm{id}, \alpha)_! (\mathrm{pr}_1^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_1^* \mathcal{E}_n) \to (\mathrm{pr}_1^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathrm{pr}_1^* \mathcal{E}_n)$$

Since the above holds for any $j' \in J'$ the exchange morphism is therefore an isomorphism by (Der2). $\square$

In the right fibered situation the analogously defined morphisms $f^{i,\bullet}$ are not expected to be made into a morphism of fibers this way. For a discussion of how this is solved, we refer the reader to Section 5.4, where a fibered multiderivator is redefined as a certain type of six-functor-formalism. This will let appear the discussion and results of this section in a much more clear fashion. However, we have:

Lemma 4.3.14 (right). *Let $\mathbb{D} \to \mathbb{S}$ be a right fibered multiderivator and let $I \in \mathrm{Dia}$ be a diagram and $f \in \mathrm{Hom}_{\mathbb{S}(J)}(S_1, \dots, S_n; T)$, for some $n \geq 1$, be a morphism. For each $J \in \mathrm{Dia}$ and for each collection $\mathcal{E}_k \in \mathbb{D}(I)$, $k \neq i$, the association*

$$\begin{array}{rcl} \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_i} \\ \mathcal{F} & \mapsto & (\mathrm{pr}_2^* f)^{\bullet,i}(\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathrm{pr}_2^* \mathcal{E}_n; \mathcal{F}) \end{array}$$

defines a morphism of derivators which is right continuous (i.e. commutes with right Kan extensions). This is the right adjoint in the pre-derivator sense to the morphism of pre-derivators in the previous lemma, as soon as $\mathbb{D} \to \mathbb{S}$ is left and right fibered.

Proof. Consider the following 2-commutative square where $I, J, J' \in \mathrm{Dia}$, $\alpha : J \to J'$ is a functor, and $j' \in J'$

$$\begin{tikzcd} I \times (J \times_{/J'} j') \arrow[r, "(\mathrm{id},\iota)"] \arrow[d, "(\mathrm{id},p)"'] \arrow[dr, phantom, "\swarrow"] & I \times J \arrow[d, "(\mathrm{id},\alpha)"] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is homotopy exact by 4.3.23, 4.
Therefore we have (using FDer3–5 right):

$$\begin{array}{rl} & (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_! (\mathrm{pr}_1^* f)^{i,\bullet} (\mathrm{pr}_1^* \mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_1^* \mathcal{E}_n; \mathcal{F}) \\ \cong & (\mathrm{id}, p)_* (\mathrm{id}, \iota)^* (\mathrm{pr}_1^* f)^{i,\bullet} (\mathrm{pr}_1^* \mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_1^* \mathcal{E}_n, \mathcal{F}) \\ \cong & (\mathrm{id}, p)_* (\mathrm{pr}_1^* f)^{i,\bullet} ((\mathrm{id}, \iota)^* \mathrm{pr}_1^* \mathcal{E}_1, \overset{\widehat{i}}{\dots}, (\mathrm{id}, \iota)^* \mathrm{pr}_1^* \mathcal{E}_n; (\mathrm{id}, \iota)^* \mathcal{F}) \\ \cong & (\mathrm{id}, p)_* (\mathrm{pr}_1^* f)^{i,\bullet} ((\mathrm{id}, p)^* \mathcal{E}_1, \overset{\widehat{i}}{\dots}, (\mathrm{id}, p)^* \mathcal{E}_n; (\mathrm{id}, \iota)^* \mathcal{F}) \\ \cong & f^{i,\bullet} (\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; (\mathrm{id}, p)_* (\mathrm{id}, \iota)^* \mathcal{F}) \\ \cong & f^{i,\bullet} (\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_* \mathcal{F}) \\ \cong & (\mathrm{id}, j')^* (\mathrm{pr}_1^* f)^{i,\bullet} (\mathrm{pr}_1^* \mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_1^* \mathcal{E}_n; (\mathrm{id}, \alpha)_! \mathcal{F}) \end{array}$$

Note that (id, ι) is an opfibration, but (id, j') is not. Hence the last step has to be justified further. Consider the 2-commutative diagram:

$$\begin{tikzcd} I \times (J \times_{/J'} j') \arrow[r, "(\mathrm{id},\iota')"] \arrow[d, "(\mathrm{id},p)"'] \arrow[dr, phantom, "\swarrow"] & I \times J' \arrow[d, equal] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is again homotopy exact by 4.3.23, 4. Therefore we have

$$\begin{aligned}
&\cong f^{i,\bullet}(\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathcal{E}_n;(\mathrm{id},j')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong f^{i,\bullet}(\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathcal{E}_n;(\mathrm{id},p)_*(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{pr}_1^* f)^{i,\bullet}((\mathrm{id},p)^*\mathcal{E}_1,\overset{\widehat{i}}{\dots},(\mathrm{id},p)^*\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{pr}_1^* f)^{i,\bullet}((\mathrm{id},\iota')^*\,\mathrm{pr}_1^*\,\mathcal{E}_1,\overset{\widehat{i}}{\dots},(\mathrm{id},\iota')^*\,\mathrm{pr}_1^*\,\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{id},\iota')^*(\mathrm{pr}_1^* f)^{i,\bullet}(\mathrm{pr}_1^*\,\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},j')^*(\mathrm{pr}_1^* f)^{i,\bullet}(\mathrm{pr}_1^*\,\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathrm{pr}_1^*\,\mathcal{E}_n;(\mathrm{id},\alpha)_!\mathcal{F})
\end{aligned}$$

Note that (id,ι') is an opfibration as well. In other words: the reason why $f^{\bullet,i}$ also commutes with $(\mathrm{id},j')^*$ in this particular case is that the other argument are constant in the J direction.
A tedious check shows the composition of the isomorphisms of the previous computations yield $(\mathrm{id},j')^*$ applied to the exchange morphism

$$(\mathrm{id},\alpha)_*(\mathrm{pr}_1^* f)^{i,\bullet}(\mathrm{pr}_1^*\,\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathrm{pr}_1^*\,\mathcal{E}_n;\mathcal{F}) \to (\mathrm{pr}_1^* f)^{i,\bullet}(\mathcal{E}_1,\overset{\widehat{i}}{\dots},\mathcal{E}_n;(\mathrm{id},\alpha)_*\mathcal{F}).$$

Since the above holds for any $j' \in J'$ it is therefore an isomorphism by (Der2). □

Axiom (FDer5 left) and Corollary 2.4.17 imply the following:

Proposition 4.3.15. *The definition of a left fibered multiderivator $\mathbb{D} \to \{\cdot\}$ is equivalent to the definition of a monoidal left derivator in the sense of Groth [Gro12]. It is also, in addition, right fibered if and only if it is a right derivator and closed monoidal in the sense of [loc. cit.].*

4.3.16. Let $p : \mathbb{D} \to \mathbb{S}$ be a (left, resp. right) fibered *multi*-derivator and $S : \{\cdot\} \to \mathbb{S}(\cdot)$ a functor of multicategories. This is equivalent to the choice of an object $S \in \mathbb{S}(\cdot)$ and a collection of morphisms $\alpha_n \in \mathrm{Hom}_{\mathbb{S}(\cdot)}(\underbrace{S,\dots,S}_{n \text{ times}};S)$ for all $n \geq 2$, compatible with composition. Then the fiber

$$I \mapsto \mathbb{D}(I)_{p^*S}$$

defines even a (left, resp. right) monoidal derivator (i.e. a fibered multiderivator over $\{\cdot\}$). The same holds analogously for a functor of multicategories $S : \{\cdot\} \to \mathbb{S}(I)$.

Definition 4.3.17. *We call a pre-derivator $\mathbb{D}$* **strong**, *if the following axiom holds:*

(Der8) For any diagram K in Dia *the 'partial underlying diagram' functor*

$$\mathrm{dia} : \mathbb{D}(K \times \Delta_1) \to \mathrm{Fun}(\Delta_1, \mathbb{D}(K))$$

is full and essentially surjective.

Definition 4.3.18. *Let $p : \mathbb{D} \to \mathbb{S}$ be a fibered (left and right) derivator. We call $\mathbb{D}$* **pointed** *(relative to p) if the following axiom holds:*

(FDer6) *For any $S \in \mathbb{S}(\cdot)$, the category $\mathbb{D}(\cdot)_S$ has a zero object.*

Definition 4.3.19. *Let $p : \mathbb{D} \to \mathbb{S}$ be a fibered (left and right) derivator. We call $\mathbb{D}$* **stable** *(relative to p) if its fibers are* strong *and the following axiom holds:*

(FDer7) *For any $S \in \mathbb{S}(\cdot)$, an object in the category $\mathbb{D}(\square)_{p^*S}$ is homotopy Cartesian if and only if it is homotopy coCartesian.*

This condition can be weakened (cf. [GS12, Corollary 8.13]).

4.3.20. Recall from [Gro13] that axiom (FDer7) implies that the fibers of a stable fibered derivator are triangulated categories in a natural way. Actually the proof shows that it suffices to have a derivator of domain Posf (finite posets).
Since, by Lemma 4.3.13 and Lemma 4.3.14 push-forward, resp. pull-back w.r.t. any slot commute with homotopy colimits, resp. homotopy limits, they induce triangulated functors between the fibers.

4.3.21 (left)**.** The following is a consequence of (FDer0): For a functor $\alpha : I \to J$ and a morphism in $f : S \to T \in \mathbb{S}(J)$, we get a natural isomorphism

$$\mathbb{S}(\alpha^* f)_\bullet \alpha^* \to \alpha^* \mathbb{S}(f)_\bullet .$$

W.r.t. this natural isomorphism we have the following:

Lemma 4.3.22 (left)**.** *Given a "pasting" diagram*

$$\begin{tikzcd} N \arrow[r, "G"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\nu}"] & L \arrow[r, "B"] \arrow[d, "a"] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ M \arrow[r, "\gamma"] & K \arrow[r, "\beta"] & J \end{tikzcd}$$

we get for the pasted natural transformation $\nu \odot \mu := (\beta * \nu) \circ (\mu * G)$ that the following diagram is commutative:

$$\begin{tikzcd} A_! \mathbb{S}(\beta * \nu)_\bullet G^* \mathbb{S}(\mu)_\bullet B^* \arrow[r] \arrow[d, "\sim"] & \gamma^* a_! \mathbb{S}(\mu)_\bullet B^* \arrow[r] & \gamma^* \beta^* \alpha_! \\ A_! \mathbb{S}(\beta * \nu)_\bullet \mathbb{S}(G * \mu)_\bullet G^* B^* \arrow[d, "\sim"] & & \\ A_! \mathbb{S}(\nu \odot \mu)_\bullet G^* B^* \arrow[uurr] & & \end{tikzcd}$$

Here the morphisms going to the right are (induced by) the various base-change morphisms. In particular, the pasted square is homotopy exact if the individual two squares are.

Proof. This is an analogue of [Gro13, Lemma 1.17] and proven similarly. ☐

Proposition 4.3.23. *1. Any square of the form*

$$\begin{tikzcd} I \times_{/J} K \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

(where $I \times_{/J} K$ is the slice category) is homotopy exact (in particular the ones from axiom FDer4 left and FDer4 right are).

2. *A Cartesian square*

$$\begin{tikzcd} I \times_J K \arrow[r, "B"] \arrow[d, "A"'] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

(where $I \times_J K$ is the fiber product) is homotopy exact, if α is an opfibration or if β is a fibration.

3. *If $\alpha : I \to J$ is a morphism of opfibrations over a diagram E, then*

$$\begin{tikzcd} I_e \arrow[r, "w_I"] \arrow[d, "\alpha_e"'] & I \arrow[d, "\alpha"] \\ J_e \arrow[r, "w_J"] & J \end{tikzcd}$$

is homotopy exact for all objects $e \in E$.

4. *If a square*

$$\begin{tikzcd} L \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

is homotopy exact then so is the square

$$\begin{tikzcd} L \times X \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \times X \arrow[d, "\alpha"] \\ K \times X \arrow[r, "\beta"] & J \times X \end{tikzcd}$$

for any diagram X.

Proof. This proof is completely analogous to the non-fibered case. We sketch the arguments here (for the left-case only, the other case follows by logical duality):

3. Let j be an object in J_e and consider the cube:

$$\begin{tikzcd}
& I_e \times_{/J_e} j \arrow[rr, "\iota_e"] \arrow[dl, "w"'] \arrow[dd, "p_e"] & & I_e \arrow[dl, "w_I"'] \arrow[dd, "\alpha_e"] \\
I \times_{/J} j \arrow[rr, "\iota"] \arrow[dd, "p"] & & I \arrow[dd, "\alpha"] \arrow[r, phantom, "\swarrow\mu"] & \\
& \cdot \arrow[rr, "j"] \arrow[dl, equal] \arrow[d, phantom, "\swarrow\mu"] & & J_e \arrow[dl, "w_J"] \\
\cdot \arrow[rr, "j"] & & J &
\end{tikzcd} \tag{28}$$

where w is given by the inclusions $\iota_{I,e}$ resp. $\iota_{J,e}$. By standard arguments on homotopy exact squares it suffices to show that the left square is homotopy exact on constant diagrams, i.e. that

$$p_{e,!}w^* \cong p_!$$

holds true for all usual derivators. By [Gro13, Proposition 1.23] it suffices to show that w has a left adjoint.
Denote $\pi_I : I \to E$ and $\pi_J : J \to E$ the given opfibrations. Consider the two functors

$$\begin{tikzcd}
I_e \times_{/J,e} j \arrow[r, shift left, "w"] & I \times_{/J} j \arrow[l, shift left, "c"]
\end{tikzcd}$$

where c is given by mapping $(i, \mu : \alpha(i) \to j)$ to $(i', \mu' : \alpha(i') \to j)$ where we chose, for any i, a coCartesian morphism $\xi_{i,\mu} : i \to i'$ over $\pi_I(\mu) : \pi_I(i) \to e$. Since α maps coCartesian morphisms to coCartesian morphisms by assumption, $\alpha(\xi_{i,\mu}) : \alpha(i) \to \alpha(i')$ is coCartesian, and therefore there is a unique factorization

$$\alpha(i) \xrightarrow{\alpha(\xi_i)} \alpha(i') \xrightarrow{\mu'} j$$

of μ. A morphism $\alpha : (i_1, \mu_1 : \alpha(i_1) \to j) \to (i_2, \mu_2 : \alpha(i_2) \to j)$, by definition of coCartesian, gives rise to a unique morphism $\alpha' : i'_1 \to i'_2$ over $\pi_I(i_1) \to \pi_I(i_2)$ such that $\alpha'\xi_{i_1,\mu_1} = \xi_{i_2,\mu_2}\alpha'$ holds, and we set $c(\alpha) := \alpha'$. We have $c \circ w = \mathrm{id}$, and a morphism $\mathrm{id}_{I \times_{/J} j} \to w \circ c$ given by $(i, \mu) \mapsto \xi_{i,\mu}$. This makes w right adjoint to c.
2. By axiom (Der2) it suffices to show that for any object k of K, the induced morphism

$$k^* A_! B^* \to k^* \beta^* \alpha_!$$

is an isomorphism. Consider the following pasting diagram

$$\begin{tikzcd}
I \times_J k \arrow[r, "j"] \arrow[d, "\pi"] & I \times_J K \times_{/K} k \arrow[r, "\iota"] \arrow[d, "p"] \arrow[dr, phantom, "\swarrow\mu"] & I \times_J K \arrow[r, "B"] \arrow[d, "A"] & I \arrow[d, "\alpha"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, "k"] & K \arrow[r, "\beta"] & J
\end{tikzcd}$$

Lemma 4.3.22 shows that the following composition

$$\pi_! \mathbb{S}(\beta \star \mu \star j)_\bullet j^* \iota^* B^* \to \pi_! j^* \mathbb{S}(\beta \star \mu)_\bullet \iota^* B^* \to p_! \mathbb{S}(\beta \star \mu)_\bullet \iota^* B^* \to k^* A_! B^* \to k^* \beta^* \alpha_!$$

is the base-change associated with the pasting of the 3 squares in the diagram. All morphisms in this sequence are isomorphisms except possibly for the rightmost one. The second from the left is an isomorphism because j is a right adjoint [Gro13, Proposition 1.23]. The base-change morphism of the pasting is an isomorphism because of 3.
1. By axiom (Der2) it suffices to show that for any object k of K the induced morphism

$$k^* A_! \mathbb{S}(\mu)_\bullet B^* \to k^* \beta^* \alpha_!$$

is an isomorphism. Consider the following pasting diagram

$$\begin{tikzcd}
I \times_{/J} k \arrow[r, "\iota"] \arrow[d, "p"] & I \times_{/J} K \arrow[r, "B"] \arrow[d, "A"] \arrow[dr, phantom, "\swArrow \mu"] & I \arrow[d, "\alpha"] \\
\cdot \arrow[r, "k"] & K \arrow[r, "\beta"] & J
\end{tikzcd}$$

Lemma 4.3.22 shows that the following diagram is commutative

$$\begin{tikzcd}
p_! \mathbb{S}(\mu\iota)_\bullet \iota^* B^* \arrow[r, "\sim"] \arrow[d, "\text{can.}"', "\sim"] & k^* \beta^* \alpha_! \\
p_! \iota^* \mathbb{S}(\mu)_\bullet B^* \arrow[r, "\sim"] & k^* A_! \mathbb{S}(\mu)_\bullet B^* \arrow[u]
\end{tikzcd}$$

where the bottom horizontal morphism is an isomorphism by 2., and the top horizontal morphism is an isomorphism by (FDer4 left). Therefore the right vertical morphism is also an isomorphism.
4. (cf. also [Gro13, Theorem 1.30]). For any $x \in X$ consider the cube

$$\begin{tikzcd}
 & L \arrow[rr, "B"] \arrow[dl, "(\mathrm{id},x)"'] \arrow[dd, "A"] & & I \arrow[dl, "(\mathrm{id},x)"'] \arrow[dd, "\alpha"] \\
L \times X \arrow[rr, crossing over, "B"'] \arrow[dd, "A"'] & & I \times X \arrow[dr, phantom, "\swArrow \mu"] & \\
 & K \arrow[rr, "\beta"] \arrow[dl, "(\mathrm{id},x)"'] \arrow[dr, phantom, "\swArrow \mu"] & & J \arrow[dl, "(\mathrm{id},x)"] \\
K \times X \arrow[rr, "\beta"] & & J \times X \arrow[from=uu, crossing over, "\alpha"] &
\end{tikzcd} \tag{29}$$

The left and right hand side squares are homotopy exact because of 3., whereas the rear one is homotopy exact by assumption. Therefore the pasting

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "A"'] & I \times X \arrow[d, "\alpha"] \\
K \arrow[r] & J \times X
\end{tikzcd}$$

is homotopy exact. Therefore we have an isomorphism

$$(\mathrm{id}, x)^* A_! \mathbb{S}(\mu)_\bullet B^* \to (\mathrm{id}, x)^* \beta^* \alpha_!$$

where the morphism is induced by the base change of the given 2-commutative square. We may then conclude by axiom (Der2). $\square$

4.3.24 (left)**.** If $\mathbb{S}$ is strong the pull-backs and push-forwards along a morphism in $\mathbb{S}(\cdot)$, or more generally along a morphism in $\mathbb{S}(I)$, can be expressed using only the relative Kan-extension functors:
Let $p : \mathbb{D} \to \mathbb{S}$ be a left fibered derivator such that $\mathbb{S}$ is strong. Consider the 2-commutative square

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swArrow \mu"] & I \arrow[d, "\iota_0"] \\ I \arrow[r, "\iota_1"'] & I \times \Delta_1 \end{tikzcd}$$

and consider a morphism $f : S \to T$ in $\mathbb{S}(I)$. By the strongness of $\mathbb{S}$, the morphism f may be lifted to an object $F \in \mathbb{S}(I \times \Delta_1)$, and this means that the morphism

$$\mathbb{S}(\mu)_\bullet : \iota_0^* F \to \iota_1^* F$$

is isomorphic to f. Since the square is homotopy exact by Proposition 4.3.23 1., we get that the natural transformation

$$f_\bullet \to \iota_1^* \iota_{0,!}$$

is an isomorphism.

4.3.25 (left)**.** Let $\alpha : I \to J$ a functor in Dia and let $f : S \to T$ be a morphism in $\mathbb{S}(J)$. Axiom (FDer0) of a left fibered derivator implies that we have a canonical isomorphism

$$(\alpha^*(f))_\bullet \alpha^* \xrightarrow{\sim} \alpha^* f_\bullet$$

which is determined by the choice of the push-forward functors. We get an associated exchange morphism

$$\alpha_! (\alpha^*(f))_\bullet \to f_\bullet \alpha_!. \tag{30}$$

Proposition 4.3.26. *If $p : \mathbb{D} \to \mathbb{S}$ is a left fibered derivator, and $\mathbb{S}$ is strong, then the natural transformation (30) is an isomorphism. The corresponding dual statement holds for a right fibered derivator.*

Proof. Consider the following 2-commutative squares (the third and fourth are even commutative on the nose):

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swArrow \mu_I"] & I \arrow[d, "p_I"] \\ I \arrow[r, "\iota_I"'] & I \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} J \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swArrow \mu_J"] & J \arrow[d, "p_J"] \\ J \arrow[r, "\iota_J"'] & J \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "\iota_I"] & J \arrow[d, "\iota_J"] \\ I \times \Delta_1 \arrow[r, "\alpha"'] & J \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} I \arrow[r, "p_I"] \arrow[d, "\alpha"] & I \times \Delta_1 \arrow[d, "\alpha"] \\ J \arrow[r, "p_J"'] & J \times \Delta_1 \end{tikzcd} \tag{31}$$

They are all homotopy exact. Consider the diagram

$$\begin{tikzcd}
\alpha_!(\alpha^*(f))_\bullet \arrow[rr] \arrow[d] & & f_\bullet \alpha_! \arrow[d] \\
\alpha_! \iota_I^* p_{I,!} \arrow[r] & \iota_J^* \alpha_! p_{I,!} \arrow[r] & \iota_J^* p_{J,!} \alpha_!
\end{tikzcd}$$

where the vertical morphisms come from (4.3.24) — these are the base change morphism for the first and second square in (31) — and the lower horizontal morphisms are respectively the base change for the third diagram in (31), and the natural morphism associated with the commutativity of the fourth diagram in (31). Repeatedly appying Lemma 4.3.22 shows that this diagram is commutative. Therefore the upper horizontal morphism is an isomorphism because all the others in the diagram are. □

4.3.27. The last proposition states that push-forward commutes with homotopy colimits (left case) and pull-back commutes with homotopy limits (right case). This is also the content of (FDer5 left/right) for fibered derivators (not multiderivators), and hence this axiom is implied by the other axioms of left fibered derivators if $\mathbb{S}$ is strong. Even in the multi-case, by Lemma 4.3.8, axiom (FDer5 left/right) also follow from both (FDer0 left) *and* (FDer0 right).

4.3.28 (left)**.** Let $\alpha : I \to J$ be a functor in Dia. Proposition 4.3.26 (or FDer5 left) allows us to extend the functor $\alpha_!$ to a functor

$$\alpha_! : \mathbb{D}(I) \times_{\mathbb{S}(I)} \mathbb{S}(J) \to \mathbb{D}(J)$$

which is still left adjoint to α^*, more precisely: to $(\alpha^*, p(J))$. Here the fiber product is formed w.r.t. $p(I)$ and α^* respectively. We sketch its construction: $\alpha_!(\mathcal{E}, S)$ is given by $\alpha_!^S \mathcal{E}$, where $\alpha_!^S$ is the functor from axiom (FDer3 left) with base S. Let a pair of a morphism $f : S \to T$ in $\mathbb{S}(J)$ and $F : \mathcal{E} \to \mathcal{F}$ in $\mathbb{D}(I)$ over $\alpha^*(f)$ be given. We define $\alpha_!(F, f)$ as follows: F corresponds to a morphism

$$(\alpha^* f)_\bullet \mathcal{E} \to \mathcal{F}.$$

Applying $\alpha_!^T$ we get a morphism

$$\alpha_!^T (\alpha^* f)_\bullet \mathcal{E} \to \alpha_!^T \mathcal{F}$$

and composition with the inverse of the morphism (30) yields

$$f_\bullet \alpha_!^S \mathcal{E} \to \alpha_!^T \mathcal{F}$$

or, equivalently, a morphism which we define to be $\alpha_!(F, f)$

$$\alpha_!^S \mathcal{E} \to \alpha_!^T \mathcal{F}$$

over f.

For the adjunction, we have to give a functorial isomorphism

$$\mathrm{Hom}_{\alpha^* f}(\mathcal{E}, \alpha^* \mathcal{F}) \cong \mathrm{Hom}_f(\alpha_!(\mathcal{E}, S), \mathcal{F}),$$

where $\mathcal{E} \in \mathbb{D}(I)_{\alpha^* S}$ and $\mathcal{F} \in \mathbb{D}(J)_T$. We define it to be the following composition of isomorphisms:

$$\begin{aligned} & \mathrm{Hom}_{\alpha^* f}(\mathcal{E}, \alpha^* \mathcal{F}) \\ \cong\ & \mathrm{Hom}_{\mathrm{id}_{\alpha^* T}}((\alpha^* f)_\bullet \mathcal{E}, \alpha^* \mathcal{F}) \\ \cong\ & \mathrm{Hom}_{\mathrm{id}_T}(\alpha_!(\alpha^* f)_\bullet \mathcal{E}, \mathcal{F}) \\ \cong\ & \mathrm{Hom}_{\mathrm{id}_T}(f_\bullet \alpha_! \mathcal{E}, \mathcal{F}) \\ \cong\ & \mathrm{Hom}_f(\alpha_! \mathcal{E}, \mathcal{F}). \end{aligned}$$

A dual statement holds for a right fibered derivator and the functor α_*.

From Proposition 4.3.26 we also get a vertical version of Lemma 4.3.22:

Lemma 4.3.29 (left)**.** *Given a "pasting" diagram*

$$\begin{tikzcd} N \arrow[r, "B"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\swarrow^{\nu}"] & M \arrow[d, "\gamma"] \\ L \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

we get for the pasted natural transformation $\mu \odot \nu := (\mu * \Gamma) \circ (\alpha * \nu)$ that the following diagram is commutative:

$$\begin{tikzcd} a_! \mathbb{S}(\mu)_\bullet \Gamma_! \mathbb{S}(\alpha * \nu)_\bullet B^* \arrow[r] \arrow[d, "\sim"] & a_! \mathbb{S}(\mu)_\bullet b^* \gamma_! \arrow[r] & \beta^* \alpha_! \gamma_! \\ a_! \Gamma_! \mathbb{S}(\mu * \Gamma)_\bullet \mathbb{S}(\alpha * \nu)_\bullet B^* \arrow[d, "\sim"] & & \\ a_! \Gamma_! \mathbb{S}(\mu \odot \nu)_\bullet B^* \arrow[uurr] & & \end{tikzcd}$$

Here the morphisms going to the right are (induced by) the various base-change morphisms and the upper horizontal morphism is the isomorphism from Proposition 4.3.26. In particular, the pasted square is homotopy exact if the two individual squares are.

4.4 Transitivity

Proposition 4.4.1. *Let*

$$\mathbb{E} \xrightarrow{p_1} \mathbb{D} \xrightarrow{p_2} \mathbb{S}$$

be two left (resp. right) fibered multiderivators. Then also the composition $p_3 = p_2 \circ p_1 : \mathbb{E} \to \mathbb{S}$ is a left (resp. right) fibered multiderivator.

Proof. We will show the statement for left fibered multiderivators. The other statement follows by logical duality.
Axiom (FDer0): For any $I \in \mathrm{Dia}$, we have a sequence

$$\mathbb{E}(I) \to \mathbb{D}(I) \to \mathbb{S}(I)$$

of fibered multicategories. It is well-known that then also the composition $\mathbb{E}(I) \to \mathbb{S}(I)$ is a fibered multicategory (see 2.4). The other statement of (FDer0) is immediate as well.
Let $\alpha : I \to J$ be a functor as in axioms (FDer3 left) and (FDer4 left). We denote the relative homotopy Kan-extension functors w.r.t. the two fibered derivators by $\alpha_!^1$, and $\alpha_!^2$, respectively. As always, the base will be understood from the context or explicitly given as extra argument as in (4.3.28).
Axiom (FDer3 left): Let $S \in \mathbb{S}(J)$ be given. We define a functor

$$\alpha_!^3 : \mathbb{E}(I)_{\alpha^* S} \to \mathbb{E}(J)_S$$

in the fiber (under p_2) of $\mathcal{E} \in \mathbb{D}(I)_{\alpha^* S}$ as the composition

$$\mathbb{E}(I)_{\alpha^* S} \xrightarrow{(\nu_\bullet, \alpha_!^2 p_1)} \mathbb{E}(I)_{\alpha^* S} \times_{\mathbb{D}(I)_{\alpha^* S}} \mathbb{D}(J)_S \xrightarrow{\alpha_!^1} \mathbb{E}(J)_S$$

where ν is the unit

$$\nu : \mathcal{E} \to \alpha^* \alpha_!^2 \mathcal{E}$$

and $\alpha_!^1$ with two arguments is the extension given in (4.3.28).
Let $\mathcal{F}_1 \in \mathbb{E}(I)_{\alpha^* S}$ and $\mathcal{F}_2 \in \mathbb{E}(J)_S$ be given with images $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively under p_1. The adjunction is given by the following composition of isomorphisms:

$$\begin{array}{lll}
 & \mathrm{Hom}_S(\alpha_!^3 \mathcal{F}_1, \mathcal{F}_2) & \\
= & \mathrm{Hom}_S(\alpha_!^1(\nu_\bullet \mathcal{F}_1, \alpha_!^2 \mathcal{E}_1), \mathcal{F}_2) & \text{Definition} \\
= & \{f \in \mathrm{Hom}_S(\alpha_!^2 \mathcal{E}_1, \mathcal{E}_2); \xi \in \mathrm{Hom}_f(\alpha_!^1(\nu_\bullet \mathcal{F}_1, \alpha_!^2 \mathcal{E}_1), \mathcal{F}_2)\} & \text{Definition} \\
\cong & \{f \in \mathrm{Hom}_S(\alpha_!^2 \mathcal{E}_1, \mathcal{E}_2); \xi \in \mathrm{Hom}_{\alpha^* f}(\nu_\bullet \mathcal{F}_1, \alpha^* \mathcal{F}_2)\} & \text{Adjunction (4.3.28)} \\
\cong & \{\widetilde{f} \in \mathrm{Hom}_{\alpha^* S}(\mathcal{E}_1, \alpha^* \mathcal{E}_2); \xi \in \mathrm{Hom}_{\widetilde{f}}(\mathcal{F}_1, \alpha^* \mathcal{F}_2)\} & \text{Note below} \\
= & \mathrm{Hom}_{\alpha^* S}(\mathcal{F}_1, \alpha^* \mathcal{F}_2)\} & \text{Definition}
\end{array}$$

Note that the composition

$$\widetilde{f} : \mathcal{E}_1 \xrightarrow{\nu} \alpha^* \alpha_!^2 \mathcal{E}_1 \xrightarrow{\alpha^* f} \alpha^* \mathcal{E}_2$$

is determined by f via the adjunction of (FDer3 left) for base S and $p_2 : \mathbb{D} \to \mathbb{S}$.
Axiom (FDer4 left): Let $\mathcal{E}$ be in $\mathbb{E}(I)_{\alpha^* S}$ and let $\mathcal{F}$ be its image under p_1. We have to show that the natural morphism

$$\alpha_{j!}^3 \mathbb{S}(\mu)_\bullet^3 \iota^* \mathcal{E} \to j^* \alpha_!^3$$

is an isomorphism. Inserting the definition of the push-forwards, resp. of the Kan extensions for p_3, we get

$$\alpha_{j!}^1 (\nu_j)_\bullet^1 \mathrm{cart}_\bullet^1 \iota^* \mathcal{E} \to j^* \alpha_!^1 \nu_\bullet^1 \mathcal{E}.$$

Here $\nu_j : \mathbb{S}(\mu)^2_\bullet \iota^*\mathcal{F} \to \alpha_j^*\alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}$ is the unit and $\nu : \mathcal{F} \to \alpha^*\alpha^2_!\mathcal{F}$ is the unit. 'cart^1' is the Cartesian morphism $\iota^*\mathcal{F} \to \mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}$. Consider the base-change isomorphism (FDer4 for p_2)

$$\mathrm{bc} : \alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F} \to j^*\alpha^2_!\mathcal{F},$$

and the morphism

$$\mathbb{D}(\mu) : \iota^*\alpha^*\alpha^2_!\mathcal{F} \to \alpha_j^* j^*\alpha^2_!\mathcal{F}.$$

Claim: We have the equality

$$(\alpha_j^*\mathrm{bc}) \circ \nu_j \circ \mathrm{cart} = \mathbb{D}(\mu) \circ \iota^*(\nu).$$

Proof of the claim: Consider the diagram (which affects only the fibered derivator $p_2 : \mathbb{D} \to \mathbb{S}$, hence we omit superscripts):

$$\begin{tikzcd}
\alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\mathcal{F} \arrow[r, "\alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\nu"'] \arrow[rrr, bend left, "\alpha_j^*\mathrm{bc}"] & \alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\alpha^*\alpha_!\mathcal{F} \arrow[r] & \alpha_j^*\alpha_{j!}\alpha_j^* j^*\alpha_!\mathcal{F} \arrow[r] & \alpha_j^* j^*\alpha_!\mathcal{F} \\
\mathbb{S}(\mu)_\bullet\iota^*\mathcal{F} \arrow[u, "\nu_j"] \arrow[r, "\mathbb{S}(\mu)_\bullet\iota^*\nu"'] & \mathbb{S}(\mu)_\bullet\iota^*\alpha^*\alpha_!\mathcal{F} \arrow[u] \arrow[r, "\text{induced"] & \alpha_j^* j^*\alpha_!\mathcal{F} \arrow[u] \arrow[ur, equal] & \\
\iota^*\mathcal{F} \arrow[u, "\mathrm{cart}"] \arrow[r, "\iota^*\nu"'] & \iota^*\alpha^*\alpha_!\mathcal{F} \arrow[u, "\mathrm{cart}"] \arrow[ur, "\mathbb{D}(\mu)"'] & &
\end{tikzcd}$$

Clearly all squares and triangles in this diagram are commutative. The two given morphisms are the compositions of the extremal paths hence they are equal.
We have a natural isomorphism induced by bc:

$$\alpha^1_{j!}(\cdots, \alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}) \cong \alpha^1_{j!}((\alpha_j^*\mathrm{bc})_\bullet(\cdots), j^*\alpha^2_!\mathcal{F})$$

(this is true for any isomorphism).
We therefore have

$$\begin{aligned}
& \alpha^1_{j!}(\nu_j)^1_\bullet \mathrm{cart}^1_\bullet \iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}(\alpha_j^*\mathrm{bc})^1_\bullet(\nu_j)^1_\bullet \mathrm{cart}^1_\bullet\iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet(\iota^*\nu)^1_\bullet\iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet\iota^*\nu^1_\bullet\mathcal{E}
\end{aligned}$$

Thus we are left to show that

$$\alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet\iota^*\nu^1_\bullet\mathcal{E} \to j^*\alpha^1_!\nu^1_\bullet\mathcal{E}$$

is an isomorphism. A tedious check shows that this *is* the base change morphism associated with p_1. It is an isomorphism by (FDer4 left) for p_1. $\square$

4.5 (Co)local morphisms

4.5.1. Let Dia be a diagram category and let $\mathbb{S}$ be a *strong* right derivator with domain Dia. Strongness implies that for each diagram

$$\begin{tikzcd} & U \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

in $\mathbb{S}(\cdot)$ there exists a homotopy pull-back "$U \times_T S$" which is well-defined up to (non-unique!) isomorphism. A Grothendieck pre-topology on $\mathbb{S}$ is basically a Grothendieck pre-topology in the usual sense on $\mathbb{S}(\cdot)$ except that pull-backs are replaced by homotopy pull-backs. We state the precise definition:

Definition 4.5.2. *A* **Grothendieck pre-topology** *on $\mathbb{S}$ is the datum consisting of, for any $S \in \mathbb{S}(\cdot)$, a collection of families $\{U_i \to S\}_{i \in \mathcal{I}}$ of morphisms in $\mathbb{S}(\cdot)$ called* **covers**, *such that*

1. *Every family consisting of isomorphisms is a cover,*
2. *If $\{U_i \to S\}_{i\in\mathcal{I}}$ is a cover and $T \to S$ is any morphism then the family $\{$"$U_i \times_S T$" $\to T\}_{i\in\mathcal{I}}$ is a cover for any choice of particular members of the family $\{$"$U_i \times_S T$"$\}$.*
3. *If $\{U_i \to S\}_{i\in\mathcal{I}}$ is a cover and for each i, the family $\{U_{i,j} \to U_i\}_{j\in\mathcal{J}_i}$ is a cover then the family of compositions $\{U_{i,j} \to U_i \to S\}_{i\in\mathcal{I}, j\in\mathcal{J}_i}$ is a cover.*

Definition 4.5.3 (left)**.** *Let $p : \mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right). Assume that pull-backs exist in $\mathbb{S}$. We say that a morphism $f : U \to X$ in $\mathbb{S}(\cdot)$ is* **$\mathbb{D}$-local** *if*

(Dloc1 left) The morphism f satisfies **base change***: for any diagram $Q \in \mathbb{D}(\Box)$ with underlying diagram*

$$\begin{tikzcd} A \arrow[r, "\widetilde{F}"] \arrow[d, "\widetilde{G}"'] & B \arrow[d, "\widetilde{g}"] \\ C \arrow[r, "\widetilde{f}"'] & D \end{tikzcd}$$

such that $p(Q)$ in $\mathbb{S}(\Box)$ is a pull-back-diagram, i.e. is (homotopy) Cartesian, the following holds true: If $\widetilde{F}$ and $\widetilde{f}$ are Cartesian, and $\widetilde{g}$ is coCartesian then also $\widetilde{G}$ is coCartesian.[20]

[20]In other words, if

$$\begin{tikzcd} \text{“}U \times_X Y\text{”} \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ U \arrow[r, "f"'] & X \end{tikzcd}$$

(Dloc2 left) *The morphism of derivators (cf. Lemma 4.3.14)*

$$f^\bullet : \mathbb{D}_X \to \mathbb{D}_U$$

commutes with homotopy colimits.

A morphism $f : U \to X$ in $\mathbb{S}(\cdot)$ is called **universally $\mathbb{D}$-local** *if any homotopy pull-back of f is $\mathbb{D}$-local.*

Definition 4.5.4 (left)**.** *Assume that $\mathbb{S}$ is equipped with a Grothendieck pre-topology (cf. 4.5.2). A left fibered derivator $p : \mathbb{D} \to \mathbb{S}$ as in Definition 4.5.3 is called* **local** *w.r.t. the pre-topology on $\mathbb{S}$, if the following conditions hold:*

1. *Every morphism $U_i \to S$ which is part of a cover is $\mathbb{D}$-local.*
2. *For a cover $\{f_i : U_i \to S\}$ the family*

$$f_i^\bullet : \mathbb{D}(S) \to \mathbb{D}(U_i)$$

is jointly conservative.

Definition 4.5.5 (right)**.** *Let $p : \mathbb{D} \to \mathbb{S}$ be a right fibered derivator satisfying also (FDer0 left). Assume that push-outs exist in $\mathbb{S}$. We say that a morphism $f : X \to U$ in $\mathbb{S}(\cdot)$ is* **$\mathbb{D}$-colocal** *if*

(Dloc1 right) *The morphism f satisfies* **base change***: for any diagram $Q \in \mathbb{D}(\square)$ with underlying diagram:*

$$\begin{tikzcd} A & B \arrow[l, "\widetilde{F}"'] \\ C \arrow[u, "\widetilde{G}"] & D \arrow[l, "\widetilde{f}"] \arrow[u, "\widetilde{g}"'] \end{tikzcd}$$

such that $p(Q)$ in $\mathbb{S}(\square)$ is a pushout-diagram, i.e. is (homotopy) coCartesian, if $\widetilde{F}$ and $\widetilde{f}$ are coCartesian, and $\widetilde{g}$ is Cartesian then also $\widetilde{G}$ is Cartesian.

(Dloc2 right) *The morphism of derivators (cf. Lemma 4.3.13)*

$$f_\bullet : \mathbb{D}_X \to \mathbb{D}_U$$

commutes with homotopy limits.

A morphism $f : X \to U$ in $\mathbb{S}(\cdot)$ is called **universally $\mathbb{D}$-colocal** *if any homotopy push-out of f is $\mathbb{D}$-colocal.*

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is the underlying diagram of $p(Q)$ then the exchange morphism

$$G_\bullet F^\bullet \to f^\bullet g_\bullet$$

is an isomorphism.

Definition 4.5.6 (right)**.** *Assume that $\mathbb{S}$ is equipped with a Grothendieck pre-cotopology, i.e. that $\mathbb{S}^{\mathrm{op}}$ is equipped with a Grothendieck pre-topology (cf. 4.5.2). A right fibered derivator $p : \mathbb{D} \to \mathbb{S}$ as in Definition 4.5.5 is called* **colocal** *w.r.t. the pre-cotopology on $\mathbb{S}$, if*

1. *Every morphism $S \to U_i$ which is part of a cocover is $\mathbb{D}$-colocal.*
2. *For a cocover $\{f_i : S \to U_i\}$ the family*
$$(f_i)_\bullet : \mathbb{D}(\cdot)_S \to \mathbb{D}(\cdot)_{U_i}$$
is jointly conservative.

4.6 The associated pseudo-functors

Let $p : \mathbb{D} \to \mathbb{S}$ be a morphism of pre-derivators with domain Dia.

4.6.1 (left)**.** Let $\mathrm{Dia}(\mathbb{S})$ be the 2-category of diagrams over $\mathbb{S}$, where the objects are pairs (I, F) such that $I \in \mathrm{Dia}$ and $F \in \mathbb{S}(I)$, the morphisms $(I, F) \to (J, G)$ are pairs (α, f) such that $\alpha : I \to J, f : F \to \alpha^* G$ and the 2-morphisms $(\alpha, f) \to (\beta, g)$ are the natural transformations $\mu : \alpha \Rightarrow \beta$ satisfying $\mathbb{S}(\mu)(G) \circ f = g$.
We call a morphism (α, f) of **fixed shape** if $\alpha = \mathrm{id}$, and of **diagram type** if f consists of identities. Every morphism is obviously a composition of one of diagram type by one of fixed shape.

4.6.2 (right)**.** There is a dual notion of a 2-category $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$. Explicitly, the objects are pairs (I, F) such that $I \in \mathrm{Dia}$ and $F \in \mathbb{S}(I)$, the morphisms $(I, F) \to (J, G)$ are pairs (α, f) such that $\alpha : I \to J, f : \alpha^* G \to F$ and the 2-morphisms $(\alpha, f) \to (\beta, g)$ are the natural transformations $\mu : \alpha \Rightarrow \beta$ satisfying $f \circ \mathbb{S}(\mu)(G) = g$.
The association $(I, F) \mapsto (I^{\mathrm{op}}, F^{\mathrm{op}})$ induces an isomorphism $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}) \to \mathrm{Dia}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}}$.

We are interested in associating to a fibered derivator a pseudo-functor like for classical fibered categories.

4.6.3 (left)**.** We associate to a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ which satisfies (FDer0 right) a (contravariant) 2-pseudo-functor
$$\mathbb{D} : \mathrm{Dia}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$$
mapping a pair (I, F) to $\mathbb{D}(I)_F$, and a morphism $(\alpha, f) : (I, F) \to (J, G)$ to $f^\bullet \circ \alpha^* : \mathbb{D}(J)_G \to \mathbb{D}(I)_F$. A natural transformation $\mu : \alpha \Rightarrow \beta$ is mapped to the natural transformation pasted from the following two 2-commutative triangles:

$$\begin{tikzcd}
 & \mathbb{D}(I)_{G\circ\alpha} \arrow[dr, "f^\bullet"] & \\
\mathbb{D}(J)_G \arrow[ur, "\alpha^*"] \arrow[dr, "\beta^*"'] \arrow[r, phantom, "\Downarrow^\mu"] & \mathbb{S}(\mu)(G)^\bullet \arrow[r, phantom, "\Downarrow"] & \mathbb{D}(I)_F \\
 & \mathbb{D}(I)_{G\circ\beta} \arrow[uu, "\mathbb{S}(\mu)(G)^\bullet"'] \arrow[ur, "g^\bullet"'] &
\end{tikzcd}$$

Proof of the pseudo-functor property. For a composition $(\beta, g)\circ(\alpha, f) = (\beta\circ\alpha, \alpha^*(g)\circ f)$ we have: $f^\bullet \circ \alpha^* \circ g^\bullet \circ \beta^* \cong f^\bullet \circ (\alpha^* g)^\bullet \circ \alpha^* \circ \beta^*$. This follows from the isomorphism $\alpha^* \circ g^\bullet \cong (\alpha^* g)^\bullet \circ \alpha^*$ (FDer0). One checks that this indeed yields a pseudo-functor. □

4.6.4 (right)**.** We associate to a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ which satisfies (FDer0 left) a (contravariant) 2-pseudo-functor

$$\mathbb{D} : \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$$

mapping a pair (I, F) to $\mathbb{D}(I)_{F(I)}$, and a morphism $(\alpha, f) : (I, F) \to (J, G)$ to $f_\bullet \circ \alpha^*$ from $\mathbb{D}(J)_G \to \mathbb{D}(I)_F$. This defines a functor by the same reason as in 4.6.3.

4.6.5 (left)**.** We assume that $\mathbb{S}$ is a strong right derivator. There is a notion of "comma object" in $\mathrm{Dia}(\mathbb{S})$ which we describe here for the case that $\mathbb{S}$ is the pre-derivator associated with a category $\mathcal{S}$ and leave it to the reader to formulate the derivator version. In that case the corresponding object will be determined up to (non-unique!) isomorphism only.
Given diagrams $D_1 = (I_1, F_1), D_2 = (I_2, F_2), D_3 = (I_3, F_3)$ in $\mathrm{Dia}(\mathbb{S})$ and morphisms $\beta_1 : D_1 \to D_3$, $\beta_2 : D_2 \to D_3$, we form the comma diagram $D_1 \times_{/D_3} D_2$ as follows: the underlying diagram $I_1 \times_{/I_3} I_2$ has objects being triples (i_1, i_2, μ) such that $i_1 \in I_1$, $i_2 \in I_2$, and $\mu : \alpha_1(i_1) \to \alpha_2(i_2)$ in I_3. A morphism is a pair $\beta_j : i_j \to i'_j$ for $j = 1, 2$ such that

$$\begin{tikzcd} \alpha_1(i_1) \arrow[r, "\alpha_1(\beta_1)"] \arrow[d, "\mu"] & \alpha_1(i'_1) \arrow[d, "\mu'"] \\ \alpha_2(i_2) \arrow[r, "\alpha_2(\beta_2)"] & \alpha_2(i'_2) \end{tikzcd}$$

commutes in I_3. The corresponding functor $\widetilde{F} \in \mathbb{S}(I_1 \times_{/I_3} I_2)$ maps a triple (i_1, i_2, μ) to

$$F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2).$$

We define P_j to be (ι_j, p_j) for $j = 1, 2$, where ι_j maps a triple (i_1, i_2, μ) to i_j, and p_j is the corresponding projection of the fiber product. We then get a 2-commutative diagram

$$\begin{tikzcd} D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"] \arrow[dr, phantom, "\swarrow^{\mu}"] & D_1 \arrow[d, "\beta_1"] \\ D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

If we are given I_2, I_3 only and two maps $I_1 \to I_3$ and $I_2 \to I_3$ we also form $D_1 \times_{/I_3} I_2$ by the same underlying category, with functor $F_1 \circ \iota_1$.

4.6.6 (right)**.** We assume that $\mathbb{S}$ is a strong left derivator. There is a dual notion of "comma object" in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ which we describe here again for the case that $\mathbb{S}$ is the pre-derivator associated with a category $\mathcal{S}$ and leave it to the reader to formulate

the derivator version. In that case the corresponding object will be determined up to (non-unique!) isomorphism only.
Given three diagrams $D_1^o = (I_1, F_1), D_2^o = (I_2, F_2)$ in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ mapping to $D_3^o = (I_3, F_3)$, we form the comma diagram $D_1^o \times_{/D_3^o} D_2^o$ as follows: the underlying diagram is $I_1 \times_{/D_3} I_2$ which has object being triples (i_1, i_2, μ) such that $i_1 \in I_1, i_2 \in I_2$ and $\mu : \alpha_1(i_1) \to \alpha_2(i_2)$ in I_3. A morphism is a pair $\beta_j : i_j \to i'_j$ for $j = 1, 2$ such that

$$\begin{tikzcd} \alpha_1(i_1) \arrow[r, "\alpha_1(\beta_1)"] \arrow[d, "\mu"] & \alpha_1(i'_1) \arrow[d, "\mu'"] \\ \alpha_2(i_2) \arrow[r, "\alpha_2(\beta_2)"] & \alpha_2(i'_2) \end{tikzcd}$$

commutes in I_3. The corresponding functor $\widetilde{F}$ maps a triple (i_1, i_2, μ) to

$$F_1(i_1) \sqcup_{F_3(\alpha_1(i_1))} F_2(i_2).$$

We then get a 2-commutative diagram

$$\begin{tikzcd} D_2^o \times_{/D_3^o} D_1^o \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\mu"] & D_1^o \arrow[d] \\ D_2^o \arrow[r] & D_3^o \end{tikzcd}$$

This language allows us to restate Lemma 4.3.22 and Lemma 4.3.29 in a more convenient way:

Lemma 4.6.7 (left). *1. Given a "pasting" diagram in* Dia($\mathbb{S}$)

$$\begin{tikzcd} D_1 \arrow[r, "\Gamma"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow\nu"] & D_3 \arrow[r, "B"] \arrow[d, "a"] \arrow[dr, phantom, "\swarrow\mu"] & D_5 \arrow[d, "\alpha"] \\ D_2 \arrow[r, "\gamma"] & D_4 \arrow[r, "\beta"] & D_6 \end{tikzcd}$$

the pasted natural transformation $\nu \odot \mu := \beta\nu \circ \mu\Gamma$ satsisfies

$$\nu_! \odot \mu_! = (\nu \odot \mu)_!.$$

2. *Given a "pasting" diagram in* Dia($\mathbb{S}$)

$$\begin{tikzcd} D_1 \arrow[r, "B"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\swarrow\nu"] & D_2 \arrow[d, "\gamma"] \\ D_3 \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow\mu"] & D_4 \arrow[d, "\alpha"] \\ D_5 \arrow[r, "\beta"] & D_6 \end{tikzcd}$$

the pasted natural transformation $\nu \odot \mu := \alpha\nu \circ \mu\Gamma$ satisfies

$$\mu_! \odot \nu_! = (\mu \odot \nu)_!.$$

Definition 4.6.8. *If $\mathbb{S}$ is equipped with a Grothendieck pre-topology (cf. 4.5.2) then we call $(\alpha, f) : (I, F) \to (J, G)$* **$\mathbb{D}$-local** *if $f_i : F(i) \to G \circ \alpha(i)$ is $\mathbb{D}$-local (cf. 4.5.3) for all $i \in I$. Likewise for the notions of universally $\mathbb{D}$-local, $\mathbb{D}$-colocal, and universally $\mathbb{D}$-colocal.*

Proposition 4.6.9 (left). *Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right) and such that $\mathbb{S}$ is a strong right derivator. Then the associated pseudo-functor satisfies the following properties:*

1. *For a morphism of diagrams $(\alpha, f) : D_1 \to D_2$ the corresponding pull-back*

$$(\alpha, f)^* : \mathbb{D}(D_2) \to \mathbb{D}(D_1)$$

   has a left-adjoint $(\alpha, f)_!$.

2. *For a diagram like in 4.6.5*

$$\begin{tikzcd} D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, phantom, "\swarrow\alpha"] & D_1 \arrow[d, "\beta_1"] \\ D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

   the corresponding exchange morphism

$$P_{2!} P_1^* \to \beta_2^* \beta_{1!}$$

   is an isomorphism in $\mathbb{D}(D_2)$ provided that β_2 is $\mathbb{D}$-local.

Proof. 1. By (FDer0 left) and (FDer3 left) we can form $(\alpha, f)_! := \alpha_! \circ f_\bullet$ which is clearly left adjoint to $(\alpha, f)^*$.
2. We first reduce to the case where I_2 is the trivial category. Indeed consider the diagram

$$\begin{tikzcd} D_1 \times_{/D_3} (\{i_2\}, F_2(i_2)) \arrow[r, "can."] \arrow[d] \arrow[dr, phantom, "\swarrow"] & D_1 \times_{/D_3} D_2 \times_{/D_2} (\{i_2\}, F_2(i_2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, phantom, "\swarrow"] & D_1 \arrow[d, "\beta_1"] \\ (\{i_2\}, F_2(i_2)) \arrow[r, equal] & (\{i_2\}, F_2(i_2)) \arrow[r] & D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

The exchange morphism of the middle square and outmost rectangle are isomorphisms by the reduced case. The morphism *can.* of the left hand square is of diagram type and its underlying diagram functor has an adjoint. The exchange morphism is therefore an isomorphism by [Gro13, 1.23]. Using Lemma 4.6.7 therefore, applying this for all $i_2 \in I_2$,

also the exchange morphism of the right square has to be an isomorphism (this uses axiom Der2).
Now we may assume $D_2 = (\{i_2\}, F_2(i_2))$. Consider the following diagram, in which we denote $\beta_1 = (\alpha_1, f_1)$, $\beta_2 = (\alpha_2, f_2)$.

$$\begin{tikzcd}
(I_1 \times_{/I_3} \{i_2\}, \widetilde{F}) \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\swarrow \; 1"] & (I_1 \times_{/I_3} \{i_2\}, F_1 \circ \iota_1) \arrow[r, "\iota_1"] \arrow[d, "\iota_1^* f_1"] \arrow[dr, phantom, "\swarrow \; 4"] & (I_1, F_1) \arrow[d, "f_1"] \\
(I_1 \times_{/I_3} \{i_2\}, \widetilde{F}') \arrow[r, "p_1'"] \arrow[d, "p_2'"] \arrow[dr, phantom, "\swarrow \; 2"] & (I_1 \times_{/I_3} \{i_2\}, F_3 \circ \alpha_1 \circ \iota_1) \arrow[r, "\iota_1"] \arrow[d, "F_3(\mu)"] \arrow[ddr, phantom, "\swarrow \; 5"] & (I_1, F_3 \circ \alpha_1) \arrow[dd, "\alpha_1"] \\
(I_1 \times_{/I_3} \{i_2\}, F_2(i_2)) \arrow[r, "\iota_2^* f_2"] \arrow[d, "\iota_2"] \arrow[dr, phantom, "\swarrow \; 3"] & (I_1 \times_{/I_3} \{i_2\}, F_3(\alpha_2(i_2))) \arrow[d, "\iota_2"] & \\
(\{i_2\}, F_2(i_2)) \arrow[r, "f_2"] & (\{i_2\}, F_3(\alpha_2(i_2))) \arrow[r, "\alpha_2"] & (I_3, F_3)
\end{tikzcd}$$

where $\widetilde{F}$ is the functor defined in 4.6.5 mapping a triple $(i_1, i_2, \mu : \alpha_1(i_1) \to \alpha_2(i_2))$ to

$$F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2)$$

and $\widetilde{F}'$ is the functor mapping a triple $(i_1, i_2, \mu : \alpha_1(i_1) \to \alpha_2(i_2))$ to

$$F_3(\alpha_1(i_1)) \times_{F_3(\alpha_2(i_2))} F_2(i_2).$$

We have to show that the exchange morphism for the outer square is an isomorphism. Using Lemma 4.6.7 it suffices to show this for the squares 1–5. That the exchange morphism for the squares 1 and 2, where the morphisms are of fixed shape, is an isomorphism can be checked point-wise by (Der2). Then it boils down to the base change condition (Dloc1 left). Note that the squares are pull-back squares in $\mathcal{S}$ by construction of $\widetilde{F}'$ resp $\widetilde{F}$. The exchange morphism for 4 is an isomorphism by (FDer0). The exchange morphism for 3 is an isomorphism because of (Dloc2 left). The exchange morphism for 5 is an isomorphism because of (FDer4 left). $\square$

Dualizing, there is a right-variant of the theorem, which uses $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ instead. We leave its formulation to the reader.

4.7 Construction of fibered multiderivators

4.7.1. The most basic situation in which a (non-trivial) fibered (multi)derivator can be constructed arises from a bifibration of (locally small) multicategories

$$p : \mathcal{D} \to \mathcal{S}$$

where we are given a class of weak equivalences $\mathcal{W}_S \subset \mathrm{Mor}(\mathcal{D}_S)$ for each object S of $\mathcal{S}$. In the examples we have in mind, the objects of $\mathcal{S}$ are spaces (or schemes), the objects of $\mathcal{D}$ are chain complexes of sheaves (coherent, etale Abelian, etc.) on them, and the morphisms in $\mathcal{W}_S$ are the quasi-isomorphisms. In these examples the multicategory-structure arises from the tensor product and it is even, in most cases, the more natural structure because no particular tensor-product is chosen a priori.

Definition 4.7.2. *In the situation above, let $\mathbb{S}$ be the pre-multiderivator associated with the multicategory $\mathcal{S}$. We define a pre-multiderivator $\mathbb{D}$ as follows (cf. 2.3.1 for localizations of multicategories):*

$$\mathbb{D}(I) = \mathrm{Fun}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

where $\mathcal{W}_I$ is the set of natural transformations which are element-wise in the union $\bigcup_S \mathcal{W}_S$. The functor p obviously induces a morphism of pre-multiderivators

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}$$

Observe that morphisms in $\mathcal{W}_I$, by definition, necessarily map to identities in $\mathrm{Fun}(I, \mathcal{S})$.

In this section we prove that the above morphism of pre-(multi)derivators is a left (resp. right) fibered (multi)derivator on directed (resp. inverse) diagrams, provided that the fibers are model categories whose structure is compatible with the structure of bifibration. We use the definition of a model category from [Hov99]. We denote the cofibrant replacement functor by Q and the fibrant replacement functor by R.

Definition 4.7.3. *A* **bifibration of (multi-)model-categories** *is a bifibration of (multi)categories $p : \mathcal{D} \to \mathcal{S}$ together with the collection of a closed model structure on the fiber*

$$(\mathcal{D}_S, \mathrm{Cof}_S, \mathrm{Fib}_S, \mathcal{W}_S)$$

for any object S in $\mathcal{S}$ such that the following two properties hold:

1. *For any $n \geq 1$ and for every multimorphism*

$$\begin{matrix} S_1 & \searrow & & \\ \vdots & & f \longrightarrow T & \\ S_n & \nearrow & & \end{matrix}$$

the push-forward $f_\bullet$ and the various pull-backs $f^{\bullet,j}$ define a Quillen adjunction in n-variables

$$\textstyle\prod_i (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{f_\bullet} (\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T)$$

$$(\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T) \times \textstyle\prod_{i \neq j} (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{f^{\bullet,j}} (\mathcal{D}_{S_j}, \mathrm{Cof}_{S_j}, \mathrm{Fib}_{S_j}, \mathcal{W}_{S_j})$$

2. *For any 0-ary morphism f in $\mathcal{S}$, let $f_\bullet()$ be the corresponding unit object (i.e. the object representing the 0-ary morphism functor* $\mathrm{Hom}_f(;-)$*) and consider the cofibrant replacement $Qf_\bullet() \to f_\bullet()$. Then the natural morphism*

$$F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_{i-1},Qf_\bullet(),\mathcal{E}_i,\dots,\mathcal{E}_n) \to F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_{i-1},f_\bullet(),\mathcal{E}_i,\dots,\mathcal{E}_n) \cong (F\circ_i f)_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)$$

is a weak equivalence if all $\mathcal{E}_i$ are cofibrant. Here F is any morphism which is composable with f.

Remark 4.7.4. *If $\mathcal{S} = \{\cdot\}$, the above notion coincides with the notion of monoidal model-category in the sense of [Hov99, Definition 4.2.6]. In this case it is enough to claim property 1. for $n = 1,2$.*

Theorem 4.7.5. *Under the conditions of Definition 4.7.3 the morphism of pre-derivators*

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}$$

(defined in 4.7.2) is a left fibered multiderivator (satisfying also FDer0 right) with domain Dir *and a right fibered multiderivator (satisfying also FDer0 left) with domain* Inv*. Furthermore for all $S \in \mathbb{S}(\cdot)$ its fiber $\mathbb{D}_S$ (c.f. 4.3.11) is just the pre-derivator associated with the pair $(\mathcal{D}_S, \mathcal{W}_S)$.*

There are techniques by Cisinski [Cis03] which allow to extend a derivator to more general diagram categories. We will explain in a forthcoming article how these can be applied to fibered (multi-)derivators.
The proof of the theorem will occupy the rest of this section. First we have:

Proposition 4.7.6. *Let $\mathcal{D} \to \mathcal{S}$ be a bifibration of multicategories with complete fibers. For any diagram category I, the functors*

$$p_I : \mathrm{Fun}(I,\mathcal{D}) \to \mathrm{Fun}(I,\mathcal{S}) = \mathbb{S}(I)$$

are bifibrations of multicategories.
Morphisms in $\mathrm{Fun}(I,\mathcal{D})$ *are Cartesian, if and only if they are point-wise Cartesian. The 1-ary morphisms in* $\mathrm{Fun}(I,\mathcal{D})$ *are coCartesian, if and only if they are point-wise coCartesian.*

Proof (Sketch). We choose push-forward functors $f_\bullet$ and pull-back functors $f^{i,\bullet}$ for $\mathcal{D} \to \mathcal{S}$ as usual. Let $f \in \mathrm{Hom}(S_1,\dots,S_n;T)$ be a morphism in $\mathrm{Fun}(I,\mathcal{S})$. We define a functor

$$f_\bullet : \mathrm{Fun}(I,\mathcal{D})_{S_1} \times \cdots \times \mathrm{Fun}(I,\mathcal{D})_{S_n} \to \mathrm{Fun}(I,\mathcal{D})_T$$

by

$$\mathcal{E}_1,\dots,\mathcal{E}_n \mapsto \{i \mapsto (f_i)_\bullet(\mathcal{E}_1(i),\dots,\mathcal{E}_n(i))\}.$$

Note that a morphism $\alpha : i \to i'$ in I induces a *well-defined* morphism

$$(f_i)_\bullet(\mathcal{E}_1(i),\dots,\mathcal{E}_n(i)) \to (f_{i'})_\bullet(\mathcal{E}_1(i'),\dots,\mathcal{E}_n(i'))$$

lying over $T(\alpha)$. The functor $f_\bullet$ comes equipped with a morphism in

$$\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n))$$

which is checked to be Cartesian in the strong form of Definition 2.4.4.
For 1-ary morphisms we can perform the same construction to produce coCartesian morphisms. For $n \geq 2$ the construction is more complicated. Let $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ be a morphism with $n \geq 2$. To ease notation, we construct a pull-back functor w.r.t. the first slot. The other constructions are completely symmetric.
For any $i_1 \in I$ consider the category (a variant of the twisted arrow category)

$$X_{i_1}(I) := \{\ (i_2, \dots, i_n, j, \{\alpha_k\}_{k=1..n}) \mid \alpha_k : i_k \to j\}$$

which is covariant in j and contravariant in $i_2, \dots, i_n$. For any $\beta : i_1 \to i_1'$ we have an induced functor $\widetilde{\beta} : X_{i_1'}(I) \to X_{i_1}(I)$. Any object $\alpha \in X_{i_1}(I)$ defines by pre-composition with $S_k(\alpha_k)$ for all $1 \leq k \leq n$ a morphism $f_\alpha \in \mathrm{Hom}(S_1(i_1), \dots, S_n(i_n); T(j))$.
We define a functor

$$f^{1,\bullet} : (\mathrm{Fun}(I, \mathcal{D})_{S_2})^{\mathrm{op}} \times \dots \times (\mathrm{Fun}(I, \mathcal{D})_{S_n})^{\mathrm{op}} \times \mathrm{Fun}(I, \mathcal{D})_T \to \mathrm{Fun}(I, \mathcal{D})_{S_1}$$

assigning to $\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F}$ the following functor $X_{i_1}(I) \to \mathcal{D}_{S_1(i_1)}$:

$$\alpha \mapsto (f_\alpha)^{1,\bullet}(\mathcal{E}_2(i_2), \dots, \mathcal{E}_n(i_n); \mathcal{F}(j))$$

and then taking $\lim_{X_{i_1}(I)}$ which exists because the fibers are required to be complete.
For the functoriality note that for $\beta : i_1 \to i_1'$ we have a natural morphism

$$\lim_{X_{i_1}(I)} \cdots \to \lim_{X_{i_1'}(I)} \cdots$$

induced by $\widetilde{\beta}$.
We define a morphism

$$\Xi \in \mathrm{Hom}_f(f^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F}), \mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F})$$

and we will show that it is coCartesian w.r.t. the first slot in a weak sense. At some object $i \in I$, the morphism Ξ is given by composing the projections from

$$\lim_{X_{i=i_1}(I)} f_\alpha^{1,\bullet}(\mathcal{E}_2(i_2), \dots, \mathcal{E}_n(i_n); \mathcal{F}(j))$$

to $f_i^{1,\bullet}(\mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i))$ (note that $f_i = f_\alpha$ for $\alpha = \{\mathrm{id}_i\}_k$) and then composing with the coCartesian morphism (in $\mathcal{D}$) in

$$\mathrm{Hom}(f_i^{1,\bullet}(\mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i)), \mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i)).$$

One checks that the so defined Ξ is functorial in i. It remains to be shown that the composition with Ξ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{id}_{S_1}}(\mathcal{E}_1; f^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F})) \to \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}). \tag{32}$$

We will give a map in the other direction which is inverse to composition with Ξ. Let

$$a \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

be a morphism. To give a morphism on the left hand side of (32), for any i_1 and $\alpha \in X_{i_1}(I)$ we have to give a morphism (functorial in i_1)

$$\mathcal{E}_1(i_1) \to f_\alpha^{1,\bullet}(\mathcal{E}_2(i_2); \dots, \mathcal{E}_n(i_n); \mathcal{F}(j))$$

or, which is the same, a morphism

$$\mathrm{Hom}_{f_\alpha}(\mathcal{E}_1(i_1), \mathcal{E}_2(i_2), \dots, \mathcal{E}_n(i_n); \mathcal{F}(j)).$$

But we have such a morphism, namely the pre-composition of a_j with the n-tuple $\{\mathcal{E}_k(\alpha_k)\}_k$. (Because we know already that $\mathrm{Fun}(I, \mathcal{D}) \to \mathrm{Fun}(I, \mathcal{S})$ is an *op*fibration of multicategories, it suffices to establish that Ξ is coCartesian in this weak form.) □

Remark 4.7.7. *The construction in the proof of the above Proposition become much clearer in the light of Definition 5.4.1 where a fibered multiderivator is (re)defined* itself *as a kind of six-functor-formalism. For example, for $\mathcal{S} = \{\cdot\}$ get an external and internal monoidal product, resp. right adjoints which a clear relation. We have in that case*

$$\boxtimes : \mathrm{Fun}(I, \mathcal{D}) \times \mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}(I \times J, \mathcal{D})$$

by applying $\otimes$ point-wise *and*

$$\mathcal{HOM}_{l/r} : \mathrm{Fun}(I, \mathcal{D}) \times \mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}(I^{\mathrm{op}} \times J, \mathcal{D})$$

by applying $\mathrm{Hom}_{l/r}$ point-wise. *The formula for the internal hom obtained in the proof of the proposition boils down to the formula*

$$\mathrm{Hom}_{l/r}(\mathcal{E}, \mathcal{F})(i_1) = \int_i \mathcal{HOM}_{l/r}(\mathcal{E}(i), \mathcal{F}(i))^{\mathrm{Hom}(i_1, i)}$$

where $\int_i$ is the categorical end. See section 6.1 for more explanations.

We will need later the following

Lemma 4.7.8. *Let $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ be a morphism in $\mathrm{Fun}(I, \mathcal{S})$ for some $n \geq 2$. Consider the pull-back functor $f^{j,\bullet}$ constructed in the proof of Proposition 4.7.6. Let $p : I \times J \to I$ be the projection and fix objects $\mathcal{E}_1, .\widehat{j}., \mathcal{E}_n, \mathcal{F}$ in $\mathcal{D}$ lying over $S_1, .\widehat{j}., S_n, T$. Then the natural morphism*

$$p^* f^{j,\bullet}(\mathcal{E}_1, .\widehat{j}., \mathcal{E}_n; \mathcal{F}) \to (p^* f)^{j,\bullet}(p^*\mathcal{E}_1, .\widehat{j}., p^*\mathcal{E}_n; p^*\mathcal{F})$$

is an isomorphism, or, in other words, the functor $p^* : \mathrm{Fun}(I, \mathcal{D}) \to \mathrm{Fun}(I \times J, \mathcal{S})$ maps Cartesian morphisms to Cartesian morphisms.

Proof. Again, we assume $j = 1$ to ease the notation. The statement concerning the other pull-backs is completely symmetric. We have by definition

$$(f^{1,\bullet}(\mathcal{E}_2,\dots,\mathcal{E}_n;\mathcal{F}))(i') = \lim_{\alpha\in X_{i_1}(I)} f_\alpha^{1,\bullet}(\mathcal{E}_2(i_1),\dots,\mathcal{E}_n(i_n);\mathcal{F}(i'))$$

and

$$((p^*f)^{1,\bullet}(\mathcal{E}_2,\dots,\mathcal{E}_n;\mathcal{F}))(i',j') = \lim_{\alpha\in X_{i_1,j_1}(I\times J)} (p^*f)_\alpha^{1,\bullet}((p^*\mathcal{E}_2)(i_1,j_1),\dots,(p^*\mathcal{E}_n)(i_n,j_n);(p^*\mathcal{F})(i',j'))$$

The natural map in question is induced by the functor $\widetilde{p} : X_{i_1,j_1}(I\times J) \to X_{i_1}(I)$ which forgets all data involving the J direction. Now there is also a functor $\widetilde{s} : X_{i_1}(I) \to X_{i_1,j_1}(I\times J)$ which is constant on the J-component with value $\{\mathrm{id}_{j_1}\}_{k=1..n}$. We have $\widetilde{p}\circ\widetilde{s} = \mathrm{id}$ and a chain of natural transformations $\widetilde{s}\circ\widetilde{p} \Leftarrow \cdots \Rightarrow \mathrm{id}$ involving only data in the J-direction. However, all the natural transformations are mapped to identities by the functor

$$\alpha \mapsto \lim_{\alpha\in X_{i_1,j_1}(I\times J)} (p^*f)_\alpha^{1,\bullet}((p^*\mathcal{E}_2)(i_1,j_1),\dots,(p^*\mathcal{E}_n)(i_n,j_n);(p^*\mathcal{F})(i',j'))$$

because everything is constant along the J-direction. This shows that the natural morphism in the statement is an isomorphism. □

If I is directed or inverse we want to show that also p_I is a bifibration of multi-model-categories in the sense of Definition 4.7.3.
Afterwards we will apply the following variant and generalization to multicategories of the results in [SGA73, Exposé XVII, §2.4].

Proposition 4.7.9. *Let $p : \mathcal{D} \to \mathcal{S}$ be a bifibration of (multi-)model-categories in the sense of 4.7.3. Let $\mathcal{W}$ be the union of the $\mathcal{W}_S$ over all objects $S \in \mathcal{S}$. Then the fibers of $\widetilde{p} : \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{S}$ (as ordinary categories) are the homotopy categories $\mathcal{D}_S[\mathcal{W}_S^{-1}]$ and $\widetilde{p}$ is again a bifibration of multicategories such that the push-forward $F_\bullet$ along any $F \in$* $\mathrm{Hom}_{\mathcal{S}}(S_1,\dots,S_n;T)$ *(for $n \geq 1$) is the left derived functor of the corresponding push-forward w.r.t. p. Similarly the pull-back w.r.t. some slot ist the right derived functor of the corresponding pull-back w.r.t. p.*

4.7.10. The above proposition and its proof have several well-known consequences which we mention, despite being all elementary, because the proof below gives a uniform treatment of all the cases.

1. The homotopy category of a model category is locally small and can be described as the category of cofibrant/fibrant objects modulo homotopy of morphisms. *Apply the proof of the proposition to the (trivial) bifibration of ordinary categories $\mathcal{D} \to \{\cdot\}$.*

2. Quillen adjunctions lead to an adjunction of the derived functors on the homotopy categories. *Apply the proposition to a bifibration of ordinary categories $\mathcal{D} \to \Delta_1$.*

3. The homotopy category of a closed monoidal model category is a closed monoidal category. *Apply the proposition to a bifibration of multicategories $\mathcal{D} \to \{\cdot\}$.*

4. Quillen adjunctions in n variables lead to an adjunction in n variables on the homotopy categories. *Apply the proposition to a bifibration of multicategories $\mathcal{D} \to \Delta_{1,n}$, where the multicategory $\Delta_{1,n}$ consists of $n+1$ objects and one n-ary morphism connecting them.*

Before proving Proposition 4.7.9, we define homotopy relations on $\mathrm{Hom}_F(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ where $F \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is a multimorphism in $\mathcal{S}$.

Definition 4.7.11. *1. Two morphisms f and g in $\mathrm{Hom}_F(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ are called* **right homotopic** *if there is a path object of $\mathcal{F}$*

$$\mathcal{F} \longrightarrow \mathcal{F}' \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathcal{F}$$

and a morphism $\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}')$ over the same multimorphism F such that the compositions with pr_1 and pr_2 are f and g, respectively.

2. For $n \geq 1$, two morphisms f and g in $\mathrm{Hom}_F(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ are called **i-left homotopic** *if there is a cylinder object $\mathcal{E}_i'$ of $\mathcal{E}_i$*

$$\mathcal{E}_i \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} \mathcal{E}_i' \longrightarrow \mathcal{E}_i$$

and a morphism $\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_i', \dots, \mathcal{E}_n; \mathcal{F})$ over F such that the compositions with ι_1 and ι_2 are f and g, respectively.

Lemma 4.7.12. *1. The condition 'right homotopic' is preserved under pre-composition, while the condition 'i-left homotopic' is preserved under post-composition.*

2. Let $n \geq 1$. If $f, g \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ are i-left homotopic and all $\mathcal{E}_i$ are cofibrant then f and g are right homotopic. If $f, g \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ are right homotopic, $\mathcal{F}$ is fibrant, and all $\mathcal{E}_j$ for $j \neq i$ are cofibrant then f and g are i-left homotopic.

3. Let $n \geq 1$. In $\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ right homotopy is an equivalence relation if all $\mathcal{E}_i$ are cofibrant. In $\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ i-left homotopy is an equivalence relation if $\mathcal{F}$ is fibrant, and all $\mathcal{E}_j$, $j \neq i$ are cofibrant

In particular on the category $\mathcal{D}^{\mathrm{Cof,Fib}}$ of fibrant/cofibrant objects, i-left homotopy=right homotopy is an equivalence relation, which is compatible with composition.

Proof. 1. is obvious.
2. If all $\mathcal{E}_i$ are cofibrant then also $F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is cofibrant and f and g correspond uniquely to morphisms $f', g' : F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \to \mathcal{F}$. Since f and g are i-left homotopic, there is a cylinder object

$$\mathcal{E}_i \rightrightarrows \mathcal{E}_i' \longrightarrow \mathcal{E}_i$$

realizing the i-left homotopy. Since $\mathcal{E}_i$ is cofibrant so is $\mathcal{E}_i'$. Hence also

$$F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \rightrightarrows F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_i',\dots,\mathcal{E}_n) \longrightarrow F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)$$

is a cylinder object because all $\mathcal{E}_j$ are cofibrant, and hence also f' and g' are left homotopic. These are therefore also right homotopic and hence so are f and g. Dually we obtain the second statement.
3. follows from [Hov99, Proposition 1.2.5, (iii)]. □

Lemma 4.7.13. *Two i-left homotopic morphisms become equal in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$.*

Proof. This follows from the fact that a cylinder object

$$\begin{tikzcd} \mathcal{E}_i \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & \mathcal{E}_i' \arrow[r, "p"] & \mathcal{E}_i \end{tikzcd}$$

automatically lies in $\mathcal{D}^{\mathrm{Cof}}$ if $\mathcal{E}_i$ does, and the two maps ι_1 and ι_2 become equal because p becomes invertible. □

We have to distinguish the easier case, in which all objects $F_\bullet()$ for 0-ary morphisms F are cofibrant. Otherwise we define a category $\mathcal{D}^{\mathrm{Cof}}\widetilde{[(\mathcal{W}^{\mathrm{Cof}})^{-1}]}$ in which we set $\mathrm{Hom}_F(;\mathcal{F}) \coloneqq \mathrm{Hom}_{\mathcal{D}_S[\mathcal{W}_S^{-1}]}(QF_\bullet();\mathcal{F})$ for all $\mathcal{F}$, where F is a 0-ary morphism with domain S. Composition is given as follows: For a morphism $f \in \mathrm{Hom}_G(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ with cofibrant $\mathcal{E}_i$ and $\mathcal{F}$ and $\xi : QF_\bullet() \to \mathcal{E}_i$, we define the composition $\xi \circ f$ as the following composition

$$\begin{tikzcd}[column sep=small] \mathcal{E}_1 \arrow[dr, dash] & & & & \\ \vdots\widehat{i} & \mathrm{cocart} \arrow[r] & (F \circ G)_\bullet(\mathcal{E}_2,\overset{\widehat{i}}{\dots},\mathcal{E}_n) \arrow[r, "\sim"] & G_\bullet(\mathcal{E}_1,\dots,F_\bullet(),\dots,\mathcal{E}_n) & \arrow[l] \\ \mathcal{E}_n \arrow[ur, dash] & & & & \end{tikzcd}$$

$$\longleftarrow G_\bullet(\mathcal{E}_1,\dots,QF_\bullet(),\dots,\mathcal{E}_n) \longrightarrow G_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \longrightarrow \mathcal{F}.$$

One checks that the so-defined composition is associative and independent of the choice of the push-forwards.

Lemma 4.7.14. *If the object $F_\bullet()$ is cofibrant for every 0-ary morphism F then the natural functor*

$$\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$$

is an equivalence of categories. Otherwise it is, if we replace $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$ by $\mathcal{D}^{\mathrm{Cof}}\widetilde{[(\mathcal{W}^{\mathrm{Cof}})^{-1}]}$.

Proof. The inclusion $\mathcal{D}^{\mathrm{Cof}} \to \mathcal{D}$ induces a functor $\Xi : \mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$. If the objects $F_\bullet()$ are not cofibrant then Ξ may be modified to a functor

$$\mathcal{D}^{\mathrm{Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Cof}}})^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$$

as follows: a 0-ary morphism $QF_\bullet() \to \mathcal{F}$ is mapped to the composition

$$\begin{tikzcd} \circ \arrow[r, "\mathrm{cocart}"] & F_\bullet() & QF_\bullet() \arrow[l] \arrow[r] & \mathcal{F} \end{tikzcd}$$

in $\mathcal{D}[\mathcal{W}^{-1}]$.
We now specify a functor Φ in the other direction. Φ maps an object $\mathcal{E}$ to a cofibrant replacement $Q\mathcal{E}$. For $n \geq 1$, a morphism $f \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ over F is mapped to the following morphism. Composing with the morphisms $Q\mathcal{E}_i \to \mathcal{E}_i$, we get a morphism $f' \in \mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; \mathcal{F})$ or equivalently a morphism $X_i \to F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; \mathcal{F})$. Now choose a lift (dotted arrow in the diagram)

$$\begin{tikzcd} & F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; Q\mathcal{F}) \arrow[d] \\ Q\mathcal{E}_i \arrow[ur, dotted] \arrow[r] & F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; \mathcal{F}) \end{tikzcd}$$

which exists because the vertical map is again a trivial fibration (because all the $Q\mathcal{E}_i$ are cofibrant). The resulting map in $\mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; P\mathcal{F})$ is actually well-defined in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$. Indeed, two different lifts are left homotopic because $Q\mathcal{E}_i$ is cofibrant [Hov99, Proposition 1.2.5. (iv)], and hence the two morphisms in $\mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; Q\mathcal{F})$ become equal in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$ as well by Lemma 4.7.13. From this it follows that Φ is indeed a functor on n-ary morphisms for $n \geq 1$.
For $n = 0$, a morphism $f \in \mathrm{Hom}(;\mathcal{F})$ over F corresponds to a morphism $F_\bullet() \to \mathcal{F}$. If $F_\bullet()$ is cofibrant, this morphism lifts (again uniquely up to right homotopy) to a morphism $F_\bullet() \to Q\mathcal{F}$, i.e. to a morphism in $\mathrm{Hom}_F(;Q\mathcal{F})$. If $F_\bullet()$ is not cofibrant then the composition lifts to a morphism: $QF_\bullet() \to P\mathcal{F}$ which is defined to be the image of Φ. Furthermore Φ is inverse to Ξ up to isomorphism. □

Lemma 4.7.15. *Right homotopic morphisms become equal in* $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$.

Proof. The assertion follows from the fact that there exists a path object

$$\begin{tikzcd} \mathcal{F} & \mathcal{F}' \arrow[l, shift left, "\mathrm{pr}_1"'] \arrow[l, shift right, "\mathrm{pr}_2"] & \mathcal{F} \arrow[l, "i"'] \end{tikzcd}$$

where $\mathcal{F}'$ is cofibrant and fibrant which realizes the right homotopy [Hov99, Proposition 1.2.6.]. This uses that all sources are cofibrant and the domain is fibrant. The two morphisms pr_1 and pr_2 become equal because i becomes invertible. □

Lemma 4.7.16. *The functor* $\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}] \to \mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$*, and the functor*
$\mathcal{D}^{\mathrm{Fib,Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}}] \to \mathcal{D}^{\mathrm{Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Cof}})^{-1}}]$*, respectively, are equivalences of multicategories.*

Proof. The proof is analogous to that of Lemma 4.7.14 but with some minor changes which require, in particular, the chosen order of restriction to cofibrant and fibrant objects. We specify again a functor Φ in the other direction. On objects, Φ maps $\mathcal{E}$ to a fibrant replacement $R\mathcal{E}$. Note that $R\mathcal{E}$ is still cofibrant. A morphism $f \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ over F corresponds to a morphism $F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \to \mathcal{F}$. Now choose a lift (dotted arrow in the diagram)

$$\begin{tikzcd}
F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & R\mathcal{F} \\
F_\bullet(R\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \arrow[d] & & \\
\vdots \arrow[d] & & \\
F_\bullet(R\mathcal{E}_1, R\mathcal{E}_2, \dots, R\mathcal{E}_n) \arrow[uuurr, dotted] & &
\end{tikzcd}$$

It exists because the vertical maps are again trivial cofibrations (because all the $\mathcal{E}_i$ and $R\mathcal{E}_i$ are cofibrant). The lift is well-defined in $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$, because two lifts in the triangle above become right homotopic (because $R\mathcal{F}$ is fibrant by [Hov99, Proposition 1.2.5. (iv)]). Therefore also the corresponding morphisms in $\mathrm{Hom}(R\mathcal{E}_1, \dots, R\mathcal{E}_n; R\mathcal{F})$ become equal in $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$ by the previous lemma. It follows that Φ is indeed a functor which is inverse to the inclusion up to isomorphism. □

Lemma 4.7.17. *If the objects $F_\bullet()$ for all 0-ary morphisms in $\mathcal{S}$ are cofibrant then the natural functor*

$$\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}] \to \mathcal{D}^{\mathrm{Fib,Cof}}/\sim$$

is an isomorphism of categories. Otherwise it is, if we modify the 0-ary morphisms as before.

Proof. The natural functor $\mathcal{D}^{\mathrm{Fib,Cof}} \to \mathcal{D}^{\mathrm{Fib,Cof}}/\sim$ takes weak equivalences to isomorphisms [Hov99, Proposition 1.2.8] and has the universal property of $\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}]$ by the same argument as in [Hov99, Proposition 1.2.9]. □

Proof of Proposition 4.7.9. The previous lemmas showed that $\mathcal{D}[\mathcal{W}^{-1}]$ is equivalent to $\mathcal{D}^{\mathrm{Fib,Cof}}/\sim$ if all objects of the form $F_\bullet()$ are cofibrant, or if we replace the second multicategory by $\mathcal{D}^{\widetilde{\mathrm{Fib,Cof}}}/\sim$, where we set $\mathrm{Hom}_{F,\mathcal{D}^{\widetilde{\mathrm{Fib,Cof}}}/\sim}(;\mathcal{F}) \coloneqq \mathrm{Hom}_{\mathcal{D}_S[\mathcal{W}_S^{-1}]}(F_\bullet(), \mathcal{F})$ for all 0-ary morphism F in $\mathcal{S}$ with domain S and for every $\mathcal{F} \in \mathcal{D}_S$.

It remains to show that the functor

$$p/\sim\colon\ \widetilde{\mathcal{D}^{\mathrm{Fib,Cof}}}/\sim\ \to \mathcal{S}$$

is bifibered if all $F_\bullet()$ are cofibrant or otherwise bifibered for $n \geq 1$ (i.e. (co)Cartesian morphisms exist for $n \geq 1$). (The modification $\widetilde{\mathcal{D}^{\mathrm{Fib,Cof}}}/\sim$ has been constructed in such a way that it has coCartesian morphisms for $n = 0$.)
We show that $p/\sim$ is opfibered, the other case being similar. Let F be a multimorphism in $\mathcal{S}$ with codomain S. The set $\mathrm{Hom}_F(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ modulo right homotopy is in bijection with the set $\mathrm{Hom}_{\mathcal{D}_Y}(F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n), \mathcal{F})$ modulo right homotopy. Since $\mathcal{F}$ is fibrant, the latter set is the same as $\mathrm{Hom}_{\mathcal{D}_S}(R(F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)), \mathcal{F})$ modulo right homotopy. Hence morphisms in $\mathrm{Hom}_F(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ uniquely decompose as the composition

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[dr, no head] & & & \\
\vdots & \text{cocart} \arrow[r] & F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \arrow[r] & R(F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)) \\
\mathcal{E}_n \arrow[ur, no head] & & &
\end{tikzcd}$$

followed by a morphism in $\mathrm{Hom}_{\mathcal{D}_S}(R(F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)), \mathcal{F})$ modulo right homotopy. More generally, by the same argument, a morphism in some $\mathrm{Hom}_{GF}(\mathcal{F}_1, \dots, \mathcal{E}_1, \dots, \mathcal{E}_n, \dots, \mathcal{F}_m; \mathcal{G})$, where G is another multimorphism in $\mathcal{S}$, modulo right homotopy factorizes uniquely into the above composition followed by a morphism in

$$\mathrm{Hom}_G(\mathcal{F}_1, \dots, R(F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)), \dots, \mathcal{F}_m; \mathcal{G})$$

modulo right homotopy.
It remains to see that the push-forward in $\mathcal{D}[\mathcal{W}^{-1}]$ corresponds to the left derived functor of $F_\bullet$. For any objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ the composition

$$\begin{tikzcd}
RQ\mathcal{E}_1 \arrow[dr, no head] & & & \\
\vdots & \text{cocart} \arrow[r] & F_\bullet(RQ\mathcal{E}_1, \dots, RQ\mathcal{E}_n) \arrow[r] & R(F_\bullet(RQ\mathcal{E}_1, \dots, RQ\mathcal{E}_n)) \\
RQ\mathcal{E}_n \arrow[ur, no head] & & &
\end{tikzcd}$$

is a coCartesian morphism lying over F, with domains isomorphic to the $\mathcal{E}_i$.
However, the object $R(F_\bullet(RQ\mathcal{E}_1, \dots, RQ\mathcal{E}_n))$ is isomorphic to the value of the left derived functor of $F_\bullet$ at $\mathcal{E}_1, \dots, \mathcal{E}_n$. □

4.7.18. We now focus on the left case. If I is a directed diagram, we proceed to construct a model structure on the fibers of the bifibration of multicategories (cf. 4.7.6):

$$\mathrm{Fun}(I, \mathcal{D}) \to \mathrm{Fun}(I, \mathcal{S}) = \mathbb{S}(I).$$

This model structure is an analogue of the classical Reedy model structure and it has the property that pull-backs w.r.t. diagrams and the corresponding relative left Kan extension functors form a Quillen adjunction.
Let $I \in \mathrm{Dir}$ and let $F : I \to \mathcal{S}$ be a functor. We will define a model-category structure

$$(\mathcal{D}_F, \mathrm{Cof}_F, \mathrm{Fib}_F, \mathcal{W}_F)$$

where $\mathcal{D}_F$ is the fiber of $\mathrm{Fun}(I, \mathcal{D})$ over F and where $\mathcal{W}_F$ is the class of morphisms which are element-wise in the corresponding $\mathcal{W}_{F(i)}$.
For any $G \in \mathcal{D}_F$, and for any $i \in I$, we define a **latching object**

$$L_i G := \mathrm{colim}_{I_i} \{F(\alpha)_\bullet G(j)\}_{\alpha: j \to i},$$

Here I_i is the full subcategory of $I \times_{/I} i$ consisting of all objects except id_i. We have a canonical morphism

$$L_i G \to G(i)$$

in $\mathcal{D}_{F(i)}$. We define Fib_F to be the class of morphisms which are element-wise in the corresponding $\mathrm{Fib}_{F(i)}$. We define Cof_F to be the class of morphisms $G \to H$ such that for any $i \in I$ the induced morphism δ in the diagram

$$\begin{tikzcd}
L_i G \arrow[r] \arrow[d] & L_i H \arrow[d] & \\
G(i) \arrow[r] & \text{push-out} \arrow[r, dotted, "\delta"] & H(i)
\end{tikzcd}$$

belongs to $\mathrm{Cof}_{F(i)}$. We call a morphism $G \to H$ in Cof_F temporarily an **acyclic cofibration** if δ is, in addition, a weak equivalence. The proof that this yields a model-category structure is completely analogous to the classical case [Hov99, §5.1] (here this is recovered if $\mathcal{S}$ is trivial). We need a couple of lemmas:

Lemma 4.7.19. *The class of cofibrations (resp. acyclic cofibrations) in $\mathcal{D}_F$ consists precisely of the morphisms which have the left lifting property w.r.t. trivial fibrations (resp. fibrations). These are stable under retracts.*

Proof. This is shown as in the classical case: we first prove that acyclic cofibrations have the lifting property w.r.t. fibrations. Consider a diagram

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "\alpha"] & H_1 \arrow[d, "\beta"] \\
G_2 \arrow[r] & H_2
\end{tikzcd}$$

where α is an acyclic cofibration and β is a fibration. We proceed by induction on n and assume that for all $i \in I$ with $\nu(i) < n$ a map $G_2(i) \to H_1(i)$ has been constructed such that it is a lift in the above diagram, evaluated at i. For each i of degree n consider the following diagram (where the morphism $L_iG_2 \to L_iH_1 \to H_1(i)$ is formed using the already constructed lifts):

$$\begin{tikzcd} G_1(i) \coprod_{L_iG_1} L_iG_2 \arrow[r] \arrow[d, "\alpha'(i)"] & H_1(i) \arrow[d, "\beta(i)"] \\ G_1(i) \arrow[r] & H_2(i) \end{tikzcd}$$

Here $\alpha'(i)$ is a trivial $\mathrm{Cof}_{F(i)}$-cofibration by definition, and $\beta(i)$ is a $\mathrm{Fib}_{F(i)}$-fibration by definition. Hence a lift exists. In the same way the statement for cofibrations and for trivial fibrations is shown. Closure under retracts is left as an exercise for the reader. The assertion that the class of acyclic cofibrations (resp. cofibrations) is *precisely* the class of morphisms that have the left lifting property w.r.t. fibrations (resp. trivial fibrations) follows from the retract argument as for model categories. □

Lemma 4.7.20. *There exists a functorial factorization of morphisms in $\mathcal{D}_F$ into a fibration followed by an acyclic cofibration and into a trivial fibration followed by a cofibration.*

Proof. We show this again by induction on n. We do the first case, the other being similar. Let $G \to K$ a morphism in $\mathcal{D}_F$. We have the following diagram:

$$\begin{tikzcd} L_iG \arrow[r] \arrow[d] & L_iH \arrow[rr] \arrow[d] & & L_iK \arrow[d] \\ G(i) \arrow[r] & G(i) \coprod_{L_iG} L_iH \arrow[r, dotted] & H(i) \arrow[r, dotted] & K(i) \end{tikzcd}$$

Here the top row is constructed using the already defined factorizations. The object $H(i)$ and the dotted maps are constructed as the factorization in the model category $\mathcal{D}_{F(i)}$ into a trivial $\mathrm{Cof}_{F(i)}$-cofibration followed by $\mathrm{Fib}_{F(i)}$-fibration. □

Lemma 4.7.21. *The classes of cofibrations, acyclic cofibrations, fibrations and weak equivalences are stable under composition.*

Proof. This follows from the characterization by a lifting property (resp. by definition for the case of the weak equivalences). □

Lemma 4.7.22. *Acyclic cofibrations are precisely the trivial cofibrations.*

Proof. We begin by showing that an acyclic cofibration is a weak equivalence. It suffices to show that in the diagram

$$\begin{tikzcd} L_iG \arrow[r] \arrow[d] & L_iH \arrow[d] \\ G(i) \arrow[r] & H(i) \end{tikzcd}$$

the top horizontal morphism is a trivial cofibration. Then the lower horizontal morphism is a composition of two trivial cofibrations and hence is a weak equivalence. The top morphism is indeed a trivial cofibration because the morphism of I_i-diagrams (cf. 4.7.18)

$$\{F(\alpha)_{\bullet}G(j)\}_{\alpha:j\to i} \to \{F(\alpha)_{\bullet}G(j)\}_{\alpha:j\to i}$$

is a trivial cofibration in the classical sense (i.e. over the constant diagram over I_i with value $F(i)$) because of Lemmas 4.7.23 and 4.7.24.
In the other direction, let f be a trivial cofibration and factor it as $f = pg$, where g is an acyclic cofibration and p is a fibration. It follows that p is a weak-equivalence. Now construct a lift in the diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "f"'] & H \arrow[d, "p"] \\ G \arrow[r, equal] & G \end{tikzcd}$$

This shows that f is a retract of g, and hence is an acyclic cofibration as well. $\square$

Lemma 4.7.23. *For each (1-ary) morphism of diagrams $f \in \mathrm{Hom}_{\mathcal{S}}(S_1;T)$ there is an associated push-forward and an associated pull-back, defined by taking the point-wise push-forward $f_\bullet$, and point-wise pull-back $f^\bullet$ (cf. 4.7.6), respectively. The push-forward $f_\bullet$ respects the classes of cofibrations and acyclic cofibrations. The pull-back $f^\bullet$ respects the classes of fibrations and trivial fibrations.*

Proof. It suffices (by the lifting property) to show that $f^\bullet$ respects fibrations and trivial fibrations. This is clear because they are defined point-wise. $\square$

A posteriori this will say that the pair of functors $f^\bullet, f_\bullet$ form a Quillen adjunction between the corresponding model categories (cf. 4.7.28).

Lemma 4.7.24. *Let $i \in I$ be an object, let $\iota : I_i \to I$ be the corresponding latching category with its natural functor to I, and let $F_i := \iota^* F : I_i \to \mathcal{S}$ be the restriction of F to I_i. The pull-back $\iota^* : \mathcal{D}_F \to \mathcal{D}_{F_i}$ respects cofibrations and acyclic cofibrations.*

Proof. It is easy to see that the pull-back induces an isomorphism of the corresponding latching objects as in the classical case. $\square$

Corollary 4.7.25. *The structure constructed in 4.7.18 defines a model category.*

Proof. This follows from the previous Lemmas. $\square$

Proposition 4.7.26. *For any morphism of directed diagrams $\alpha : I \to J$, and for any functor $F : J \to \mathcal{S}$, the functor*

$$\alpha^* : \mathcal{D}_F \to \mathcal{D}_{\alpha^* F}$$

has a left adjoint $\alpha_!^F$. The pair $\alpha^*, \alpha_!^F$ define a Quillen adjunction.

Proof. That the two functors define a Quillen adjunction is clear once we have shown that $\alpha_!$ exists because α^* preserves fibrations and weak equivalences. Let G be an object of $\mathcal{D}_F$. We define

$$(\alpha_! G)(j) := \mathrm{colim}_{I\times_{/J}j}\, \mathbb{S}(\mu)_\bullet \iota_j^* G.$$

For each morphism $\mu : j \to j'$ we get a functor

$$\widetilde{\mu} : I \times_{/J} j \to I \times_{/J} j'$$

and hence an induced morphism

$$F(\mu)_\bullet \mathbb{S}(\mu)_\bullet \iota_j^* G \to \widetilde{\mu}^* \mathbb{S}(\mu')_\bullet \iota_{j'}^*.$$

Sincd $F(\mu)_\bullet$ commutes with colimits we get a morphism

$$F(\mu)_\bullet\, \mathrm{colim}_{I\times_{/J}j}\, \mathbb{S}(\mu)_\bullet \iota_j^* G \to \mathrm{colim}_{I\times_{/J}j'}\, \mathbb{S}(\mu')_\bullet \iota_{j'}^*$$

which we define to be $(\alpha_! G)(\mu)$. We now proceed to show that the functor we have constructed is indeed adjoint to α^*. A morphism $\mu : G \to \alpha^* H$ is given by a collection of maps $a(i) : G(i) \to H(\alpha(i))$ for all objects $i \in I$, subject to the condition that the diagram

$$\begin{tikzcd}
F(\alpha(\lambda))_\bullet G(i) \arrow[r, "F(\alpha(\lambda))_\bullet a(i)"] \arrow[d, "\overline{G(\lambda)}"] & F(\alpha(\lambda))_\bullet H(\alpha(i)) \arrow[d, "\overline{H(\alpha(\lambda))}"] \\
G(i') \arrow[r, "a(i')"] & H(\alpha(i'))
\end{tikzcd}$$

commutes for each morphism $\lambda : i \to i'$ in I. For each $j \in J$ and morphism $\mu : \alpha(i) \to j$ we get a morphism

$$\overline{H(\mu)} \circ (F(\mu)_\bullet a(i)) : F(\mu)_\bullet G(i) \to H(j)$$

and therefore for fixed j a morphism

$$\mathrm{colim}_{I\times_{/J}j}\, \mathbb{S}(\mu)_\bullet \iota_j^* G \to H(j).$$

One checks that this yields a morphism $\alpha_! G \to H$. On the other hand, let $b : \alpha_! G \to H$ be a morphism given by

$$b(j) : \mathrm{colim}_{I\times_{/J}j}\, \mathbb{S}(\mu)_\bullet \iota_j^* G \to H(j)$$

or equivalently for all $\mu : \alpha(i) \to j$ by morphisms

$$F(\mu)_\bullet G(i) \to H(j).$$

In particular, if μ is the identity of $\alpha(i)$, we get morphisms

$$G(i) \to H(\alpha(i))$$

which constitute a morphism of diagrams $G \to \alpha^* H$. One checks that these associations are inverse to each other. $\square$

Lemma 4.7.27. *Let $\alpha : I \to J$ be a morphism of directed diagrams and let j be an object of J. The functor $\iota_j^* : \mathcal{D}_I \to \mathcal{D}_{I \times_{/J} j}$ respects cofibrations and trivial cofibrations.*

Proof. This follows easily from the fact that ι_j induces a canonical identification

$$I_i = (I \times_{/J} j)_\mu$$

for any $\mu = (i, \alpha(i) \to j)$. For this implies that we have a canonical isomorphism $L_i G \cong L_\mu \iota_j^* G$. □

Lemma 4.7.28. *The bifibration of multicategories, defined in 4.7.6*

$$\mathrm{Fun}(I, \mathcal{D}) \to \mathrm{Fun}(I, \mathcal{S}) = \mathbb{S}(I)$$

equipped with the model-category structures constructed in 4.7.18 is a bifibration of multi-model-categories in the sense of 4.7.3.

Proof. First for each multimorphism of diagrams $f \in \mathrm{Hom}_{\mathcal{S}}(S_1, \dots, S_n; T)$ we have to see that the push-forward and the various pull-backs form a Quillen adjunction in n variables. The case $n = 1$ has been treated above. We only work out the case $n = 2$, the proof for higher n being similar. It suffices to check the following: for any cofibration $\mathcal{E}_1 \to \mathcal{E}_1'$ and for any fibration $\mathcal{F} \to \mathcal{F}'$ the dotted induced morphism in the following diagram

$$\begin{tikzcd} f^{\bullet,2}(\mathcal{E}_1'; \mathcal{F}) \arrow[r, dotted] & \text{pull-back} \arrow[r] \arrow[d] & f^{\bullet,2}(\mathcal{E}_1'; \mathcal{F}') \arrow[d] \\ & f^{\bullet,2}(\mathcal{E}_1; \mathcal{F}) \arrow[r] & f^{\bullet,2}(\mathcal{E}_1; \mathcal{F}') \end{tikzcd}$$

is a fibration. Since fibrations are defined point-wise and fibered products are computed point-wise, we have only to see that the assertion holds point-wise. Now $\mathcal{F} \to \mathcal{F}'$ is a point-wise fibration and $\mathcal{E}_1 \to \mathcal{E}_1'$ is a Reedy cofibration, so by the reasoning in the proof of Lemma 4.7.22 it is in particular a point-wise cofibration. Hence the assertion holds because of the assumption that $\mathcal{D} \to \mathcal{S}$ is a bifibration of multi-model-categories (4.7.3). The requested property for the 0-ary push-forward is easier and is left to the reader. □

Proposition 4.7.29. *The functor $\mathbb{D}(I) \to \mathbb{S}(I)$ defined in 4.7.2 is a bifibration of multicategories whose fibers are equivalent to $\mathcal{D}_F[\mathcal{W}_F^{-1}]$. The pull-back and push-forward functors are given by the left derived functors of $f_\bullet$, and by the right derived functors of $f^{\bullet,j}$, respectively.*

Proof. We have seen in 4.7.28 that the fibers of $\mathrm{Fun}(I, \mathcal{D}) \to \mathbb{S}(I)$ are a bifibration of multi-model-categories in the sense of 4.7.3. Therefore by Proposition 4.7.9 we get that $\mathbb{D}(I) \to \mathbb{S}(I)$ are bifibered multicategories with the requested properties. □

Proof of Theorem 4.7.5. (Der1) and (Der2) for $\mathbb{D}$ and $\mathbb{S}$ are obvious.
(FDer0 left) and the first part of (FDer0 right) follow from Theorem 4.7.29.
(FDer3 left) follows from 4.7.26.

(FDer4 left): By construction of $\alpha_!$ the natural base-change

$$\operatorname{colim} \mathbb{S}(\mu)_\bullet \iota_j^* G \to j^* \alpha_! G \tag{33}$$

is an isomorphism for the non-derived functors. For the derived functors the same follows because all functors in the equation respect cofibrations and trivial cofibrations and all functors which have to be derived in (33) are left Quillen functors and hence can be derived by composing them with cofibrant replacement.
(FDer3 right) and (FDer4 right) are shown precisely the same way.
(FDer5 left): Fixing a morphism $f \in \operatorname{Hom}(S_1, \dots, S_n; T)$ in $\mathcal{S}$ and objects $\mathcal{E}_2, \dots, \mathcal{E}_n$ over $S_2, \dots, S_n$ we have by Theorem 4.7.29 a push-forward functor

$$\begin{array}{ccc} \mathbb{D}(I \times J)_{p^* S_1} & \to & \mathbb{D}(I \times J)_{p^* T} \\ \mathcal{E}_1 & \mapsto & (p^* f)_\bullet(\mathcal{E}_1, p^*\mathcal{E}_2, \dots, p^*\mathcal{E}_n) \end{array}$$

(we denote it with the same letter as the underived version) which, by (FDer0 left), defines a morphism of pre-derivators

$$\mathbb{D}_{S_1} \to \mathbb{D}_T.$$

We first show that it preserves colimits, i.e. that for $p : J \to \cdot$ we have that for all $\mathcal{E}_1 \in \mathcal{D}_{p^* S_1}(I \times J)$ the natural morphism

$$f_\bullet(p_* \mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n) \to p_* (p^* f)_\bullet(\mathcal{E}_1, p^*\mathcal{E}_2, \cdots, p^*\mathcal{E}_n)$$

(where we wrote p also for the projection $p : I \times J \to I$) is an isomorphism. This is the same as showing that

$$p^* f^{1,\bullet}(\mathcal{E}_1, \dots, \mathcal{E}_n) \to (p^* f)^{1,\bullet}(p^*\mathcal{E}_1, \dots, p^*\mathcal{E}_n)$$

is an isomorphism. This follows from Lemma 4.7.8 because it suffices to check this for the underived functors. Now let $\alpha : I \to J$ be an opfibration. To show that

$$f_\bullet(\alpha_* \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \to \alpha_* (\alpha^* f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

is an isomorphism we may show this point-wise. Indeed, after applying j^* we get

$$(j^* f)_\bullet(j^* \alpha_* \mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to j^* \alpha_* (\alpha^* f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

$$(j^* f)_\bullet(p_* \iota_j^* \mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to p_* \iota_j^* (\alpha^* f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

where $\iota_j : I_j \to I$ is the inclusion of the fiber. Note that the commutative diagram

$$\begin{tikzcd} I_j \arrow[r, "\iota_j"] \arrow[d, "p"'] & I \arrow[d, "\alpha"] \\ j \arrow[r] & J \end{tikzcd}$$

is homotopy exact by Lemma 4.3.23, 2. because α is an opfibration. Finally we get the morphism

$$(j^*f)_\bullet(p_*\iota_j^*\mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to p_*(j^*f)_\bullet(\iota_j^*\mathcal{E}_1, p^*j^*\mathcal{E}_2, \dots, p^*j^*\mathcal{E}_n)$$

which is an isomorphism by the above reasoning.
By Lemma 4.3.8 the full content of (FDer0 right) follows from (FDer5 left) while (FDer5 right) follows from (FDer0 left). □

5 Fibered (2-)multiderivators as (op)fibrations of 2-multicategories

The purpose of this chapter is twofold. Firstly, we explain how a fibered derivator can alternatively also be seen as a certain fibration of 2-multicategories (more precisely as a kind of Wirthmüller context). Secondly, we will define **2-pre-multiderivators**, a 2-categorical analogue of pre-multiderivators, and fibered derivators over such. This will be essential to define **derivator six-functor-formalisms**. In contrast to the articles underlying this thesis, we will work with the 2-categorical version from the beginning for efficiency.

5.1 2-pre-multiderivators

We fix a diagram category Dia (cf. Definition 4.1.1) once and for all. If one wants to specify Dia, one would speak about e.g. 2-pre-multiderivators, or fibered multiderivators, **with domain** Dia. For better readability we omit this. This is justified because all arguments of this chapter are completely formal, not depending on the choice of Dia at all.

Definition 5.1.1. *A* **2-pre-multiderivator** *is a functor* $\mathbb{S} : \mathrm{Dia}^{1-\mathrm{op}} \to$ *2-$\mathcal{MCAT}$ which is strict in 1-morphisms (functors) and pseudo-functorial in 2-morphisms (natural transformations). More precisely, it associates with a diagram I a 2-multicategory $\mathbb{S}(I)$, with a functor $\alpha : I \to J$ a strict functor*

$$\mathbb{S}(\alpha) : \mathbb{S}(J) \to \mathbb{S}(I)$$

denoted also α^*, if $\mathbb{S}$ is understood, and with a natural transformation $\mu : \alpha \Rightarrow \alpha'$ a pseudo-natural transformation

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

such that the following holds:

1. *The association*

$$\mathrm{Fun}(I, J) \to \mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$$

given by $\alpha \mapsto \alpha^*$, resp. $\mu \mapsto \mathbb{S}(\mu)$, is a pseudo-functor (this involves, of course, the choice of further data). Here $\mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$ *is the 2-category of strict 2-functors, pseudo-natural transformations, and modifications.*

2. *(Strict functoriality w.r.t. compositons of 1-morphisms) For functors $\alpha : I \to J$ and $\beta : J \to K$, we have an* equality *of pseudo-functors* $\mathrm{Fun}(I, J) \to \mathrm{Fun}(\mathbb{S}(I), \mathbb{S}(K))$

$$\beta^* \circ \mathbb{S}(-) = \mathbb{S}(\beta \circ -).$$

A **symmetric, resp. braided 2-pre-multiderivator** *is given by the structure of strictly symmetric (resp. braided) 2-multicategory on $\mathbb{S}(I)$ such that the strict functors α^* are equivariant w.r.t. the action of the symmetric groups (resp. braid groups). Similarly we define a* **lax, resp. oplax, 2-pre-multiderivator** *where the same as before holds but where the*

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

are lax (resp. oplax) natural transformations and in 1. "pseudo-natural transformations" is replaced by "lax (resp. oplax) natural transformations".

Definition 5.1.2. *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of 2-pre-multiderivators (resp. lax/oplax 2-pre-multiderivators) is given by a collection of strict 2-functors*

$$p(I) : \mathbb{D}(I) \to \mathbb{S}(I)$$

for each $I \in \mathrm{Dia}$ such that we have $\mathbb{S}(\alpha) \circ p(J) = p(I) \circ \mathbb{D}(\alpha)$ and $\mathbb{S}(\mu) * p(J) = p(I) * \mathbb{D}(\mu)$ for all functors $\alpha : I \to J$, $\alpha' : I \to J$ and natural transformations $\mu : \alpha \Rightarrow \alpha'$ as illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "p(J)"] \arrow[d, bend right, "\mathbb{D}(\alpha)"'] \arrow[d, bend left, "\mathbb{D}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{D}(\mu)}{\Rightarrow}"] & \mathbb{S}(J) \arrow[d, bend right, "\mathbb{S}(\alpha)"'] \arrow[d, bend left, "\mathbb{S}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{S}(\mu)}{\Rightarrow}"] \\
\mathbb{D}(I) \arrow[r, "p(I)"] & \mathbb{S}(I)
\end{tikzcd}$$

Definition 5.1.3. *Given a (lax/oplax) 2-pre-derivator $\mathbb{S}$, we define*

$$\mathbb{S}^{1-\mathrm{op}} : I \mapsto \mathbb{S}(I^{\mathrm{op}})^{1-\mathrm{op}}$$

and given a (lax/oplax) 2-pre-multiderivator $\mathbb{S}$, we define

$$\mathbb{S}^{2-\mathrm{op}} : I \mapsto \mathbb{S}(I)^{2-\mathrm{op}}$$

reversing the arrow in the (lax/oplax) pseudo-natural transformations. I.e. the second operation interchanges lax and oplax 2-pre-multiderivators.

5.1.4. As with usual pre-multiderivators we consider the following axioms:

(Der1) For $I, J \in \mathrm{Dia}$, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence of 2-multicategories. Moreover $\mathbb{D}(\varnothing)$ is not empty.

(Der2) For $I \in \mathrm{Dia}$ the 'underlying diagram' functor

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Fun}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{lax}}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{oplax}}(I, \mathbb{D}(\cdot))$$

is 2-conservative (this means that it is conservative on 2-morphisms and that a 1-morphism α is an equivalence if $\mathrm{dia}(\alpha)$ is an equivalence).

5.1.5. Let $\mathcal{D}$ be a 2-multicategory. We define some (lax/oplax) 2-pre-multiderivators which are called **representable**.
We define a 2-pre-multiderivator associated with $\mathcal{D}$ as

$$\begin{array}{rcl} \mathbb{D} : \mathrm{Dia} & \to & 2\text{-}\mathcal{MCAT} \\ I & \mapsto & \mathrm{Fun}(I, \mathcal{D}) \end{array}$$

where $\mathrm{Fun}(I, \mathcal{D})$ is the 2-multicategory of pseudo-functors, pseudo-natural transformations, and modifications. This is usually considered only if all 2-morphisms in $\mathcal{D}$ are invertible.
We define a lax 2-pre-multiderivator as

$$\begin{array}{rcl} \mathbb{D}^{\mathrm{lax}} : \mathrm{Dia} & \to & 2\text{-}\mathcal{MCAT} \\ I & \mapsto & \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{D}) \end{array}$$

where $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{D})$ is the 2-multicategory of pseudo-functors, *lax* natural transformations, and modifications.
We similarly define an oplax 2-pre-multiderivator $\mathbb{D}^{\mathrm{oplax}}$.

Proposition 5.1.6. *1. If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) of 2-categories then $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) of 2-categories.*

2. *If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration and 2-opfibration of 2-categories then $\mathbb{D}^{\mathrm{lax}}(I) \to \mathbb{S}^{\mathrm{lax}}(I)$ is a 1-fibration and 2-opfibration of 2-categories. If $\mathcal{D} \to \mathcal{S}$ is a 1-opfibration and 2-fibration of 2-multicategories then $\mathbb{D}^{\mathrm{lax}}(I) \to \mathbb{S}^{\mathrm{lax}}(I)$ is a 1-opfibration and 2-fibration of 2-multicategories.*

   If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration and 2-fibration of 2-categories then $\mathbb{D}^{\mathrm{oplax}}(I) \to \mathbb{S}^{\mathrm{oplax}}(I)$ is a 1-fibration and 2-fibration of 2-categories. If $\mathcal{D} \to \mathcal{S}$ is a 1-opfibration and 2-opfibration of 2-multicategories then $\mathbb{D}^{\mathrm{oplax}}(I) \to \mathbb{S}^{\mathrm{oplax}}(I)$ is a 1-opfibration and 2-opfibration of 2-multicategories.

3. *If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-isofibration of 2-multicategories with* complete *1-categorical fibers then $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-bifibration and 2-isofibration of 2-multicategories.*

The proof will be sketched in section 5.6.

5.2 Fibered (2-)multiderivators as (op)fibrations of 2-multicategories

Let Dia be a diagram category (cf. Definition 4.1.1). Assume that strictly associative fiber products have been chosen in Dia. Assume also for this section that Dia permits arbitrary Grothendieck constructions, i.e. if I is in Dia and $F : I \to \text{Dia}$ is a pseudo-functor, then $\int F$ is in Dia.
In this section we will define a category Dia^{cor} of correspondences in Dia similarly to the category of correspondences in a usual category considered in Section 3.1. A Wirthmüller context over Dia^{cor} in a similar way as defined in Section 3.1 will be essentially equivalent to a closed monoidal derivator with domain Dia (without the axioms (Der1) and (Der2)). Also the more general notion of fibered multiderivator developed in Chapter 4 can be easily encoded as a certain (op)fibration of 2-multicategories. Since Dia is a 2-category, the definition of Dia^{cor} is a bit more involved.

Definition 5.2.1. *Let $I_1, \dots, I_n, J$ be diagrams in* Dia*. Define* $\text{Cor}(I_1, \dots, I_n; J)$ *to be the following strict 2-category:*

1. *The objects are diagrams of the form*

$$\begin{tikzcd} & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\ I_1 & \cdots & I_n & ; & J \end{tikzcd}$$

   with $A \in \text{Dia}$.

2. *The 1-morphisms $(A, \alpha_1, \dots, \alpha_n, \beta) \Rightarrow (A', \alpha_1', \dots, \alpha_n', \beta')$ are functors $\gamma : A \to A'$ and natural transformations $\nu_1, \dots, \nu_n, \mu$:*

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[rd, "\alpha_i"'] & \Rightarrow^{\nu_i} & A' \arrow[ld, "\alpha_i'"] \\ & I_i & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[rd, "\beta"'] & \Leftarrow^{\mu} & A' \arrow[ld, "\beta'"] \\ & J & \end{tikzcd}$$

3. *The 2-morphisms are natural transformations $\eta : \gamma \Rightarrow \gamma'$ such that $(\alpha_i' * \eta) \circ \nu_i = \nu_i'$ and $(\beta' * \eta) \circ \mu' = \mu$ hold.*

We define also the full subcategory $\text{Cor}^F(I_1, \dots, I_n; J)$ of those objects for which $\alpha_1 \times \cdots \times \alpha_n : A \to I_1 \times \cdots \times I_n$ is a fibration and β is an opfibration. The γ's do not need to be morphisms of fibrations, respectively of opfibrations.

5.2.2. For a 2-category $\mathcal{C}$, denote by $\tau_1(\mathcal{C})$ the 1-category in which the morphism sets or classes are the π_0 (sets or classes of connected components) of the respective categories of 1-morphisms in $\mathcal{C}$.

Definition 5.2.3. *We define the* **2-multicategory of correspondences of diagrams** Dia^{cor} *as the following 2-multicategory:*

1. *The objects are diagrams $I \in \mathrm{Dia}$.*

2. *For every $I_1, \dots, I_n, J$ diagrams in* Dia*, the category* $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}}(I_1, \dots, I_n; J)$ *of 1-morphisms of* $\mathrm{Dia}^{\mathrm{cor}}$ *is the truncated category* $\tau_1(\mathrm{Cor}^F(I_1, \dots, I_n; J))$.

Composition is defined by taking fiber products. The diagram (forgetting the functor to J_i)

$$\begin{tikzcd}
 & & & & & A \times_{J_i} B \arrow[dll] \arrow[drr] & & & & & \\
 & & & A \arrow[dlll] \arrow[dl] \arrow[drrr, "\beta_A"] & & & & B \arrow[dlll] \arrow[dl, "\alpha_{B,i}"'] \arrow[dr] \arrow[drrr] & & & \\
I_1 & \cdots & I_n & ; & J_1 & \cdots & J_i & \cdots & J_m & ; & K
\end{tikzcd}$$

is defined to be the composition of the left hand side correspondence in $\mathrm{Hom}(I_1, \dots, I_n; J_i)$ with the right hand side correspondence in $\mathrm{Hom}(J_1, \dots, J_m; K)$. One checks that $A \times_{J_i} B \to J_1 \times \cdots \times J_{i-1} \times I_1 \times \cdots \times I_n \times J_{i+1} \times \cdots \times J_m$ is again a opfibration and that $A \times_{J_i} B \to K$ is again a fibration. It remains to be seen that the composition is functorial in 2-morphisms and that the relations in π_0 are respected. This follows from the following

Lemma 5.2.4. *The fiber product construction above defines a pseudo-functor of 2-categories*

$$\mathrm{Cor}^F(I_1, \dots, I_n; J_i) \times \mathrm{Cor}^F(J_1, \dots, J_m; K) \to \mathrm{Cor}^F(J_1, \dots, J_{i-1}, I_1, \dots, I_n, J_{i+1}, \dots, J_m; K)$$

Proof. By assumption the functor β_A is an opfibration and the functor $\alpha_{B,1} \times \cdots \times \alpha_{B,m}$ is a fibration for all objects $(A, \alpha_{A,1}, \dots, \alpha_{A,n}, \beta_A) \times (B, \alpha_{B,1}, \dots, \alpha_{B,m}, \beta_B)$ of the source 2-category. We choose associated pseudo-functors denoted by $- \mapsto (-)_\bullet$, resp. $- \mapsto (-)^\bullet$. A 1-morphism $(\gamma_A, \nu_{A,1}, \dots, \nu_{A,n}, \mu_A) \times (\gamma_B, \nu_{B,1}, \dots, \nu_{B,n}, \mu_B)$ is sent to the following 1-morphism: We have a well-defined coCartesian morphism (in the first row lying over the second row) w.r.t. $\beta_A : A \to J_i$

$$\begin{tikzcd}
\gamma_A(a) \arrow[r] \arrow[d, mapsto] & \mu_A(a)_\bullet \gamma_A(a) \arrow[d, mapsto] \\
\beta_{A'}(\gamma_A(a)) \arrow[r, "\mu_A(a)"] & \beta_A(a) = \alpha_{B,i}(b)
\end{tikzcd}$$

and a well-defined Cartesian morphism (in the first row lying over the second row) w.r.t. $(\alpha_{B,1}, \dots, \alpha_{B,m}) : B \to J_1 \times \cdots \times J_m$:

$$\begin{tikzcd}
(\mathrm{id}, \dots, \nu_{B,i}(b), \dots, \mathrm{id})^\bullet \gamma_B(b) \arrow[r] \arrow[d, mapsto] & \gamma_B(b) \arrow[d, mapsto] \\
\alpha_{B',1}(b)\gamma_B(b), \dots, \alpha_{B,i}(b), \dots, \alpha_{B',m}(b)\gamma_B(b) \arrow[r, "{\mathrm{id}, \dots, \nu_{B,i}(b), \dots, \mathrm{id}}"] & \alpha_{B',1}(b)\gamma_B(b), \dots, \alpha_{B',m}(b)\gamma_B(b)
\end{tikzcd}$$

Using these (co)Cartesian morphisms we define a functor

$$\gamma_A \times \gamma_B : A \times_{J_i} B \to A' \times_{J_i} B'$$

given by

$$(a, b) \mapsto (\mu_A(a)_{\bullet}\gamma_A(a), (\mathrm{id}, \dots, \nu_{B,i}(b), \dots, \mathrm{id})^{\bullet}\gamma_B(b)).$$

The required natural transformations $\nu_1, \dots, \nu_{m+n-1}, \mu$ are given as follows: We have a 2-commutative diagram

$$\begin{tikzcd}
A \times_{J_j} B \arrow[r] \arrow[dd, "\gamma_A \times \gamma_B"'] & A \arrow[dr, "\alpha_{A,j}"] \arrow[dd, phantom, "\Downarrow"] & \\
& & I_j \\
A' \times_{J_j} B' \arrow[r] & A' \arrow[ur, "\alpha_{A',j}"'] &
\end{tikzcd}$$

where the 2-morphism is given by the composition

$$\alpha_{A,i}(a) \to \alpha_{A',i}(\gamma_A(a)) \to \alpha_{A',j}(\mu_A(a)_{\bullet}\gamma_A(a))$$

We have a 2-commutative diagram for $j \neq i$:

$$\begin{tikzcd}
A \times_{J_i} B \arrow[r] \arrow[dd, "\gamma_A \times \gamma_B"'] & B \arrow[dr, "\alpha_{B,j}"] \arrow[dd, phantom, "\Downarrow"] & \\
& & J_j \\
A' \times_{J_i} B' \arrow[r] & B' \arrow[ur, "\alpha_{B',j}"'] &
\end{tikzcd}$$

where the 2-morphism is given by

$$\alpha_{B,j}(b) \to \alpha_{B',j}(\gamma_B(b)) = \alpha_{B',j}((\mathrm{id}, \dots, \nu_{B,i}(b), \dots, \mathrm{id})^{\bullet}\gamma_B(b))$$

We have a 2-commutative diagram:

$$\begin{tikzcd}
A \times_{J_i} B \arrow[r] \arrow[dd, "\gamma_A \times \gamma_B"'] & B \arrow[dr, "\beta_{B'}"] \arrow[dd, phantom, "\Uparrow"] & \\
& & K \\
A' \times_{J_i} B' \arrow[r] & B' \arrow[ur, "\beta_{B'}"'] &
\end{tikzcd}$$

where the 2-morphism is given by the composition

$$\beta_{B'}((\mathrm{id}, \dots, \nu_{B,i}(b), \dots, \mathrm{id})^{\bullet}\gamma_B(b)) \to \beta_{B'}(\gamma_B(b)) \to \beta_B(b).$$

A 2-morphism given by a pair $\kappa_A : \gamma_A \Rightarrow \gamma'_A$ and $\kappa_B : \gamma_B \Rightarrow \gamma'_B$ is sent to the natural transformation

$$\gamma_A \times \gamma_B \Rightarrow \gamma'_A \times \gamma'_B$$

given by the dotted maps in the *commuting* diagrams

$$\begin{tikzcd}
\gamma_A(a) \arrow[r] \arrow[ddd, "\kappa_A(a)"'] & \mu_A(a)_\bullet\gamma_A(a) \arrow[d] \\
 & \kappa_A(a)_\bullet\mu_A(a)_\bullet\gamma_A(a) \arrow[d, "\sim"] \\
 & \mu'_A(a)_\bullet\gamma_A(a) \arrow[d, dotted] \\
\gamma'_A(a) \arrow[r] & \mu'_A(a)_\bullet\gamma'_A(a)
\end{tikzcd}
\qquad
\begin{tikzcd}
\gamma_B(b) \arrow[ddd, "\kappa_B(b)"'] & (\mathrm{id},\dots,\nu_{B,i}(b),\dots,\mathrm{id})^\bullet\gamma_B(b) \arrow[l] \\
 & (\mathrm{id},\dots,\kappa_B(b),\dots,\mathrm{id})^\bullet(\mathrm{id},\dots,\nu_{B,i}(b),\dots,\mathrm{id})^\bullet\gamma_B(b) \arrow[u] \\
 & (\mathrm{id},\dots,\nu'_{B,i}(b),\dots,\mathrm{id})^\bullet\gamma_B(b) \arrow[u, "\sim"'] \arrow[d, dotted] \\
\gamma'_B(b) & (\mathrm{id},\dots,\nu'_{B,i}(b),\dots,\mathrm{id})^\bullet\gamma'_B(b) \arrow[l]
\end{tikzcd}$$

We leave it to the reader to check that this defines indeed a pseudo-functor (this follows easily because the used push-forward and pull-back functors form a pseudo-functor with source J_i, resp. $J_1 \times \cdots \times J_m$) and that all relevant diagrams commute. □

We could also have used $\tau_1(\mathrm{Cor}(I_1,\dots,I_n;J))$ (without the restriction ^{F}) in the definition of $\mathrm{Dia}^{\mathrm{cor}}$ and defined composition involving the comma category. This leads only to a bicategory which, however, is equivalent to the present strict one (cf. Corollary 5.2.7 and the discussion thereafter). The composition pseudo-functor is a bit easier to describe in that case.

5.2.5. Recall the procedure from [Cis03, §1.3.1] to associate with a pseudo-functor $F : I^{\mathrm{op}} \times J \to \mathrm{Dia}$, a category

$$\begin{tikzcd}
 & \int \nabla F = \nabla \int F \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
I & & J
\end{tikzcd}$$

such that α is a fibration and β is an opfibration. This is done by applying the Grothendieck construction, and its dual, respectively, to the two variables separately (cf. 2.4.14, 2.4.15). Explicitly, the category $\int \nabla F$ has the objects $(i, j, X \in F(i,j))$ and the morphisms $(i, j, X \in F(i,j)) \to (i', j', X' \in F(i',j'))$ are triples consisting of morphisms $a : i \to i'$ and $b : j \to j'$ and a morphism $F(\mathrm{id}_i, b)X \to F(a, \mathrm{id}_j)X'$. The pseudo-functors $F : I^{\mathrm{op}} \times J \to \mathrm{Dia}$ form a 2-category $\mathrm{Fun}(I^{\mathrm{op}} \times J, \mathrm{Dia})$ consisting of pseudo-functors, pseudo-natural transformations and modifications.

Proposition 5.2.6. *There is a pair of pseudo-functors*

$$\mathrm{Fun}(I_1^{\mathrm{op}} \times \cdots \times I_n^{\mathrm{op}} \times J, \mathrm{Dia}) \underset{\Pi}{\overset{\Xi}{\rightleftarrows}} \mathrm{Cor}(I_1,\dots,I_n;J)$$

such that there are morphisms in the 2-category of endofunctors of $\mathrm{Cor}(I_1,\dots,I_n;J)$

$$\Xi\circ\Pi \rightleftarrows \mathrm{id}_{\mathrm{Cor}(I_1,\dots,I_n;J)}$$

which are inverse to each other up to chains of 2-morphisms, and such that there are morphisms in the 2-category of endofunctors of $\mathrm{Fun}(I_1^{\mathrm{op}}\times\cdots\times I_n^{\mathrm{op}}\times J,\mathrm{Dia})$

$$\Pi\circ\Xi \rightleftarrows \mathrm{id}_{\mathrm{Fun}(I_1^{\mathrm{op}}\times\cdots\times I_n^{\mathrm{op}}\times J,\mathrm{Dia})}$$

which are inverse to each other up to chains of 2-morphisms.

Proof. The pseudo-functor Ξ is defined as follows: A pseudo-functor $F\in\mathrm{Fun}(I_1^{\mathrm{op}}\times\cdots\times I_n^{\mathrm{op}}\times J,\mathrm{Dia})$ is sent to the category $\int\nabla F$ defined above, which comes equipped with a fibration to $I_1\times\cdots\times I_n$ and an opfibration to J. The fact that these are a fibration, and an opfibration, respectively, does not play any role for this proposition, however. A natural transformation $\mu:F\to G$ is sent to the obvious functor $\widetilde{\mu}:\int\nabla F\to\int\nabla G$. A modification $\mu\Rightarrow\mu'$ induces a natural transformation $\widetilde{\mu}\Rightarrow\widetilde{\mu}'$ which whiskered with any of the projections to the I_k or to J gives an identity.
Π is defined as follows: A correspondence $(A,\alpha_1,\dots,\alpha_n,\beta)$ in $\mathrm{Cor}(I_1,\dots,I_n;J)$ is sent to the following functor:

$$\begin{array}{rcl} I_1^{\mathrm{op}}\times\cdots\times I_n^{\mathrm{op}}\times J & \to & \mathrm{Dia} \\ (i_1,\dots,i_n,j) & \mapsto & \{(i_1,\dots,i_n)\}\times_{/(I_1\times\cdots\times I_n)}A\times_{/J}\{j\} \end{array}$$

A 1-morphism given by $\gamma:A\to A'$ and $\nu_1,\dots,\nu_n,\mu$, respectively, induces functors

$$\widetilde{\gamma}(i_1,\dots,i_n;j):\{(i_1,\dots,i_n)\}\times_{/(I_1\times\cdots\times I_n)}A\times_{/J}\{j\}\to\{(i_1,\dots,i_n)\}\times_{/(I_1\times\cdots\times I_n)}A'\times_{/J}\{j\}$$

which assemble to a pseudo-natural transformation. A 2-morphism $\mu:\gamma\Rightarrow\gamma'$ induces a natural transformation between the corresponding functors $\widetilde{\gamma}(i_1,\dots,i_n;j)\Rightarrow\widetilde{\gamma}(i_1,\dots,i_n;j)$ which assemble to a modification.
We now proceed to construct the required 1-morphisms: $\Pi\circ\Xi$ maps a functor F to the functor

$$F:(i_1,\dots,i_n,j)\mapsto\{(i_1,\dots,i_n)\}\times_{/(I_1\times\cdots\times I_n)}(\int\nabla F)\times_{/J}\{j\}.$$

Pointwise the required natural transformation $\mathrm{id}\to\Pi\circ\Xi$ is given by sending an object X of $F(i_1,\dots,i_n;j)$ to the object $(i_1,\dots,i_n,j,X)$ of $\int\nabla F$ together with the various identities $\mathrm{id}_{i_1},\dots,\mathrm{id}_{i_n},\mathrm{id}_j$. Pointwise the required natural transformation $\Pi\circ\Xi\to\mathrm{id}$ is given by sending an object $(i_1',\dots,i_n',j',X\in F(i_1',\dots,i_n',j'))$ of $\int\nabla F$ together with $\alpha_k:i_k\to i_k'$ and $\beta:j'\to j$ to $F(\alpha_1,\dots,\alpha_n;\beta)X\in F(i_1,\dots,i_n;j)$. One easily checks that these natural transformations even constitute an adjunction in the 2-category of endofunctors of $\mathrm{Fun}(I_1^{\mathrm{op}}\times\dots\times I_n^{\mathrm{op}}\times J,\mathrm{Dia})$.
The pseudo-functor $\Xi\circ\Pi$ is given by

$$(A,\alpha_1,\dots,\alpha_n,\beta)\mapsto(I_1\times\cdots\times I_n\times_{/(I_1\times\cdots\times I_n)}A\times_{/J}J;\mathrm{pr}_{I_1},\dots,\mathrm{pr}_{I_n},\mathrm{pr}_J)$$

together with the various projections. First we will construct an adjunction of $\Xi \circ \Pi$ with the pseudo-functor

$$(A, \alpha_1, \dots, \alpha_n, \beta) \mapsto (I_1 \times \cdots \times I_n \times_{/(I_1 \times \cdots \times I_n)} A; \mathrm{pr}_{I_1}, \dots, \mathrm{pr}_{I_n}, \beta).$$

In one direction we have the functor which complements an object $(a, \dots)$ by the identity $\mathrm{id}_{\beta(a)}$. In the other direction we have the forgetful functor, forgetting $\beta(a) \to j$. Those two functors form an adjunction in the 2-category of endofunctors of $\mathrm{Cor}(I_1, \dots, I_n; J)$. Similarly we have an adjunction between

$$(A, \alpha_1, \dots, \alpha_n, \beta) \mapsto (I_1 \times \cdots \times I_n \times_{/(I_1 \times \cdots \times I_n)} A; \mathrm{pr}_{I_1}, \dots, \mathrm{pr}_{I_n}, \beta)$$

and the identity

$$(A, \alpha_1, \dots, \alpha_n, \beta) \mapsto (A, \alpha_1, \dots, \alpha_n, \beta).$$

$\square$

Observe that the functor Ξ actually has values in the full subcategory $\mathrm{Cor}^F(I_1, \dots, I_n; J)$.

Corollary 5.2.7. *We have equivalences of categories (cf. 5.2.2):*

$$\tau_1(\mathrm{Fun}(I_1^{\mathrm{op}} \times \cdots \times I_n^{\mathrm{op}} \times J, \mathrm{Dia})) \cong \tau_1(\mathrm{Cor}(I_1, \dots, I_n; J)) \cong \tau_1(\mathrm{Cor}^F(I_1, \dots, I_n; J)),$$

Hence we could have defined the 2-multicategory $\mathrm{Dia}^{\mathrm{cor}}$ (as a bimulticategory) using any of these three models for the categories of 1-morphisms. The composition of 1-morphisms looks as follows in these three models:

1. Using $\tau_1(\mathrm{Cor}^F(I_1, \dots, I_n; J))$ we get the composition as defined before:

$$\begin{tikzcd}[column sep=small]
 & & A \arrow[lld] \arrow[ld] \arrow[rd] & & & & B \arrow[lld] \arrow[ld] \arrow[rd] & \\
I_1 & \cdots \quad I_n & ; & J_i \ \circ_i \ J_1 & \cdots & J_m & ; & K
\end{tikzcd}$$

$$= \begin{tikzcd}[column sep=small]
 & & & & A \times_{J_i} B \arrow[llll] \arrow[lll] \arrow[ld] \arrow[rd] \arrow[rrr] & & & \\
J_1 & \cdots \ I_1 & \dots & I_n & & \dots & J_m \ ; & K
\end{tikzcd}$$

2. Using $\tau_1(\mathrm{Cor}(I_1, \dots, I_n; J))$ the composition involves the comma category:

$$\begin{tikzcd}[column sep=small]
 & & A \arrow[lld] \arrow[ld] \arrow[rd] & & & & B \arrow[lld] \arrow[ld] \arrow[rd] & \\
I_1 & \cdots \quad I_n & ; & J_i \ \circ_i \ J_1 & \cdots & J_m & ; & K
\end{tikzcd}$$

$$= \begin{tikzcd}[column sep=small]
 & & & & A \times_{/J_i} B \arrow[llll] \arrow[lll] \arrow[ld] \arrow[rd] \arrow[rrr] & & & \\
J_1 & \cdots \ I_1 & \dots & I_n & & \dots & J_m \ ; & K
\end{tikzcd}$$

3. Using $\tau_1(\mathrm{Fun}(I_1^{\mathrm{op}} \times \cdots \times I_n^{\mathrm{op}} \times J, \mathrm{Dia}))$ we get for pseudo-functors

$$F : I_1^{\mathrm{op}} \times \cdots \times I_n^{\mathrm{op}} \times J_i \to \mathrm{Dia} \qquad G : J_1^{\mathrm{op}} \times \cdots \times J_m^{\mathrm{op}} \times K \to \mathrm{Dia}$$

that

$$G \circ_i F = \mathrm{hocoend}_{J_i} G \times F,$$

where hocoend is defined in Definition 5.2.8 below.

All these compositions are compatible with the equivalences of Corollary 5.2.7. However, only using the model $\mathrm{Cor}^F(I_1, \dots, I_n; J)$ we get strict associativity and the existence of identities.

Definition 5.2.8. *Let J be a diagram and let $F : J^{\mathrm{op}} \times J \to \mathrm{Dia}$ be a pseudo-functor. We define the diagram $\mathrm{hocoend}_J F$ as the category whose objects are the pairs (j, x) with $j \in J$ and $x \in F(j, j)$ and whose morphisms $(\alpha, \gamma); (j, x) \to (j', x')$ are the pairs consisting of a morphisms $\alpha : j \to j'$ and a morphism $\gamma : F(\mathrm{id}_j, \alpha)x \to F(\alpha, \mathrm{id}_{j'})x'$. The composition of two morphisms $(\alpha, \gamma); (j, x) \to (j', x')$ and $(\alpha', \gamma'); (j', x') \to (j'', x'')$ is defined by $(\alpha', \gamma') \circ (\alpha, \gamma) = (\alpha'\alpha, (F(\alpha, \mathrm{id}_{j''})\gamma') \circ (F(\mathrm{id}_j, \alpha')\gamma))$.*

Proposition 5.2.9. *1. There is a pseudo-functor of 2-multicategories*

$$\mathrm{Dia}^{2-\mathrm{op}} \to \mathrm{Dia}^{\mathrm{cor}}$$

where $\mathrm{Dia}^{2-\mathrm{op}}$ is turned into a 2-multicategory by setting

$$\mathrm{Hom}_{\mathrm{Dia}^{2-\mathrm{op}}}(I_1, \dots, I_n; J) := \mathrm{Hom}(I_1 \times \cdots \times I_n, J)^{\mathrm{op}}.$$

2. There is a pseudo-functor of 2-multicategories

$$\mathrm{Dia}^{1-\mathrm{op}} \to \mathrm{Dia}^{\mathrm{cor}}$$

where $\mathrm{Dia}^{1-\mathrm{op}}$ is turned into a 2-multicategory by setting

$$\mathrm{Hom}_{\mathrm{Dia}^{1-\mathrm{op}}}(I_1, \dots, I_n; J) := \mathrm{Fun}(J, I_1) \times \cdots \times \mathrm{Fun}(J, I_n).$$

In particular for any $I \in \mathrm{Dia}$ there is a natural pseudofunctor of 2-multicategories

$$\{\cdot\} \to \mathrm{Dia}^{\mathrm{cor}}$$

with value I.

Proof. The functor

$$\mathrm{Dia}^{2-\mathrm{op}} \to \mathrm{Dia}^{\mathrm{cor}}$$

is the identity on objects. A 1-morphism $\alpha \in \mathrm{Hom}(I_1 \times \cdots \times I_n, J)$ is mapped to the correspondence

$$\begin{tikzcd} & I_1 \times \cdots \times I_n \times_{/J} J \arrow[dl] \arrow[d] \arrow[dr] & \\ I_1 \quad \cdots & I_n \quad ; & J \end{tikzcd}$$

and a 2-morphism $\mu : \alpha \to \alpha'$ to the morphism $(I_1 \times \cdots \times I_n) \times_{/\alpha',J} J \to (I_1 \times \cdots \times I_n) \times_{/\alpha,J} J$ induced by μ. Note that the projections from $I_1 \times \cdots \times I_n \times_{/J} J$ to $I_1 \times \cdots \times I_n$, and to J, are respectively an opfibration, and a fibration.
To establish the pseudo-functoriality, we have to show that there is a natural isomorphism of correspondences between

$$(I_1 \times \cdots \times I_n) \times_{/J_i} J_i \times_{J_i} (J_1 \times \cdots \times J_m) \times_{/K} K$$
$$= (I_1 \times \cdots \times I_n) \times_{/J_i} (J_1 \times \cdots \times J_m) \times_{/K} K$$

and

$$(J_1 \times \cdots \times J_{i-1} \times I_1 \times \cdots \times I_n \times J_{i+1} \times \cdots \times J_m) \times_{/K} K$$

in $\tau_1(\mathrm{Cor}^F(J_1, \dots, J_{i-1}, I_1, \dots, I_n, J_{i+1}, \dots, J_m; K))$. One checks that there is even an adjunction between the two categories which establishes this isomorphism.
The pseudo-functor

$$\mathrm{Dia}^{1-\mathrm{op}} \to \mathrm{Dia}^{\mathrm{cor}}$$

sends a multimorphism given by $\{\alpha_k : J \to I_k\}$ to the correspondence

$$\begin{tikzcd}
& & I_1 \times \cdots \times I_n \times_{/(I_1 \times \cdots \times I_n)} J \arrow[dll] \arrow[d] \arrow[drr] & & \\
I_1 & \cdots & I_n & ; & J
\end{tikzcd}$$

To establish the pseudo-functoriality, we have to show that there is a natural isomorphism of correspondences between

$$(I_1 \times \cdots \times I_n) \times_{/(I_1 \times \cdots \times I_n)} J_i \times_{J_i} (J_1 \times \cdots \times J_m) \times_{/(J_1 \times \cdots \times J_m)} K$$
$$= (I_1 \times \cdots \times I_n) \times_{/(I_1 \times \cdots \times I_n)} (J_1 \times \cdots \times J_m) \times_{/(J_1 \times \cdots \times J_m)} K$$

and

$$(J_1 \times \cdots \times J_{i-1} \times I_1 \times \cdots \times I_n \times J_{i+1} \times \cdots \times J_m) \times_{(J_1 \times \cdots \times J_{i-1} \times I_1 \times \cdots \times I_n \times J_{i+1} \times \cdots \times J_m)} K$$

in $\tau_1(\mathrm{Cor}^F(J_1, \dots, J_{i-1}, I_1, \dots, I_n, J_{i+1}, \dots, J_m; K))$. One checks that there is even an adjunction between the two categories which establishes this isomorphism.
The requested pseudo-functor

$$\{\cdot\} \to \mathrm{Dia}^{2-\mathrm{op}}$$

with value I is given by the composition of the obvious pseudo-functor $\{\cdot\} \to \mathrm{Dia}^{1-\mathrm{op}}$, sending the unique multimorphism in $\mathrm{Hom}(\cdot, \dots, \cdot; \cdot)$ to $\{\mathrm{id}_I\}_{i=1..n}$, with the previous pseudo-functor $\mathrm{Dia}^{1-\mathrm{op}} \to \mathrm{Dia}^{\mathrm{cor}}$. $\square$

Proposition 5.2.10. *The 2-multicategory* $\mathrm{Dia}^{\mathrm{cor}}$ *is (strictly symmetric) 1-bifibered and (trivially) 2-bifibered over* $\{\cdot\}$ *hence it is a (strictly symmetric) monoidal 2-category with monoidal structure represented by*

$$I \otimes J = I \times J$$

and internal hom

$$\mathcal{HOM}(I, J) = I^{\mathrm{op}} \times J$$

with unit given by the final diagram $\{\cdot\}$*. In particular every object is dualizable w.r.t. the final diagram and the duality functor is* $I \mapsto I^{\mathrm{op}}$ *on the objects, while on the morphism categories it is given by the composition of equivalences:*

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}}(J^{\mathrm{op}}, I^{\mathrm{op}}) \cong \tau_1(\mathrm{Fun}(J \times I^{\mathrm{op}}, \mathrm{Dia})) = \tau_1(\mathrm{Fun}(I^{\mathrm{op}} \times J, \mathrm{Dia})) \cong \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}}(I, J).$$

Proof. By Corollary 5.2.7 we have equivalences

$$\tau_1(\mathrm{Cor}(I_1, I_2; J)) \cong \tau_1(\mathrm{Fun}(I_1^{\mathrm{op}} \times I_2^{\mathrm{op}} \times J, \mathrm{Dia})) \tag{34}$$

and also

$$\tau_1(\mathrm{Cor}(I_1 \times I_2; J)) \cong \tau_1(\mathrm{Fun}(I_1^{\mathrm{op}} \times I_2^{\mathrm{op}} \times J, \mathrm{Dia})). \tag{35}$$

Obviously the composition of (34) with the inverse of (35) is isomorphic to the canonical equivalence

$$\tau_1(\mathrm{Cor}(I_1, I_2; J)) \to \tau_1(\mathrm{Cor}(I_1 \times I_2; J))$$

given by

$$\left(\begin{tikzcd} & & A \arrow[dll, "\alpha_1"'] \arrow[dl, "\alpha_2"] \arrow[dr, "\beta"] & \\ I_1 & I_2 & ; & J \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & A \arrow[dl, "(\alpha_1,\alpha_2)"'] \arrow[dr, "\beta"] & \\ I_1 \times I_2 & ; & J \end{tikzcd} \right).$$

Furthermore this canonical equivalence preserves the Cor^F-subcategories and is compatible with composition, by definition of the composition by fiber products.
Similarly, by Corollary 5.2.7 again, we have an equivalence

$$\tau_1(\mathrm{Hom}(I_1; I_2^{\mathrm{op}} \times J)) \cong \tau_1(\mathrm{Fun}(I_1^{\mathrm{op}} \times I_2^{\mathrm{op}} \times J, \mathrm{Dia})). \tag{36}$$

Explicitly the equivalence (34) maps a correspondence

$$\begin{tikzcd} & & A \arrow[dll, "\alpha_1"'] \arrow[dl, "\alpha_2"] \arrow[dr, "\beta"] & \\ I_1 & I_2 & ; & J \end{tikzcd}$$

to the functor

$$\begin{aligned} F_\xi : I_1^{\mathrm{op}} \times I_2^{\mathrm{op}} \times J &\to \mathrm{Dia} \\ (i_1, i_2, j) &\mapsto (i_1, i_2) \times_{/I_1 \times I_2} A \times_{/J} j \end{aligned}$$

and the inverse of (36) maps this to

$$\begin{tikzcd} & \int \nabla F_\xi \arrow[dl] \arrow[dr] & \\ I_1 & ; & I_2^{\mathrm{op}} \times J \end{tikzcd}$$

Explicitly the category

$$\int \nabla F_\xi$$

has objects $(i_1, i_2, j, a, \mu_1, \mu_2, \nu)$ where $\mu_1 : i_1 \to \alpha_1(a), \mu_2 : i_2 \to \alpha_2(a), \nu : \beta(a) \to j$. Morphisms $(i_1, i_2, j, a, \mu_1, \mu_2, \nu) \to (i_1', i_2', j', a', \mu_1', \mu_2', \nu')$ are morphisms $i_1 \to i_1', i_2' \to i_2, j \to j', a \to a'$ such that the obvious diagrams commute. This again preserves the Cor^F-subcategories and is compatible with composition. □

5.2.11. We can also investigate how the corresponding Cartesian resp. coCartesian morphisms look like: The trivial correspondence

$$\begin{tikzcd} & I_1 \times I_2 \arrow[dl, equal] \arrow[dr, equal] & \\ I_1 \times I_2 & ; & I_1 \times I_2 \end{tikzcd}$$

corresponds, by the explicit description given in the proof, to the morphism

$$\begin{tikzcd} & I_1 \times I_2 \arrow[dl] \arrow[d] \arrow[drr, equal] & & \\ I_1 & I_2 & ; & I_1 \times I_2 \end{tikzcd}$$

which therefore constitutes the corresponding coCartesian morphism.
The trivial correspondence

$$\begin{tikzcd} & I_1^{\mathrm{op}} \times I_2 \arrow[dl, equal] \arrow[dr, equal] & \\ I_1^{\mathrm{op}} \times I_2 & ; & I_1^{\mathrm{op}} \times I_2 \end{tikzcd}$$

corresponds (up to 2-isomorphism) to the functor

$$\begin{array}{rcl} I_1 \times I_2^{\mathrm{op}} \times I_1^{\mathrm{op}} \times I_2 & \to & \mathrm{Dia} \\ (i_1, i_2', i_1', i_2) & \mapsto & \mathrm{Hom}(i_1', i_1) \times \mathrm{Hom}(i_2', i_2) \end{array}$$

where the image consists of discrete categories. It corresponds (up to 2-isomorphism) to the 1-morphism

$$\begin{tikzcd} & \mathrm{tw}(I_1^{\mathrm{op}}) \times I_2 \times_{/I_2} I_2 \arrow[dl] \arrow[d] \arrow[drr] & & \\ I_1 & I_2 \times I_1^{\mathrm{op}} & ; & I_2 \end{tikzcd}$$

or simply to the 1-morphism

$$\begin{tikzcd} & \mathrm{tw}(I_1^{\mathrm{op}}) \times I_2 \arrow[dl] \arrow[d] \arrow[drr] & & \\ I_1 & I_2 \times I_1^{\mathrm{op}} & ; & I_2 \end{tikzcd}$$

which therefore is the corresponding Cartesian morphism. Here for a category I, the category $\mathrm{tw}(I) = \int \mathrm{Hom}_I(-,-)$ is the twisted arrow category. In particular, the duality morphism in $\mathrm{Hom}(I, I^{\mathrm{op}}; \cdot)$ is given by the multicorrespondence of diagrams:

$$\begin{tikzcd} & \mathrm{tw}(I_1^{\mathrm{op}}) \arrow[dl] \arrow[d] \arrow[drr] & & \\ I_1 & I_1^{\mathrm{op}} & ; & \{\cdot\} \end{tikzcd}$$

5.3 Correspondences of diagrams in a (2-)pre-multiderivator

In the next two sections it is proven that the axioms of a fibered multiderivator can be encoded as a fibration over the category $\mathrm{Dia}^{\mathrm{cor}}$ defined in Section 5.2, and at the same time the notion of fibered multiderivator is extended to bases which are 2-pre-multiderivators instead of pre-multiderivators.
Recall Definition 5.2.1, where $\mathrm{Cor}(I_1, \dots, I_n; J)$ was defined.

Definition 5.3.1. *Let $\mathbb{S}$ be a (lax/oplax) 2-pre-multiderivator (cf. Definition 5.1.1). For each collection $(I_1, S_1), \dots, (I_n, S_n); (J, T)$, where $I_1, \dots, I_n, J$ are diagrams in* Dia *and $S_i \in \mathbb{S}(I_i), T \in \mathbb{S}(J)$ are objects, we define a pseudo-functor*

$$\mathrm{Cor}_{\mathbb{S}} : \mathrm{Cor}(I_1, \dots, I_n; J)^{1-\mathrm{op}} \to \mathcal{CAT}$$

in the oplax case and

$$\mathrm{Cor}_{\mathbb{S}} : \mathrm{Cor}(I_1, \dots, I_n; J)^{1-\mathrm{op},2-\mathrm{op}} \to \mathcal{CAT}$$

in the lax case. $\mathrm{Cor}_{\mathbb{S}}$ *maps a multicorrespondence of diagrams in* Dia

$$\begin{tikzcd} & & & A \arrow[dlll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\ I_1 & \cdots & I_n & & J \end{tikzcd}$$

to the category

$$\mathrm{Hom}_{\mathbb{S}(A)}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T),$$

maps a 1-morphism $(\gamma, \nu_1, \dots, \nu_n, \mu)$ to the functor

$$\rho \mapsto \mathbb{S}(\mu)(T) \circ (\gamma^* \rho) \circ (\mathbb{S}(\nu_1)(S_1), \dots, \mathbb{S}(\nu_n)(S_n))$$

and maps a 2-morphism represented by $\eta : \gamma \Rightarrow \gamma'$ (and such that $(\alpha_i' * \eta) \circ \nu_i = \nu_i'$ and $\mu' \circ (\beta' * \eta) = \mu$) to the morphism

$$\mathbb{S}(\mu)(T)\circ(\gamma^*\rho)\circ(\mathbb{S}(\nu_1)(S_1),\dots,\mathbb{S}(\nu_n)(S_n)) \leftrightarrow \mathbb{S}(\mu')\circ((\gamma')^*\rho)\circ(\mathbb{S}(\nu_1)(S_1),\dots,\mathbb{S}(\nu_n)(S_n)) \tag{37}$$

given as the composition of the isomorphisms

$$\mathbb{S}(\mu)(T) \xrightarrow{\sim} \mathbb{S}(\mu')(T) \circ \underbrace{\mathbb{S}(\beta' * \eta)(T)}_{=\mathbb{S}(\eta)((\beta')^*T)}$$

$$\mathbb{S}(\nu_i)(S_i) \xrightarrow{\sim} \underbrace{\mathbb{S}(\alpha_i' * \eta)(S_i)}_{=\mathbb{S}(\eta)((\alpha_i')^*S_i)} \circ \mathbb{S}(\nu_i)(S_i)$$

with the morphism

$$\mathbb{S}(\eta)((\beta')^*T) \circ (\gamma^*\rho) \leftrightarrow ((\gamma')^*\rho) \circ (\mathbb{S}(\eta)((\alpha_1')^*S_1),\dots,\mathbb{S}(\eta)((\alpha_n')^*S_n)) \tag{38}$$

coming from the fact that $\mathbb{S}(\eta)$ is a (lax/oplax) pseudo-natural transformation $\gamma^* \Rightarrow (\gamma')^*$. The morphisms (37) and (38) point to the left in the lax case and to the right in the oplax case.

Definition 5.3.2. *Let $\mathbb{S}$ be a (lax/oplax) 2-pre-multiderivator. Let $S_i \in \mathbb{S}(I_i)$, for $i = 1,\dots,n$ and $T \in \mathbb{S}(J)$ be objects. Let*

$$\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$$

be the strict 2-category obtained from the pseudo-functor $\mathrm{Cor}_{\mathbb{S}}$ defined in 5.3.1 by the 2-categorical Grothendieck construction (Definition 2.4.14).

Both definitions depend on the choice of Dia, but we do not specify it explicitly.

5.3.3. The category $\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$ defined in 5.3.2 is very important to understand fibered multiderivators. Therefore we explicitly spell out the definition in detail:

1. Objects are a multicorrespondence of diagrams in Dia

$$\begin{tikzcd} & & & A \arrow[dlll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\ I_1 & \cdots & I_n & & J \end{tikzcd}$$

   together with a 1-morphism

$$\rho \in \mathrm{Hom}(\alpha_1^*S_1,\dots,\alpha_n^*S_n;\beta^*T)$$

   in $\mathbb{S}(A)$.

2. The 1-morphisms $(A, \alpha_1, \dots, \alpha_n, \beta, \rho) \to (A', \alpha'_1, \dots, \alpha'_n, \beta', \rho')$ are tuples $(\gamma, \nu_1, \dots, \nu_n, \mu, \Xi)$, where $\gamma : A \to A'$ is a functor, ν_i is a natural transformation in

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "\alpha_i"'] & \arrow[d, phantom, "\overset{\nu_i}{\Rightarrow}"] & A' \arrow[dl, "\alpha'_i"] \\ & I_i & \end{tikzcd}$$

and μ is a natural transformation in

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\overset{\mu}{\Leftarrow}"] & A' \arrow[dl, "\beta'"] \\ & J & \end{tikzcd}$$

and Ξ is a 2-morphism in

$$\begin{tikzcd} \gamma^*(\alpha')^*S \arrow[r, "\gamma^*\rho'"] \arrow[dr, phantom, "\Uparrow\Xi"] & \gamma^*(\beta')^*T \arrow[d, "\mathbb{S}(\mu)"] \\ \alpha^*S \arrow[u, "\mathbb{S}(\nu)"] \arrow[r, "\rho"] & \beta^*T \end{tikzcd}$$

3. The 2-morphisms are the natural transformations $\eta : \gamma \Rightarrow \gamma'$ such that $(\alpha'_i * \eta) \circ \nu_i = \nu'_i$ and $(\beta' * \eta) \circ \mu' = \mu$ and such that following prism-shaped diagram

$$\begin{tikzcd}[column sep=large] \gamma^*(\alpha')^*S \arrow[rrr, "\gamma^*\rho'"] \arrow[dr, "\mathbb{S}(\alpha'*\eta)(S)=\mathbb{S}(\eta)((\alpha')^*S)"] \arrow[drr, phantom, "\Updownarrow\mathbb{S}(\alpha'*\eta)(\rho')"] & & & \gamma^*(\beta')^*T \arrow[dl, "\mathbb{S}(\eta)((\beta')^*T)=\mathbb{S}(\beta'*\eta)(T)"'] \arrow[dd, "\mathbb{S}(\mu)(T)"] \\ \arrow[r, phantom, "\overset{\mathbb{S}_{\eta,\nu}(S)}{\Rightarrow}"] & (\gamma')^*(\alpha')^*S \arrow[r, "(\gamma')^*\rho'"] \arrow[dr, phantom, "\Uparrow\Xi'"] & (\gamma')^*(\beta')^*T \arrow[dr, "\mathbb{S}(\mu')(T)"] \arrow[r, phantom, "\overset{\mathbb{S}_{\mu',\eta}(T)}{\Rightarrow}"] & \\ \alpha^*S \arrow[uu, "\mathbb{S}(\nu)(S)"] \arrow[ur, "\mathbb{S}(\nu')(S)"] \arrow[rrr, "\rho"] & & & \beta^*T \end{tikzcd}$$

is 2-commutative, where the 2-morphism in the front face (not depicted) points upwards and is Ξ. We assumed here $n = 1$ for simplicity. Note that we have $\mathbb{S}(\alpha' * \eta)(S) = \mathbb{S}(\eta)((\alpha')^*S)$ because $\mathbb{S}$ is strictly compatible with composition of 1-morphisms (cf. Definition 5.1.1). Note that the 2-morphism denoted $\Updownarrow$ goes up in the lax case and down in the oplax (and plain) case while the two 'horizontal' 2-morphisms are invertible.

We again define the full subcategory $\mathrm{Cor}^F_{\mathbb{S}}$ insisting that $\alpha_1 \times \cdots \times \alpha_n : A \to I_1 \times \cdots \times I_n$ is a fibration and β is an opfibration.

Lemma 5.3.4. *Let $\mathcal{D} \to \mathcal{S}$ be a 1-fibration and 2-(op)fibration of 2-categories with 1-categorical fibers. Given an adjunction in $\mathcal{S}$*

$$\begin{tikzcd} S \arrow[r, bend left, "F"] & T \arrow[l, bend left, "G"] \end{tikzcd}$$

with counit $G \circ F = \mathrm{id}_S$ being the identity and unit $F \circ G \Rightarrow \mathrm{id}_T$, for any object $\mathcal{E} \in \mathcal{D}_S$ there is an adjunction

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "\widetilde{F}"] & \mathcal{F} \arrow[l, bend left, "\widetilde{G}"] \end{tikzcd}$$

in $\mathcal{D}$, lying over the previous one, where $\widetilde{F}$ and $\widetilde{G}$ are Cartesian.

Proof. We concentrate on the 2-opfibered case and may assume by Proposition 2.4.16 that $\mathcal{D}$ is equal to the Grothendieck construction applied to a pseudo-functor $\Psi : \mathcal{S}^{1-\mathrm{op}} \to \mathcal{CAT}$. We then have corresponding pullback functors $F^\bullet := \Psi(F)$, $G^\bullet := \Psi(G)$ and a 2-isomorphism $\eta : \mathrm{id}_{\Psi(S)} \cong F^\bullet \circ G^\bullet$ and a 2-morphism $\mu : G^\bullet \circ F^\bullet \Rightarrow \mathrm{id}_{\Psi(T)}$ given by the pseudo-functoriality and the contravariant functoriality on 2-morphisms.
We define $\widetilde{G} := (G, \mathrm{id}_{G^\bullet \mathcal{E}}) : G^\bullet \mathcal{E} \to \mathcal{E}$, the canonical Cartesian morphism, and $\widetilde{F} := (F, \eta(\mathcal{E})) : \mathcal{E} \to G^\bullet \mathcal{E}$, which is Cartesian as well, $\eta(\mathcal{E})$ being an isomorphism. There is a 2-isomorphism $\widetilde{G} \circ \widetilde{F} \cong \mathrm{id}_{\mathcal{E}}$, and a 2-morphism $\widetilde{F} \circ \widetilde{G} \to \mathrm{id}_{G^\bullet \mathcal{E}}$ given by $\mu(G^\bullet \mathcal{E})$. One checks that those define unit and counit of an adjunction again.
In the 2-fibered case we set $\widetilde{F} := (F, \eta(\mathcal{E})^{-1}) : \mathcal{E} \to G^\bullet \mathcal{E}$ and may reason analogously. $\square$

Lemma 5.3.5. *Let $p : \mathbb{D} \to \mathbb{S}$ be a strict morphism of (lax/oplax) 2-pre-multiderivators (cf. Definition 5.1.2).*
Consider the strictly commuting diagram of 2-categories and strict 2-functors

$$\begin{tikzcd} \mathrm{Cor}^F_{\mathbb{D}}((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F})) \arrow[r, hook] \arrow[d] & \mathrm{Cor}_{\mathbb{D}}((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F})) \arrow[d] \\ \mathrm{Cor}^F_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))) \arrow[r, hook] \arrow[d] & \mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)) \arrow[d] \\ \mathrm{Cor}^F(I_1,\dots,I_n;J) \arrow[r, hook] & \mathrm{Cor}(I_1,\dots,I_n;J) \end{tikzcd}$$

1. *If the functors $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ induced by p are fibrations, the vertical 2-functors are 1-fibrations with 1-categorical fibers. They are 2-fibrations in the lax case and 2-opfibrations in the oplax case.*

2. *If the functors* $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ *induced by p are fibrations* with discrete fibers, *then the upper vertical 2-functors have discrete fibers.*

3. *Every object in a 2-category on the right hand side is in the image of the corresponding horizontal 2-functor* up to a chain of adjunctions.

Proof. 1. and 2. follow directly from the definition.
3. We first embed the left hand side category, say $\mathrm{Cor}^F_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$, into the full subcategory of $\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$ consisting of those objects $(A,\alpha_1,\dots,\alpha_n,\beta,\rho)$, in which β is an opfibration but the α_i are arbitrary. We will show that every object is connected by an adjunction with an object of this bigger subcategory. By a similar argument one shows that this holds also for the second inclusion.
Consider an arbitrary correspondence ξ' of diagrams in Dia

$$\begin{tikzcd}
 & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\
I_1 & \cdots & I_n & & J
\end{tikzcd}$$

and the 1-morphisms in $\mathrm{Cor}(I_1,\dots,I_n;J)$

$$\begin{tikzcd}
 & & & A \times_{/J} J \arrow[llld, "\alpha_1 \mathrm{pr}_1"'] \arrow[ld, "\alpha_n \mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_1"] \arrow[rd, "\mathrm{pr}_2"] & \\
I_1 & \cdots & I_n & \arrow[r, phantom, "\Rightarrow^{\mu}"] & J \\
 & & & A \arrow[lllu, "\alpha_1"] \arrow[lu, "\alpha_n"'] \arrow[ru, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[dd, "\Delta"] \arrow[rd, "\beta"] & \\
I_1 & \cdots & I_n & & J \\
 & & & A \times_{/J} J \arrow[lllu, "\alpha_1 \mathrm{pr}_1"] \arrow[lu, "\alpha_n \mathrm{pr}_1"'] \arrow[ru, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

One easily checks that $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_A$ and that the obvious 2-morphism $\Delta \circ \mathrm{pr}_1 \Rightarrow \mathrm{id}_{A \times_{/J} J}$ induced by μ define an adjunction in the 2-category $\mathrm{Cor}(I_1,\dots,I_n;J)$. Using Lemma 5.3.4, we get a corresponding adjunction also in the 2-category

$$\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)).$$

□

Lemma 5.3.6. *Let $p: \mathbb{D} \to \mathbb{S}$ be a morphism of (lax/oplax) 2-pre-multiderivators. Consider the following strictly commuting diagram of functors obtained from the one of Lemma 5.3.5 by 1-truncation (cf. 5.2.2):*

$$\begin{tikzcd}
\tau_1(\mathrm{Cor}^F_{\mathbb{D}}((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F}))) \arrow[r, hook] \arrow[d] & \tau_1(\mathrm{Cor}_{\mathbb{D}}((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F}))) \arrow[d] \\
\tau_1(\mathrm{Cor}^F_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))) \arrow[r, hook] \arrow[d] & \tau_1(\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))) \arrow[d] \\
\tau_1(\mathrm{Cor}^F(I_1,\dots,I_n;J)) \arrow[r, hook] & \tau_1(\mathrm{Cor}(I_1,\dots,I_n;J))
\end{tikzcd}$$

1. *The horizontal functors are equivalences.*

2. *If the functors* $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ *induced by* p *are fibrations* with discrete fibers, *then the upper vertical morphisms are fibrations with discrete fibers. Furthermore the top-most horizontal functor maps Cartesian morphisms to Cartesian morphisms.*

Proof. That the horizontal morphisms are equivalences follows from the definition of the truncation and Lemma 5.3.5, 3. If we have a 1-fibration and 2-fibration of 2-categories $\mathcal{D} \to \mathcal{C}$ with *discrete* fibers then the truncation $\tau_1(\mathcal{D}) \to \tau_1(\mathcal{C})$ is again fibered (in the 1-categorical sense). Hence the second assertion follows from Lemma 5.3.5, 2. □

Definition 5.3.7. *Let* $\mathbb{S}$ *be a 2-pre-multiderivator. We define a 2-multicategory* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *equipped with a strict functor*

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathrm{Dia}^{\mathrm{cor}}$$

as follows

1. *The objects of* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *are pairs* (I,S) *consisting of* $I \in \mathrm{Dia}$ *and* $S \in \mathbb{S}(I)$.

2. *The category* $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_1,S_1),\dots,(I_n,S_n);(J,T))$ *of 1-morphisms of* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *is the truncated category* $\tau_1(\mathrm{Cor}^F_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)))$.

Composition is given by the composition of correspondences of diagrams

$$\begin{tikzcd}
 & & & & & A \times_{J_i} B \arrow[dlll, "\mathrm{pr}_1"'] \arrow[drrr, "\mathrm{pr}_2"] & & & & & \\
 & & A \arrow[dll] \arrow[d] \arrow[drrrr, "\beta_A"] & & & & & & B \arrow[dllll] \arrow[dll, "\alpha_{B,i}"'] \arrow[d] \arrow[drr] & & \\
I_1 & \cdots & I_n & ; & J_1 & \cdots & J_i & \cdots & J_m & ; & K
\end{tikzcd}$$

and composing $\rho_A \in \mathrm{Hom}(\alpha_{A,1}^*S_1,\dots,\alpha_{A,n}^*S_n;\beta_A^*T_i)$ *with* $\rho_B \in \mathrm{Hom}(\alpha_{B,1}^*T_1,\dots,\alpha_{B,m}^*T_m;\beta_B^*U)$ *to*

$$(\mathrm{pr}_2^*\,\rho_B) \circ_i (\mathrm{pr}_1^*\,\rho_A).$$

If $\mathbb{S}$ *is symmetric or braided, then there is a natural action of the symmetric, resp. braid groups:*

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_1,S_1),\dots,(I_n,S_n);(J,T)) \to \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_{\sigma(1)},S_{\sigma(1)}),\dots,(I_{\sigma(n)},S_{\sigma(n)});(J,T))$$

involving the corresponding action in $\mathbb{S}$. *This turns* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *into a symmetric, resp. braided 2-multicategory.*

Note that because of the brute-force truncation this category is in general not 2-fibered anymore over Dia^{cor}.
For any strict morphism of 2-pre-multiderivators $p : \mathbb{D} \to \mathbb{S}$ we get an induced strict functor

$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}).$$

5.3.8. Let $\mathbb{S}$ be a pre-multiderivator (not a 2-pre-multiderivator). Recall the definition of $\mathrm{Dia}(\mathbb{S})$ from 4.6.1:

1. The objects of $\mathrm{Dia}(\mathbb{S})$ are the pairs (I, S) where $I \in \mathrm{Dia}$ and $S \in \mathbb{S}(S)$.
2. The 1-morphisms in $\mathrm{Hom}_{\mathrm{Dia}(\mathbb{S})}((I, S); (J, T))$ are pairs (α, f), where $\alpha : I \to J$ is a functor in Dia together with a morphism
$$f : S \to \alpha^* T.$$
3. The 2-morphisms $(\alpha, f) \Rightarrow (\alpha', f')$ are given by natural transformations $\delta : \alpha \to \alpha'$ such that the diagram
$$\begin{tikzcd} \alpha^* S \arrow[r, "f"] \arrow[d, "\mathbb{S}(\delta)"'] & T \\ (\alpha')^* S \arrow[ru, "f'"'] & \end{tikzcd}$$
commutes.

This category is 1-fibered and 2-fibered over Dia. There is a commutative diagram of pseudo-functors of 2-categories (not of 2-multicategories)
$$\begin{tikzcd} \mathrm{Dia}(\mathbb{S})^{2-\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \arrow[d] \\ \mathrm{Dia}^{2-\mathrm{op}} \arrow[r] & \mathrm{Dia}^{\mathrm{cor}} \end{tikzcd}$$
where the bottom horizontal pseudo-functor is the one of Proposition 5.2.9, 1.

5.3.9. Let $\mathbb{S}$ be a pre-multiderivator (not a 2-pre-multiderivator). Recall the definition of $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ from 4.6.2. We define here the category $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}}$ even as a 2-multicategory:

1. The objects of $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}}$ are the pairs (I, S) where $I \in \mathrm{Dia}$ and $S \in \mathbb{S}(S)$.
2. The 1-morphisms in $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}}}((I_1, S_1), \dots, (I_n, S_n); (J, T))$ are collections $\{\alpha_i : J \to I_i\}$ together with a morphism
$$f \in \mathrm{Hom}_{\mathbb{S}(J)}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; T).$$
3. The 2-morphisms are given by collections $\{\delta_i : \alpha_i \to \alpha_i'\}$ such that the diagram
$$\begin{tikzcd} (\alpha_1^* S_1, \dots, \alpha_n^* S_n) \arrow[r] \arrow[d] & T \\ ((\alpha_1')^* S_1, \dots, (\alpha_n')^* S_n) \arrow[ru] & \end{tikzcd}$$
commutes.

There is a commutative diagram of pseudo-functors of 2-multicategories

$$\begin{tikzcd} \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \arrow[d] \\ \mathrm{Dia}^{1-\mathrm{op}} \arrow[r] & \mathrm{Dia}^{\mathrm{cor}} \end{tikzcd}$$

where the bottom horizontal pseudo-functor is the one of Proposition 5.2.9, 2.

5.4 Fibered multiderivators over 2-pre-multiderivators

The definition of a fibered multiderivator over 2-pre-multiderivators is a straightforward generalization of the notion of *fibered multiderivator* from chapter 4. In this section we give a much slicker definition which comprises the previous Definition 4.3.6 of fibered multiderivator. See Corollary 5.4.3 for the equivalence of the two formulations for fibered multiderivator over usual pre-multiderivators.

Definition 5.4.1. *A strict morphism $\mathbb{D} \to \mathbb{S}$ of (lax/oplax) 2-pre-multiderivators (Definition 5.1.2) such that $\mathbb{D}$ and $\mathbb{S}$ each satisfy (Der1) and (Der2) (cf. 5.1.4) is a*

1. **lax left (resp. oplax right) fibered multiderivator** *if the corresponding strict functor of 2-multicategories*

$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$$

of Definition 5.3.7 is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

2. **oplax left (resp. lax right) fibered multiderivator** *if the corresponding strict functor of 2-multicategories*

$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

of Definition 5.3.7 is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

Similarly, we define **symmetric**, *resp.* **braided** *fibered multiderivators where everything is, in addition, equipped in a compatible way with the action of the symmetric, resp. braid groups.*

If in $\mathbb{S}$ all 2-morphisms are invertible then left oplax=left lax and right oplax=right lax. In that case we omit the adjectives "lax" and "oplax".
It seems that, in the definition, one could release the assumption on 1-categorical fibers, to get an apparently more general definition. However, then the 1-truncation involved in the definition of $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is probably not the right thing to work with. In particular one does not get any generalized definition of a 2-derivator (or monoidal 2-derivator) as 2-fibered (multi)derivator over $\{\cdot\}$.
The following Theorem 5.4.2 gives an alternative definition of a left/right fibered multiderivator over a 2-pre-multiderivator $\mathbb{S}$ more in the spirit of the original (1-categorical) Definition 4.3.6.

Theorem 5.4.2. *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of (lax/oplax) 2-pre-multiderivators such that $\mathbb{D}$ and $\mathbb{S}$ both satisfy (Der1) and (Der2) is a* left (resp. right) fibered multiderivator *if and only if the following axioms (FDer0 left/right) and (FDer3–5 left/right) hold true*[21]*. Here (FDer3–4 left/right) can be replaced by the weaker (FDer3–4 left/right').*

(FDer0 left) For each I in Dia the morphism p specializes to an 1-opfibered 2-multicategory with 1-categorical fibers. It is, in addition, 2-fibered in the lax case and 2-opfibered in the oplax case. Moreover any functor $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of 1-opfibered and 2-(op)fibered 2-multicategories, i.e. the top horizontal functor maps coCartesian 1-morphisms to coCartesian 1-morphisms and (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

(FDer3 left) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 left))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left adjoint $\alpha_!^{(S)}$.

(FDer4 left) For each functor $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swArrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $\alpha_{j!}(\mathbb{S}(\mu))_\bullet \iota^* \to j^* \alpha_!$ is an isomorphism[22].

(FDer5 left) For any opfibration $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-, \underbrace{-}_{\text{at } i}, \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -, \underbrace{\alpha_!-}_{\text{at } i}, -, \cdots, -)$$

are isomorphisms for all $i = 1, \dots, n$.

[21] where (FDer3–5 left), resp. (FDer3–5 right), only make sense in the presence of (FDer3–5 left), resp. (FDer0 right)

[22] This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

Instead of (FDer3/4 left) the following axioms are sufficient:

(FDer3 left') For each *opfibration* $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 left))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!^{(S)}$.

(FDer4 left') For each *opfibration* $\alpha : I \to J$ in Dia, and for any object $j \in J$, the induced natural transformation of functors $\mathrm{pr}_{2,!}\,\mathrm{pr}_1^* \to j^*\alpha_!$ is an isomorphism for any base. Here pr_1 and pr_2 are defined by the Cartesian square

$$\begin{tikzcd} I \times_J j \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J. \end{tikzcd}$$

We use the same notation for the axioms as in the case of usual fibered multiderivators because, in case that $\mathbb{S}$ is a usual 1-pre-multiderivator they specialize to the familiar ones. Dually, we have the following axioms:

(FDer0 right) For each I in Dia the morphism p specializes to a 1-fibered multicategory with 1-categorical fibers. It is, in addition, 2-fibered in the lax case and 2-opfibered in the oplax case. Furthermore, any *opfibration* $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of 1-fibered and 2-(op)fibered multicategories, i.e. the top horizontal functor maps Cartesian 1-morphisms w.r.t. the i-th slot to Cartesian 1-morphisms w.r.t. the i-th slot for any i and maps (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

(FDer3 right) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 right))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a right adjoint $\alpha_*^{(S)}$.

(FDer4 right) For each morphism $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $j^*\alpha_* \to \alpha_{j\,*}(\mathbb{S}(\mu))^\bullet\iota^*$ is an isomorphism[23].

(FDer5 right) For *any* functor $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_*(\alpha^*\xi)^{\bullet,i}(\alpha^*-, \overset{\widehat{i}}{\cdots}, \alpha^*- \ ; \ -) \cong \xi^{\bullet,i}(-, \overset{\widehat{i}}{\cdots}, - \ ; \ \alpha_*-)$$

are isomorphisms for all $i = 1, \dots, n$.

There is similarly a weaker version of (FDer3/4 right) in which α has to be a fibration. In particular Theorem 5.4.2 shows that the new Definition 5.4.1 agrees with the old Definiton 4.3.6 in case that the base 2-pre-multiderivator is a usual pre-multiderivator:

Corollary 5.4.3. *Let $\mathbb{D}$ and $\mathbb{S}$ be pre-multiderivators (not 2-pre-multiderivators) satisfying (Der1) and (Der2) (cf. Definition 4.3.5). A strict morphism of pre-multiderivators $\mathbb{D} \to \mathbb{S}$ is a left (resp. right) fibered multiderivator (in the sense of Definition 4.3.6) if and only if* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *is a 1-opfibration (resp. 1-fibration) of 2-multicategories.*

The proof of Theorem 5.4.2 will be given in the next section.
For *representable* 2-pre-multiderivators we have the following:

Proposition 5.4.4. *If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-fibration of 2-multicategories with 1-categorical and bicomplete fibers then*

1. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{lax}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{lax}})$ *is a 1-opfibration and 2-fibration,*
2. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{oplax}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{oplax}})$ *is a 1-fibration and 2-fibration.*

If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-opfibration of 2-multicategories with 1-categorical and bicomplete fibers then

1. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{lax,2-op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{lax,2-op}})$ *is a 1-fibration and 2-fibration.*
2. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{oplax,2-op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{oplax,2-op}})$ *is a 1-opfibration and 2-fibration.*

Proof. This follows from Proposition 5.1.6 doing the same constructions as Proposition 4.7.26. □

Definition 5.4.5. *For (lax/oplax) fibered derivators over an (lax/oplax) 2-pre-derivator $p : \mathbb{D} \to \mathbb{S}$ and an object $S \in \mathbb{S}(I)$ we have that*

$$\mathbb{D}_{I,S} : J \mapsto \mathbb{D}(I \times J)_{\mathrm{pr}_2^* S}$$

is a usual derivator. We call p **stable** *if $\mathbb{D}_{I,S}$ is stable for all $S \in \mathbb{S}(I)$ and for all I.*

[23]This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

5.5 Yoga of correspondences of diagrams in a (2-)pre-multiderivator

To prove Theorem 5.4.2 we need some preparation to improve our understanding of the category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. Let $\mathbb{S}$ be a 2-pre-multiderivator. The constructions in this section are already interesting if $\mathbb{S}$ is a usual pre-multiderivator. In fact, the 2-categorical structure of $\mathbb{S}$ hardly influences the constructions.
We will define three types of generating 1-morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. We first define them as objects in the categories $\mathrm{Cor}_{\mathbb{S}}(\dots)$ (without the restriction ^{F}).

$[\beta^{(S)}]$ for a functor $\beta : I \to J$ in Dia and an object $S \in \mathbb{S}(J)$, consists of the correspondence of diagrams

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "\beta"] & \\ I & & J \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I, \beta^* S); (J, S)))$ the canonical correspondence given by the identity $\mathrm{id}_{\beta^* S}$.

$[\alpha^{(S)}]'$ for a functor $\alpha : I \to J$ in Dia and an object $S \in \mathbb{S}(J)$, consists of the correspondence of diagrams

$$\begin{tikzcd} & I \arrow[dl, "\alpha"'] \arrow[dr, equal] & \\ J & & I \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((J, S); (I, \alpha^* S)))$ the canonical correspondence given by the identity $\mathrm{id}_{\alpha^* S}$.

$[f]$ for a morphism $f \in \mathrm{Hom}_{\mathbb{S}(A)}(S_1, \dots, S_n; T)$, where A is any diagram in Dia, and $S_1, \dots, S_n, T$ are objects in $\mathbb{S}(A)$, is defined by the trivial correspondence of diagrams

$$\begin{tikzcd} & & A \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ A & A & & A \end{tikzcd}$$

together with f.

5.5.1. Note that the correspondences of the last paragraph do not define 1-morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ yet, as we defined it, because they are not always objects in the Cor^F subcategory ($[\alpha^{(S)}]'$ is already, if α is a fibration; $[\beta^{(S)}]$ is, if β is an opfibration; and $[f]$ is, if $n = 0, 1$, respectively).
From now on, we denote by the same symbols $[\alpha^{(S)}], [\beta^{(S)}]', [f]$ morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ which are isomorphic to those defined above in the τ_1-categories (cf. Lemma 5.3.6). Those are are determined only up to 2-isomorphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.

For definiteness, we choose $[\beta^{(S)}]$ to be the correspondence

$$\begin{tikzcd} & I \times_{/J} J \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ I & & J \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I, \beta^* S); (J, S)))$ the 1-morphism $\mathrm{pr}_1^* \beta^* S \to \mathrm{pr}_2^* S$ given by the natural transformation $\mu_\beta : \beta \circ \mathrm{pr}_1 \Rightarrow \mathrm{pr}_2$. Similarly, we choose $[\alpha^{(S)}]'$ to be the correspondence

$$\begin{tikzcd} & J \times_{/J} I \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ J & & I \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((J, S); (I, \alpha^* S)))$ the 1-morphism $\mathrm{pr}_1^* S \to \mathrm{pr}_2^* \alpha^* S$ given by the natural transformation $\mu_\alpha : \mathrm{pr}_1 \Rightarrow \alpha \circ \mathrm{pr}_2$.

5.5.2. For any $\alpha : I \to J$, we define a 2-morphism

$$\epsilon : \mathrm{id} \Rightarrow [\alpha^{(S)}] \circ [\alpha^{(S)}]'$$

given by the diagrams

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta"] & \\ I & & I \\ & I \times_{/J} J \times_{/J} I \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_3"'] & \end{tikzcd} \qquad \begin{tikzcd} \Delta^* \mathrm{pr}_1^* \alpha^* S \arrow[r, equal, "\Delta^*(\mathbb{S}(\mu_2 \circ \mu_1)(S)) = \mathrm{id}_{\alpha^* S}"] \arrow[d, equal] & \Delta^* \mathrm{pr}_3^* \alpha^* S \arrow[d, equal] \\ \alpha^* S \arrow[r, equal] & \alpha^* S \end{tikzcd}$$

and we define a 2-morphism

$$\mu : [\alpha^{(S)}]' \circ [\alpha^{(S)}] \Rightarrow \mathrm{id}$$

given by the diagrams

$$\begin{tikzcd} & J \times_{/J} I \times_{/J} J \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_3"] \arrow[dd, "\alpha\,\mathrm{pr}_2"] & \\ J \arrow[r, phantom, "\overset{\mu_2}{\Rightarrow}"] \arrow[dr, equal] & {} \arrow[r, phantom, "\overset{\mu_1}{\Rightarrow}"] & J \arrow[dl, equal] \\ & J & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pr}_2^* \alpha^* S \arrow[r, equal] & \mathrm{pr}_2^* \alpha^* S \arrow[d, "\mathbb{S}(\mu_1)(S)"] \\ \mathrm{pr}_1^* S \arrow[u, "\mathbb{S}(\mu_2)(S)"] \arrow[r, "\mathbb{S}(\mu_2 \circ \mu_1)(S)"'] \arrow[ur, phantom, "\Uparrow"] & \mathrm{pr}_3^* S \end{tikzcd}$$

where the 2-isomorphism from the pseudo-functoriality of $\mathbb{S}$ is taken.

5.5.3. A natural transformation $\nu : \alpha \Rightarrow \beta$ establishes a morphism

$$[\nu] : [\mathbb{S}(\nu)(S)] \circ [\alpha^{(S)}] \Rightarrow [\beta^{(S)}]$$

given by the diagrams:

$$\begin{tikzcd}
 & J \times_{/J,\beta} I \arrow[dl, "\mathrm{pr}_1'"'] \arrow[dr, "\mathrm{pr}_2'"] \arrow[dd, "\widetilde{\nu}"] & \\
J & & I \\
 & J \times_{/J,\alpha} I \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\widetilde{\nu})^* \mathrm{pr}_1^* S \arrow[r, "\widetilde{\nu}^* \mathbb{S}(\mu_\alpha)"] \arrow[d, equal] & (\widetilde{\nu})^* \mathrm{pr}_2^* \alpha^* S \arrow[r, "\widetilde{\nu}^* \mathrm{pr}_2^* \mathbb{S}(\nu)"] \arrow[d, phantom, "\Uparrow"] & \widetilde{\nu}^* \mathrm{pr}_2^* \beta^* S \arrow[d, equal] \\
(\mathrm{pr}_1')^* S \arrow[rr, "\mathbb{S}(\mu_\beta)(S)"'] & & (\mathrm{pr}_2')^* \beta^* S
\end{tikzcd}$$

where the 2-isomorphism from the pseudo-functoriality of $\mathbb{S}$ is taken. Note that we have the equation of natural transformations $(\nu * \mathrm{pr}_2') \circ (\mu_\alpha * \widetilde{\nu}) = \mu_\beta$. Here μ_α and μ_β are as in 5.5.1.
Similarly, a natural transformation $\nu : \alpha \Rightarrow \beta$ establishes a morphism

$$[\nu] : [\beta^{(S)}]' \circ [\mathbb{S}(\nu)(S)] \Rightarrow [\alpha^{(S)}]'.$$

5.5.4. Consider the diagrams from axiom (FDer3 left/right)

$$\begin{tikzcd}
I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow\mu"] & I \arrow[d, "\alpha"] \\
j \arrow[r, hook] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow\mu"] & I \arrow[d, "\alpha"] \\
j \arrow[r, hook] & J
\end{tikzcd}$$

By the constructions in 5.5.3, we get a canonical 2-morphism

$$[\mathbb{S}(\mu)(S)] \circ [\iota^{(\alpha^* S)}] \circ [\alpha^{(S)}] \Rightarrow [p^{(S_j)}] \circ [j^{(S)}]. \tag{39}$$

and a canonical 2-morphism

$$[\alpha^{(S)}]' \circ [\iota^{(\alpha^* S)}]' \circ [\mathbb{S}(\mu)(S)] \Rightarrow [j^{(S)}]' \circ [p^{(S_j)}]'. \tag{40}$$

respectively. Here S_j denotes $j^* S$ where j, by abuse of notation, also denotes the inclusion of the one-element category j into J.

5.5.5. Let ξ be any 1-morphism $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ given by

$$\begin{tikzcd}
 & & & A \arrow[lll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\
I_1 & \cdots & I_n & & J
\end{tikzcd}$$

and a 1-morphism

$$f_\xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T).$$

We define a 1-morphism $\xi \times K$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ by

$$\begin{tikzcd} & & & A \times K \arrow[llld, "\alpha_1 \times \mathrm{id}"'] \arrow[ld, "\alpha_n \times \mathrm{id}"] \arrow[rd, "\beta \times \mathrm{id}"] & \\ I_1 \times K & \cdots & I_n \times K & & J \times K \end{tikzcd}$$

and

$$f_{\xi \times K} := \mathrm{pr}_1^* f_\xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\mathrm{pr}_1^* \alpha_1^* S_1, \dots, \mathrm{pr}_1^* \alpha_n^* S_n; \mathrm{pr}_1^* \beta^* T).$$

Note that the here defined $\xi \times K$ does not necessarily lie in the category $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$. Hence we denote by $\xi \times K$ any isomorphic correspondence which does lie in $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$. We also define a correspondence $\xi \times_j K$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ by

$$\begin{tikzcd} & & & & & A \times K \arrow[llllld, "\alpha_1 \mathrm{pr}_1"'] \arrow[llld, "\alpha_j \times \mathrm{id}"] \arrow[ld, "\alpha_n \mathrm{pr}_1"] \arrow[rd, "\beta \times \mathrm{id}"] & \\ I_1 & \cdots & I_j \times K & \cdots & I_n & & J \times K \end{tikzcd}$$

and

$$f_{\xi \times_j K} := \mathrm{pr}_1^* \xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\mathrm{pr}_1^* \alpha_1^* S_1, \dots, \mathrm{pr}_1^* \alpha_n^* S_n; \mathrm{pr}_1^* \beta^* T).$$

The here defined $\xi \times_j K$ does already lie in the category $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$.

Lemma 5.5.6. *1. The 2-morphisms of 5.5.2*

$$\epsilon : \mathrm{id} \Rightarrow [\alpha^{(S)}] \circ [\alpha^{(S)}]' \qquad \mu : [\alpha^{(S)}]' \circ [\alpha^{(S)}] \Rightarrow \mathrm{id}$$

establish an adjunction between $[\alpha^{(S)}]$ and $[\alpha^{(S)}]'$ in the 2-category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.

2. *The exchange 2-morphisms of (39) and and of (40) w.r.t. the adjunction of 1., namely*

$$[p^{(S_j)}]' \circ [\mathbb{S}(\mu)(S)] \circ [\iota^{(\alpha^* S)}] \Rightarrow [j^{(S)}] \circ [\alpha^{(S)}]'$$

and

$$[\iota^{(\alpha^* S)}]' \circ [\mathbb{S}(\mu)(S)] \circ [p^{(S_j)}] \Rightarrow [\alpha^{(S)}] \circ [j^{(S)}]'$$

are 2-isomorphisms.

3. *For any $\alpha : K \to L$ there are natural isomorphisms*

$$[\alpha^{(\mathrm{pr}_1^* T)}] \circ (\xi \times L) \cong (\xi \times K) \circ ([\alpha^{(\mathrm{pr}_1^* S_1)}], \dots, [\alpha^{(\mathrm{pr}_1^* S_n)}]) \tag{41}$$

and

$$[\alpha^{(\mathrm{pr}_1^* T)}] \circ (\xi \times_j L) \cong (\xi \times_j K) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}] \tag{42}$$

4. *The exchange of (41) w.r.t. the adjunction of 1., namely*

$$[\alpha^{(\mathrm{pr}_1^* T)}]' \circ (\xi \times K) \circ ([\alpha^{(\mathrm{pr}_1^* S_1)}], \dots, \mathrm{id}, \dots, [\alpha^{(\mathrm{pr}_1^* S_n)}]) \cong (\xi \times L) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}]'$$

is an isomorphism if α is an opfibration. The exchange of (42) w.r.t. the adjunction of 1., namely

$$[\alpha^{(\mathrm{pr}_1^* T)}]' \circ (\xi \times_j K) \cong (\xi \times_j L) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}]'$$

is an isomorphism for any α.

5. *For any $f \in \mathrm{Hom}_{\mathbb{S}(J)}(S_1, \dots, S_n; T)$ and $\alpha : I \to J$ there is a natural isomorphism*

$$[\alpha^{(T)}] \circ [f] \cong [\alpha^* f] \circ ([\alpha^{(S_1)}], \dots, [\alpha^{(S_n)}]) \tag{43}$$

6. *The exchange of (43) w.r.t. the adjunction of 1., namely*

$$[\alpha^{(T)}]' \circ [\alpha^* f] \circ ([\alpha^{(S_1)}], \dots, \mathrm{id}, \dots, [\alpha^{(S_n)}]) \cong [f] \circ_j [\alpha^{(S_j)}]'$$

is an isomorphism if α is an opfibration.

Proof. A purely algebraic manipulation that we leave to the reader. □

5.5.7. Let $\mathbb{D} \to \mathbb{S}$ be a morphism of (lax/oplax) 2-pre-multiderivators satisfying (Der1) and (Der2). Consider the strict 2-functor

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \text{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

and assume that it is a 1-opfibration, and 2-fibration with 1-categorical fibers. The fiber over a pair (I, S) is just the fiber $\mathbb{D}(I)_S$ of the strict 2-functor $\mathbb{D}(I) \to \mathbb{S}(I)$ over S and hence this is a 1-category. The 1-opfibration and 2-fibration can be seen (via the construction of Proposition 2.4.16) as a pseudo-functor of 2-multicategories

$$\Psi : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{(2-\mathrm{op})}) \to \mathcal{CAT}.$$

5.5.8. If

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \text{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

is a *1-fibration*, and 2-fibration with 1-categorical fibers there is still an associated pseudo-functor of 2-categories (not 2-multicategories)

$$\Psi' : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{(2-\mathrm{op})})^{1-\mathrm{op},2-\mathrm{op}} \to \mathcal{CAT}.$$

Proposition 5.5.9. *1. Assume that*

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \textit{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

is a 1-opfibration*, and 2-fibration with 1-categorical fibers. Then the functor* Ψ *of 5.5.7 maps (up to isomorphism of functors)*

$$\begin{array}{rcl} [\alpha^{(S)}] & \mapsto & (\alpha^S)^* \\ [\beta^{(S)}]' & \mapsto & \beta_!^S \\ [f] & \mapsto & f_\bullet \end{array}$$

where $\beta_!^S$ *is a left adjoint of* $(\beta^S)^*$ *and* $f_\bullet$ *is a functor determined by*

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}) \cong \mathrm{Hom}_{\mathbb{D}(I)_T}(f_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n),\mathcal{F}).$$

2. *Assume that*

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \textit{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

is a 1-fibration*, and 2-fibration with 1-categorical fibers.*

Then pullback functors[24] *w.r.t. the following 1-morphisms in* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *are given by*

$$\begin{array}{rcll} [\alpha^{(S)}] & \mapsto & \alpha_*^S & \\ [\beta^{(S)}]' & \mapsto & (\beta^S)^* & \\ [f] & \mapsto & f^{\bullet,j} & \textit{pullback w.r.t. the } j\textit{-th slot.} \end{array}$$

where α_*^S *is a right adjoint of* $(\alpha^S)^*$ *and* $f^{\bullet,j}$ *is a functor determined by*

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}) \cong \mathrm{Hom}_{\mathbb{D}(I)_T}(\mathcal{E}_j, f^{\bullet,j}(\mathcal{E}_1,\overset{\widehat{j}}{\dots},\mathcal{E}_n;\mathcal{F})).$$

Proof. 1. We have an isomorphism of sets[25]

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),[\alpha^{(S)}]}((J,\mathcal{E}),(I,\mathcal{F})) \cong \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(I,S)}}(\Psi([\alpha^{(S)}])\mathcal{E},\mathcal{F}).$$

On the other hand, by definition and by Lemma 5.3.6, the left hand side is isomorphic to the set

$$\mathrm{Hom}_{\mathbb{D}(I)_S}(\alpha^*\mathcal{E},\mathcal{F}).$$

The first assertion follows from the fact that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(I,S)} = \mathbb{D}(I)_S$.
The second assertion follows from the first because by Lemma 5.5.6, 1. the 1-morphisms $[\alpha^{(S)}]$ and $[\alpha^{(S)}]'$ are adjoint in the 2-category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. Note that a pseudo-functor like Ψ preserves adjunctions.
We have an isomorphism of sets

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),[f]}((A,\mathcal{E}_1),\dots,(A,\mathcal{E}_n);(A,\mathcal{F})) \cong \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(A,T)}}(\Psi([f])(\mathcal{E}_1,\dots,\mathcal{E}_n),\mathcal{F}).$$

[24] In the case of $[\alpha^{(S)}]$ and $[\beta^{(S)}]'$ these are $\Psi'([\alpha^{(S)}])$ and $\Psi'([\beta^{(S)}]')$.

[25] We identify a discrete category with its set of isomorphism classes.

On the other hand, by definition and by Lemma 5.3.6, the left hand side is isomorphic to the set

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$$

and the third assertion follows from the fact that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(A,T)} = \mathbb{D}(A)_T$.
The proof of 2. is completely analogous. □

Corollary 5.5.10. *Assuming the conditions of 5.5.7, consider any correspondence*

$$\xi' \in \mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$$

consisting of

$$\begin{tikzcd}[column sep=small] & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\ I_1 & \cdots & I_n & & J \end{tikzcd}$$

and a 1-morphism

$$f \in \mathrm{Hom}(\alpha_1^* S_1,\dots,\alpha_n^* S_n;\beta^* T)$$

in $\mathbb{S}(A)$.

1. *Over any 1-morphism ξ in* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$*, which is isomorphic to ξ', a corresponding push-forward functor between fibers (which is $\Psi(\xi')$ in the discussion 5.5.7) is given (up to isomorphism) by the composition:*

$$\beta_!^T \circ f_\bullet \circ (\alpha_1^*,\dots,\alpha_n^*).$$

2. *Over any 1-morphism ξ in* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$*, which is isomorphic to ξ', a pull-back functor w.r.t. any slot j between fibers (which is $\Psi'(\xi')$ in the discussion 5.5.8 if ξ is a 1-ary 1-morphism) is given (up to isomorphism) by the composition:*

$$\alpha_{j,*}^{S_j} \circ f^{\bullet,j} \circ (\alpha_1^*, \overset{\widehat{j}}{\dots}, \alpha_n^*;\beta^*).$$

Proof. Because of Proposition 5.5.9, in both cases, we only have to show that there is an isomorphism

$$\xi \cong [\beta^{(T)}]' \circ [f] \circ ([\alpha_1^{(S_1)}],\dots,[\alpha_n^{(S_n)}])$$

in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$, which is an easy and purely algebraic manipulation. □

We are now ready to give the

Proof of Theorem 5.4.2. We concentrate on the lax left case, the other cases are shown completely analogously.
We first show that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a 1-opfibration and a 2-fibration, if $\mathbb{D} \to \mathbb{S}$ satisfies (FDer0 left), (FDer3–4 left'), and (FDer5 left). By (FDer0 left) $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-opfibration and 2-fibration with 1-categorical fibers and we have already seen that this implies that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is 2-fibered as well (cf. Lemma 5.3.6).

Let $x = (A; \alpha_{A,1}, \dots, \alpha_{A,n}; \beta_A)$ be a correspondence in $\mathrm{Cor}^F(I_1, \dots, I_n; J)$ and let

$$f \in \mathrm{Hom}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T)$$

be a 1-morphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ lying over x. In $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ we have the following composition of isomorphisms of sets (because of Lemma 5.3.6, 2.)[26]:

$$\begin{aligned}
& \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \mathcal{F})) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),f}(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n; \beta^* \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),\mathrm{id}_{\beta^* T}}(f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n); \beta^* \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),\mathrm{id}_T}(\beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n); \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),\mathrm{id}_{(J,T)}}((J, \beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n)); (J, \mathcal{F}))
\end{aligned}$$

using (FDer0 left) and (FDer3 left'). One checks that this composition is induced by the composition in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ with a 1-morphism in

$$\mathrm{Hom}_f\,((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n)))$$

which is thus weakly coCartesian.
Note that we write $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}$ for the category of 1-morphisms which map to f in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ and those 2-morphisms that map to id_f in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.
By Proposition 2.4.6, it remains to be shown that the composition of weakly coCartesian 1-morphisms is weakly coCartesian. Let

$$g \in \mathrm{Hom}(\alpha_{B,1}^* T_1, \dots, \alpha_{B,m}^* T_m; \beta_B^* U)$$

be another 1-morphism in $\mathbb{S}(B)$, composable with f, and lying over a correspondence $y = (B; \alpha_{B,1}, \dots, \alpha_{B,m}; \beta_B)$ in $\mathrm{Cor}^F(J_1, \dots, J_m; K)$. Setting $J_i := J$ and $T_i := T$, the composition of x and y w.r.t. the i-th slot is the correspondence

$$\begin{tikzcd}
 & & & & & A \times_{J_i} B \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & & \\
 & & & A \arrow[dlll] \arrow[dl] \arrow[drrr, "\beta_A"] & & & & B \arrow[dlll] \arrow[dl, "\alpha_{B,i}"'] \arrow[dr] \arrow[drrr] & & & \\
I_1 & \cdots & I_n & ; & J_1 & \cdots & J_i & \cdots & J_m & ; & K
\end{tikzcd}$$

The composition of g and f is determined by the morphism

$$\mathrm{pr}_2^*\, g \circ_i \mathrm{pr}_1^*\, f$$

lying in

$$\mathrm{Hom}(\mathrm{pr}_2^*\, \alpha_{B,1}^* T_1, \dots, \underbrace{\mathrm{pr}_1^*\, \alpha_{A,1}^* S_1, \dots, \mathrm{pr}_1^*\, \alpha_{A,n}^* S_n}_{\text{at } i}, \dots, \mathrm{pr}_2^*\, \alpha_{B,m}^* T_m; \mathrm{pr}_2^*\, \beta_B^* U).$$

[26] We identify a discrete category with its set of isomorphism classes.

We have to show that the natural map

$$\begin{aligned}
&\beta_{B,!}g_\bullet(\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\alpha^*_{B,i}\beta_{A,!}f_\bullet(\alpha^*_{A,1}\mathcal{E}_1,\dots,\alpha^*_{A,n}\mathcal{E}_n)}_{\text{at } i},\dots\alpha^*_{B,m}\mathcal{F}_m) \\
\to\ &\beta_{B,!}\operatorname{pr}_{2,!}(\operatorname{pr}_2^* g \circ_i \operatorname{pr}_1^* f)_\bullet(\operatorname{pr}_2^*\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\operatorname{pr}_1^*\alpha^*_{A,1}\mathcal{E}_1,\dots,\operatorname{pr}_1^*\alpha^*_{A,n}\mathcal{E}_n}_{\text{at } i},\dots\operatorname{pr}_2^*\alpha^*_{B,m}\mathcal{F}_m)
\end{aligned}$$

is an isomorphism. It is the composition of the following morphisms which are all isomorphisms respectively by (FDer4 left'), (FDer5 left) observing that pr_2 is an opfibration, the second part of (FDer0 left) for pr_1, and the first part of (FDer0 left) in the form that the composition of coCartesian morphisms is coCartesian:

$$\begin{aligned}
&\beta_{B,!}g_\bullet(\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\alpha^*_{B,i}\beta_{A,!}f_\bullet(\alpha^*_{A,1}\mathcal{E}_1,\dots,\alpha^*_{A,n}\mathcal{E}_n)}_{\text{at } i},\dots,\alpha^*_{B,m}\mathcal{F}_m) \\
\to\ &\beta_{B,!}g_\bullet(\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\operatorname{pr}_{2,!}\operatorname{pr}_1^* f_\bullet(\alpha^*_{A,1}\mathcal{E}_1,\dots,\alpha^*_{A,n}\mathcal{E}_n)}_{\text{at } i},\dots,\alpha^*_{B,m}\mathcal{F}_m) \\
\to\ &\beta_{B,!}\operatorname{pr}_{2,!}(\operatorname{pr}_2^* g)_\bullet(\operatorname{pr}_2^*\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\operatorname{pr}_1^* f_\bullet(\alpha^*_{A,1}\mathcal{E}_1,\dots,\alpha^*_{A,n}\mathcal{E}_n)}_{\text{at } i},\dots,\operatorname{pr}_2^*\alpha^*_{B,m}\mathcal{F}_m) \\
\to\ &\beta_{B,!}\operatorname{pr}_{2,!}(\operatorname{pr}_2^* g)_\bullet(\operatorname{pr}_2^*\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{(\operatorname{pr}_1^* f)_\bullet(\operatorname{pr}_1^*\alpha^*_{A,1}\mathcal{E}_1,\dots,\operatorname{pr}_1^*\alpha^*_{A,n}\mathcal{E}_n)}_{\text{at } i},\dots,\operatorname{pr}_2^*\alpha^*_{B,m}\mathcal{F}_m) \\
\to\ &\beta_{B,!}\operatorname{pr}_{2,!}(\operatorname{pr}_2^* g \circ_i \operatorname{pr}_1^* f)_\bullet(\operatorname{pr}_2^*\alpha^*_{B,1}\mathcal{F}_1,\dots,\underbrace{\operatorname{pr}_1^*\alpha^*_{A,1}\mathcal{E}_1,\dots,\operatorname{pr}_1^*\alpha^*_{A,n}\mathcal{E}_n}_{\text{at } i},\dots,\operatorname{pr}_2^*\alpha^*_{B,m}\mathcal{F}_m).
\end{aligned}$$

Now we proceed to prove the converse, hence we assume that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a 1-opfibration and show that the axioms (FDer0 left, FDer3–5 left) are satisfied.
(FDer0 left) First we have an obvious pseudo-functor of 2-multicategories

$$\begin{array}{rcl}
F : \mathbb{S}(A) & \hookrightarrow & \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \\
S & \mapsto & (A, S) \\
f & \mapsto & [f]
\end{array}$$

By Proposition 2.4.24 the pull-back $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathbb{S}(A)$ in the sense of 2.4.23 is 1-opfibered and 2-fibered if $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is 1-opfibered and 2-fibered. To show that $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-opfibration and 2-fibration of multicategories, it thus suffices to show that the pull-back $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ is equivalent to $\mathbb{D}(I)$ over $\mathbb{S}(I)$. The class of objects of $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ is by definition isomorphic to the class of objects of $\mathbb{D}(I)$. Therefore we are left to show that there are equivalences of categories (compatible with composition)

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}) \to \mathrm{Hom}_{F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$$

for any 1-morphism $f \in \mathrm{Hom}_{\mathbb{S}(I)}(S_1,\dots,S_n;T)$, where $\mathcal{E}_i$ is an object of $\mathbb{D}(I)$ over S_i and $\mathcal{F}$ is an object over T. Note that the left-hand side is a discrete category. We have a 2-Cartesian diagram of categories

$$\begin{tikzcd}
\mathrm{Hom}_{F^*\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F})) \arrow[d] \\
\{f\} \arrow[r, "F"] & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I,S_1),\dots,(I,S_n);(I,T))
\end{tikzcd}$$

Since the right vertical morphism is a fibration (cf. Lemma 5.3.6) the diagram is also Cartesian (cf. Lemma 2.4.2). Futhermore by Lemma 5.3.6 the right vertical morphism is equivalent to

$$\begin{tikzcd}
\tau_1(\mathrm{Cor}_{\mathbb{D}}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F}))) \arrow[d] \\
\tau_1(\mathrm{Cor}_{\mathbb{S}}((I,S_1),\dots,(I,S_n);(I,T)))
\end{tikzcd}$$

(Here $\mathrm{Cor}_{\mathbb{D}}^F(\dots)$ was changed to $\mathrm{Cor}_{\mathbb{D}}(\dots)$ and similarly for $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$.)
In $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)))$, the object $F(f)$ is isomorphic to f over the trivial correspondence $(\mathrm{id}_I,\dots,\mathrm{id}_I;\mathrm{id}_I)$ whose fiber in $\tau_1(\mathrm{Cor}_{\mathbb{D}}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F})))$ is precisely the discrete category $\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$. The remaining part of (FDer0 left) will be shown below.
Since we have a 1-opfibration and 2-fibration we can equivalently see the given datum as a pseudo-functor

$$\Psi : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathcal{CAT}$$

and we have seen by Proposition 5.5.9 that this morphism maps $[\alpha^{(S)}]$ to a functor which is isomorphic to the functor $\alpha^* : \mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^*S}$. We have the freedom to choose Ψ in such a way that it maps $[\alpha^{(S)}]$ precisely to α^*.
Axiom (FDer3 left) follows from Lemma 5.5.6, 1. stating that $[\alpha^{(S)}]$ has a left adjoint $[\alpha^{(S)}]'$ in the category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ (cf. also Proposition 5.5.9).
Axiom (FDer4 left) follows by applying Ψ to the (first) 2-isomorphism of Lemma 5.5.6, 2.
Axiom (FDer5 left) follows by applying Ψ to the 2-isomorphism of Lemma 5.5.6, 4.
The remaining part of (FDer0 left), i.e. that α^* maps coCartesian arrows to coCartesian arrows follows by applying Ψ to the 2-isomorphism of Lemma 5.5.6, 3. □

5.6 Representable 2-pre-multiderivators

Proof (sketch) of Proposition 5.1.6: We show exemplarily 1. for 1-fibrations of 2-categories and 2. for 1-fibrations of 2-categories. The same proof works for 1-opfibrations (even of 2-multicategories). If we have a 1-bifibration of 2-multicategories *with 1-categorical fibers* then a slight extension of the proof of Proposition 4.7.6 shows 3. For 1-opfibrations and 2-fibrations of 2-multicategories with 1-categorical fibers this is actually easier to prove using the encoding by a pseudo-functor as follows: The 1-opfibration $\mathcal{D} \to \mathcal{S}$ with 1-categorical fibers is encoded in a pseudo-functor

$$\mathcal{S} \to \mathcal{MCAT}$$

The category $\mathbb{D}(I) \to \mathbb{S}(I)$ is encoded in the pseudo-functor

$$\mathbb{S}(I) \to \mathcal{MCAT}$$

which maps a pseudo-functor $F : I \to \mathcal{S}$ to the multicategory of natural transformations and modifications

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{MCAT})}(\cdot, F),$$

where $\cdot$ is the constant functor with value the 1-point category.
This construction may be adapted to 2-categorical fibers by using "pseudo-functors" of 3-categories.
The problem with *1-fibrations* of (1- or 2-)*multi*categories comes from the fact that the internal Hom cannot be computed point-wise but involves a limit construction (cf. Proposition 4.7.6). The difference between external and internal monoidal product in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ gives a theoretical explanation of this phenomenon (cf. Example 6.1.5).
1-fibration of 2-categories $\mathcal{D} \to \mathcal{S}$ implies 1-fibration of 2-categories $\mathbb{D}(I) \to \mathbb{S}(I)$:
Let I be a diagram in Dia. Let $Y, Z : I \to \mathcal{S}$ be pseudo-functors, $f : Y \Rightarrow Z$ be a pseudo-natural transformation and $\mathcal{E} : I \to \mathcal{D}$ be a pseudo-functor over Z. For each morphism $\alpha : i \to i'$ we are given a 2-commutative diagram

$$\begin{tikzcd} Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"'] \arrow[dr, phantom, "\nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \\ Y(i') \arrow[r, "f(i')"] & Z(i') \end{tikzcd}$$

Since f is assumed to be a pseudo-functor, the morphism f_α is invertible. We will construct a pseudo-functor $\mathcal{G} : I \to \mathcal{D}$ over Y and a 1-coCartesian morphism $\xi : \mathcal{G} \to \mathcal{E}$ over f. For each i, we choose a 1-coCartesian morphism

$$\xi(i) : \mathcal{G}(i) \to \mathcal{E}(i)$$

over $f(i) : Y(i) \to Z(i)$. For each α, we look at the 2-Cartesian diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i')) \arrow[r, "\xi(i')\circ"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \arrow[d] \\ \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i')) \arrow[r, "f(i')\circ"] & \mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i')) \end{tikzcd}$$

The triple $(\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha)$ is an object in the category

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \times^{\sim}_{/\mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i'))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i'))$$

Define $\mathcal{G}(\alpha)$ to be an object in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i'))$ such that there exists a 2-isomorphism

$$\Xi_\alpha : (\xi(i') \circ \mathcal{G}_\alpha, \mathrm{id}, p(\mathcal{G}_\alpha)) \Rightarrow (\mathcal{E}_\alpha \circ \xi(i), f_\alpha^{-1}, Y_\alpha).$$

Such an object exists because the above square is 2-Cartesian.
We get a 2-commutative square

$$\begin{tikzcd} \mathcal{G}(i) \arrow[r, "\xi(i)"] \arrow[d, "\mathcal{G}_\alpha"'] \arrow[dr, phantom, "\nearrow \xi_\alpha"] & \mathcal{E}(i) \arrow[d, "\mathcal{E}_\alpha"] \\ \mathcal{G}(i') \arrow[r, "\xi(i')"] & \mathcal{E}(i') \end{tikzcd}$$

Here ξ_α is the first component of Ξ_α.
This defines a pseudo-functor $\mathcal{G} : I \to \mathcal{D}$ as follows. Let $\alpha : i \to i'$ and $\beta : i' \to i''$ be two morphisms in I. We need to define a 2-isomorphism $G_{\beta\alpha} \Rightarrow G_\beta \circ G_\alpha$. It suffices to define the 2-isomorphism after applying the embedding

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i'')) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i'')) \times^{\sim}_{/\mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i''))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i''))$$

which maps $G_\beta \circ G_\alpha$ to

$$(\xi(i'') \circ \mathcal{G}_\beta \circ \mathcal{G}_\alpha, \mathrm{id}, p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha))$$

and $G_{\beta\alpha}$ to

$$(\xi(i'') \circ \mathcal{G}_{\beta\alpha}, \mathrm{id}, p(\mathcal{G}_{\beta\alpha})).$$

We have the chains of 2-isomorphisms

$$\begin{tikzcd} \xi(i'') \circ \mathcal{G}_\beta \circ \mathcal{G}_\alpha \arrow[d, "\xi_\beta * \mathcal{G}_\alpha"] & p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha) \arrow[d, "\Xi_{\beta,2} * p(\mathcal{G}_\alpha)"] \\ \mathcal{E}_\beta \circ \xi(i') \circ \mathcal{G}_\alpha \arrow[d, "\mathcal{E}_\beta * \xi_\alpha"] & Y_\beta \circ p(\mathcal{G}_\alpha) \arrow[d, "Y_\beta * \Xi_{\alpha,2}"] \\ \mathcal{E}_\beta \circ \mathcal{E}_\alpha \circ \xi(i) \arrow[d] & Y_\beta \circ Y_\alpha \arrow[d] \\ \mathcal{E}_{\beta\alpha} \circ \xi(i) & Y_{\beta\alpha} \\ \xi(i'') \circ \mathcal{G}_{\beta\alpha} \arrow[u] & p(\mathcal{G}_{\beta\alpha}) \arrow[u] \end{tikzcd}$$

Applying p to the first chain and $f(i'')\circ$ to the second chain, we get the commutative diagram

$$\begin{tikzcd}
 & f(i'')\circ p(\mathcal{G}_\beta)\circ p(\mathcal{G}_\alpha) \arrow[r, equal] \arrow[d] & f(i'')\circ p(\mathcal{G}_\beta)\circ p(\mathcal{G}_\alpha) \arrow[d] \\
 & Z_\beta\circ f(i')\circ p(\mathcal{G}_\alpha) \arrow[r] \arrow[d] \arrow[dl] & f(i'')\circ Y_\beta\circ p(\mathcal{G}_\alpha) \arrow[d] \\
Z_\beta\circ Z_\alpha\circ f(i) \arrow[r] \arrow[d] & Z_\beta\circ f(i')\circ Y_\alpha \arrow[r] & f(i'')\circ Y_\beta\circ Y_\alpha \arrow[d] \\
Z_{\beta\alpha}\circ f(i) \arrow[rr, "f_{\beta\alpha}"] & & f(i'')\circ Y_{\beta\alpha} \\
f(i'')\circ p(\mathcal{G}_{\beta\alpha}) \arrow[rr, equal] \arrow[u] & & f(i'')\circ p(\mathcal{G}_{\beta\alpha}) \arrow[u]
\end{tikzcd}$$

hence a valid isomorphism in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i),\mathcal{E}(i'')) \times^{\sim}_{/\,\mathrm{Hom}_{\mathcal{S}}(Y(i),Z(i''))} \mathrm{Hom}_{\mathcal{S}}(Y(i),Y(i''))$. One checks that this satisfies the axioms of a pseudo-functor (Definition 2.2.3) and that ξ is indeed a pseudo-natural transformation (Definition 2.2.4).
Now assume that we have a lax natural transfomation, i.e. the f_α go into the opposite direction and are no longer invertible. We assume that we have a 2-fibration as well. Then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i),\mathcal{G}(i')) \arrow[r, "\xi(i')\circ"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i),\mathcal{E}(i')) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(Y(i),Y(i')) \arrow[r, "f(i')\circ"] & \mathrm{Hom}_{\mathcal{S}}(Y(i),Z(i'))
\end{tikzcd}$$

is Cartesian as well. Moreover we have an adjunction with the full comma category

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i),\mathcal{G}(i')) \underset{\mathrm{can}}{\overset{\rho_{\xi(i'),\mathcal{G}(i)}}{\leftrightarrows}} \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i),\mathcal{E}(i')) \times_{/\,\mathrm{Hom}_{\mathcal{S}}(Y(i),Z(i'))} \mathrm{Hom}_{\mathcal{S}}(Y(i),Y(i'))$$

with $\rho\circ\mathrm{can}=\mathrm{id}$, in particular with the morphism 'can' fully faithful. See below for the precise definition of ρ. Hence we define

$$\mathcal{G}(\alpha) \coloneqq \rho_{\xi(i'),\mathcal{G}(i)}(\mathcal{E}_\alpha\circ\xi(i), f_\alpha, Y_\alpha)$$

and get at least a morphism (coming from the unit of the adjunction):

$$\Xi_\alpha : (\xi(i')\circ\mathcal{G}(\alpha), \mathrm{id}, p(\mathcal{G}(\alpha))) \Rightarrow (\mathcal{E}_\alpha\circ\xi(i), f_\alpha, Y_\alpha).$$

The first component of Ξ_α this time (potentially) define a lax-natural transformation $\xi:\mathcal{G}\to\mathcal{E}$ only. To turn $\mathcal{G}$ into a pseudo-functor, we have to see that ρ is functorial.

For a Cartesian arrow $\xi : \mathcal{E} \to \mathcal{F}$, we define

$$\rho_{\xi,\mathcal{G}} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{G},\mathcal{F}) \times_{/\, \mathrm{Hom}_{\mathcal{S}}(U,T)} \mathrm{Hom}_{\mathcal{S}}(U,S) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{G},\mathcal{E})$$

as follows: Let (τ,μ,g) be a tuple with $g \in \mathrm{Hom}_{\mathcal{S}}(U,S)$, $\tau \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{G},\mathcal{F})$ and

$$\mu : f \circ g \Rightarrow p(\tau)$$

a 2-morphism. We may choose a coCartesian 2-morphism

$$\widetilde{\mu} : X \Rightarrow \tau$$

above μ. We set $\rho_\xi(\tau,\mu,g)$ equal to an object with an isomorphism

$$(\xi \circ \rho_\xi(\tau,\mu,g), \mathrm{id}, p(\rho_\xi(\tau,\mu,g))) \xrightarrow{\sim} (X, \mathrm{id}, g).$$

Together with the morphism

$$\widetilde{\mu} : (X, \mathrm{id}, g) \longrightarrow (\tau,\mu,g)$$

we get the counit

$$\mathrm{can} \circ \rho_\xi \Rightarrow \mathrm{id}\,.$$

We need to define a 2-isomorphism $G_{\beta\alpha} \Rightarrow G_\beta \circ G_\alpha$. i.e.

$$\rho_{\xi(i''),\mathcal{G}(i)}(\mathcal{E}_{\beta\alpha} \circ \xi(i), f_{\beta\alpha}, Y_{\beta\alpha}) \to$$
$$\rho_{\xi(i''),\mathcal{G}(i')}(\mathcal{E}_\beta \circ \xi(i'), f_\beta, Y_\beta) \circ \rho_{\xi(i'),\mathcal{G}(i)}(\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha)$$

First of all, we get three Cartesian 2-morphisms

$$\begin{array}{rlcll} \widetilde{f_{\beta\alpha}}: & X_{\beta\alpha} & \Rightarrow & \mathcal{E}_{\beta\alpha} \circ \xi(i) & \text{over } f_{\beta\alpha} \\ \widetilde{f_\alpha}: & X_\alpha & \Rightarrow & \mathcal{E}_\alpha \circ \xi(i) & \text{over } f_\alpha \\ \widetilde{f_\beta}: & X_\beta & \Rightarrow & \mathcal{E}_\beta \circ \xi(i') & \text{over } f_\beta \end{array}$$

and have to define an isomorphism (after applying can)

$$(X_{\beta\alpha}, \mathrm{id}, Y(\beta\alpha)) \xrightarrow{\sim} (X_\beta, \mathrm{id}, Y(\beta)) \circ (X_\alpha, \mathrm{id}, Y(\alpha)).$$

We have the diagram

$$\begin{tikzcd}
\mathcal{G}(i) \arrow[r, "f(i)"] \arrow[d, "\mathcal{G}(\alpha)"'] \arrow[dr, "X_\alpha"] \arrow[dr, phantom, shift left=3ex, "\nearrow^{\mu_\alpha}"] \arrow[dr, phantom, shift right=3ex, "\nearrow^{\sim}"] & \mathcal{E}(i) \arrow[d, "\mathcal{E}(\alpha)"] \arrow[dd, bend left=60, "\mathcal{E}(\beta\alpha)"] \\
\mathcal{G}(i') \arrow[r, "f(i')"] \arrow[d, "\mathcal{G}(\beta)"'] \arrow[dr, "X_\beta"] \arrow[dr, phantom, shift left=3ex, "\nearrow^{\mu_\beta}"] \arrow[dr, phantom, shift right=3ex, "\nearrow^{\sim}"] & \mathcal{E}(i') \arrow[d, "\mathcal{E}(\beta)"] \arrow[r, phantom, "\nearrow^{\sim}"] & {} \\
\mathcal{G}(i'') \arrow[r, "f(i'')"] & \mathcal{E}(i'')
\end{tikzcd} \tag{44}$$

and the diagram

$$
\begin{tikzcd}
\mathcal{G}(i) \arrow[rr, "f(i)"] \arrow[dd, "\mathcal{G}(\beta\alpha)"'] \arrow[ddrr, "X_{\beta\alpha}" description] & & \mathcal{E}(i) \arrow[dd, "\mathcal{E}(\beta\alpha)"] \arrow[ddl, phantom, "\Nearrow \mu_{\beta\alpha}"] \\
& & \\
\mathcal{G}(i'') \arrow[rr, "f(i'')"'] \arrow[uur, phantom, "\Nearrow \sim"] & & \mathcal{E}(i'')
\end{tikzcd}
\tag{45}
$$

The two pastings are both Cartesian (using Lemma 5.6.1 below) over the pastings in the diagram

$$
\begin{tikzcd}
Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"'] \arrow[dr, phantom, "\Nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \arrow[dd, bend left=60, "Z(\beta\alpha)"] \\
Y(i') \arrow[r, "f(i')"] \arrow[d, "Y(\beta)"'] \arrow[dr, phantom, "\Nearrow f_\beta"] & Z(i') \arrow[r, phantom, "\Nearrow Z_{\beta,\alpha}"] \arrow[d, "Z(\beta)"] & {} \\
Y(i'') \arrow[r, "f(i'')"] & Z(i'')
\end{tikzcd}
\quad \text{resp.} \quad
\begin{tikzcd}
{} & Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"'] \arrow[dd, bend right=60, "Y(\beta\alpha)"'] \arrow[dr, phantom, "\Nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \arrow[dd, bend left=60, "Z(\beta\alpha)"] \\
{} \arrow[r, phantom, "\Nearrow Y_{\beta,\alpha}"] & Y(i') \arrow[r, "f(i')"] \arrow[d, "Y(\beta)"'] \arrow[dr, phantom, "\Nearrow f_\beta"] & Z(i') \arrow[r, phantom, "\Nearrow Z_{\beta,\alpha}"] \arrow[d, "Z(\beta)"] & {} \\
{} & Y(i'') \arrow[r, "f(i'')"] & Z(i'')
\end{tikzcd}
$$

The pasting in the second diagram is just $f_{\beta,\alpha}$ by definition of lax natural transformation for f. This yields an isomorphism between the pastings in diagram (44) and (45) *over* $Y_{\beta,\alpha}$ which we define to be $\mathcal{G}_{\beta,\alpha}$. One checks that this defines indeed a pseudo-functor $\mathcal{G}$ such that $\xi : \mathcal{G} \to \mathcal{E}$ is a lax natural transformation which is 1-Cartesian. $\square$

Lemma 5.6.1. *Let $\mathcal{D} \to \mathcal{S}$ be a 2-(op)fibration of 2-categories. Let $\mu : \alpha \Rightarrow \beta$ be a 2-(co)Cartesian morphism, where $\alpha, \beta : \mathcal{E} \to \mathcal{F}$ are 1-morphisms. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a 1-morphism then $\gamma * \mu$ is 2-(co)Cartesian. Similarly, if $\gamma' : \mathcal{G} \to \mathcal{E}$ is a 1-morphism then $\mu * \gamma'$ is 2-(co)Cartesian.*

Proof. This follows immediately from the axiom that composition is a morphism of (op)fibrations. $\square$

6 Common features of fibered multiderivators and six-functor-formalisms

6.1 Internal and external monoidal structure.

6.1.1. Let $\mathcal{D}$, $\mathcal{S}$ be symmetric (for simplicity) 2-multicategories with all 2-morphisms invertible. Let $\mathcal{D} \to \mathcal{S}$ be a symmetric 1-bifibered and 2-isofibered 2-multicategory such that also $\mathcal{S} \to \{\cdot\}$ is 1-bifibered. Then any pseudo-functor of 2-multicategories $s : \{\cdot\} \to \mathcal{S}$ with value S gives rise to a symmetric closed monoidal structure $\otimes$ on the 2-category $\mathcal{D}_S$. Moreover $\mathcal{D} \to \{\cdot\}$ is also fibered by transitivity (cf. Proposition 2.4.25). Therefore the whole 2-category $\mathcal{D}$ carries a closed monoidal structure $\boxtimes$ as well. We call $\otimes$ the **internal product**, and $\boxtimes$ the **external product**, and write $\mathcal{HOM}$, and $\mathbf{HOM}$, respectively for the adjoints. We also denote by $\otimes$ the monoidal product in $\mathcal{S}$ itself and by $\mathcal{HOM}$ its adjoint.

6.1.2. The functor s specifies, in particular, a distinguished 1-multimorphism $\Delta \in \mathrm{Hom}(S, S; S)$. By abuse of notation, we denote by Δ (resp. Δ') the corresponding 1-morphisms

$$\Delta : S \otimes S \to S \qquad \Delta' : S \to \mathcal{HOM}(S, S).$$

By the arguments in the proof of the transitivity of bifibrations of multicategories (cf. Proposition 2.4.25), we see that we actually have

Corollary 6.1.3.

$$\begin{aligned} \mathcal{E} \otimes \mathcal{F} &\cong \Delta_\bullet(\mathcal{E} \boxtimes \mathcal{F}), \\ \mathcal{HOM}(\mathcal{E}, \mathcal{F}) &\cong (\Delta')^\bullet(\mathbf{HOM}(\mathcal{E}, \mathcal{F})). \end{aligned}$$

Example 6.1.4. *Let us investigate the internal and external monoidal structure in the case $\mathcal{S} = \mathcal{S}^{\mathrm{cor}}$ (cf. Definition 3.1.1). Here the 1-morphisms Δ and Δ' are respectively given by the correspondences*

$$\begin{tikzcd} & S \arrow[dl, "\mathrm{diag}"'] \arrow[dr, equal] & \\ S \times S & ; & S \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & S \arrow[dl, equal] \arrow[dr, "\mathrm{diag}"] & \\ S & ; & S \times S. \end{tikzcd}$$

From this we see that $\Delta_\bullet \cong \Delta^*$ and $(\Delta')^\bullet \cong \Delta^!$ hold.
In the other direction, we can also reconstruct the external monoidal product and its adjoint from the internal one. The functor $\boxtimes$ is the push-forward along the coCartesian 1-morphism

$$\begin{tikzcd} & S \times S \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[drr, equal] & & \\ S & S & ; & S \times S, \end{tikzcd}$$

hence we have $(- \boxtimes -) \cong (\mathrm{pr}_1^* - \otimes \mathrm{pr}_2^* -)$. *The functor* $\mathbf{HOM}$ *is the pull-back w.r.t. the first slot (say) along the Cartesian (w.r.t. the first slot) 1-morphism:*

$$\begin{tikzcd} & S \times S \arrow[dl, equal] \arrow[d, "\mathrm{pr}_1"] \arrow[drr, "\mathrm{pr}_2"] & & \\ S \times S & S & ; & S, \end{tikzcd}$$

hence we have $\mathbf{HOM}(-,-) \cong \mathcal{HOM}(\mathrm{pr}_1^* -, \mathrm{pr}_2^! -)$.

Example 6.1.5. *Let us investigate the internal and external monoidal structure in the case* $\mathcal{S} = \mathrm{Dia}^{\mathrm{cor}}$ *(cf. Definition 5.2.3). By Proposition 5.2.10 we know* $I \otimes I \cong I \times I$ *and* $\mathcal{HOM}(I, I) \cong I^{\mathrm{op}} \times I$. *The 1-morphism* Δ *is given by the correspondence*

$$\begin{tikzcd} & I \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ I \times I & ; & I. \end{tikzcd}$$

To determine Δ', *observe that the correspondence*

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[d, equal] \arrow[drr, equal] & & \\ I & I & ; & I \end{tikzcd}$$

belongs (via 5.2.7) to the following functor in $\mathrm{Fun}(I^{\mathrm{op}} \times I^{\mathrm{op}} \times I, \mathrm{Dia})$*:*

$$\begin{array}{rcl} F_I : I^{\mathrm{op}} \times I^{\mathrm{op}} \times I & \to & \mathrm{Dia} \\ i, i', i'' & \mapsto & \mathrm{Hom}(i, i'') \times \mathrm{Hom}(i', i'') \end{array}$$

which yields (via 5.2.7 again) the correspondence Δ'*:*

$$\begin{tikzcd} & \int \nabla F_I \arrow[dl, "p"'] \arrow[dr, "q"] & \\ I & & I^{\mathrm{op}} \times I \end{tikzcd}$$

and we have $\int \nabla F_I = I \times_{/I} \mathrm{tw}(I)$.
We see that $\Delta_\bullet \cong \Delta^*$ *and* $(\Delta')^\bullet \cong p_* q^*$ *hold. The latter is also the same as* $\mathrm{pr}_{2,*} \pi_* \pi^*$ *for the following functors:*

$$\mathrm{tw}(I) \xrightarrow{\pi} I^{\mathrm{op}} \times I \xrightarrow{\mathrm{pr}_2} I.$$

Given a bifibration of 1-multicategories $\mathcal{D} \to \mathcal{S}$, *this explains more conceptually the construction of the "multi-pull-back" in the multicategory of functors* $\mathrm{Fun}(I, \mathcal{D})$ *in Proposition 4.7.6. Using Proposition 2.4.26 one can even give an alternative proof of Proposition 4.7.6 in case that* $\mathcal{S}$ *is closed monoidal (i.e. bifibered over* $\{\cdot\}$*). Applying Propositions 2.4.25 and 2.4.26 to the composition*

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}} \to \{\cdot\}$$

we can show that for a derivator $\mathbb{D}$ it is the same

1. *to define an absolute monoidal product and absolute Hom which commute with left, resp. right Kan extensions in the correct way (conditions 1.–3. of 2.4.26) or*
2. *to give $\mathbb{D}$ the structure of a closed monoidal derivator.*

6.2 Grothendieck and Wirthmüller contexts

6.2.1. Let $\mathcal{S}$ be a category with fiber products and final object and let $\mathcal{S}_0$ be a class of morphisms in $\mathcal{S}$. We can define a submulticategory $\mathcal{S}^{\mathrm{cor},0}$ of $\mathcal{S}^{\mathrm{cor}}$ where the 2-morphisms are those

$$\begin{tikzcd} & & & A \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[dd, "\gamma"] & \\ S_1 & \cdots & S_n & & T \\ & & & A' \arrow[ulll] \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in which $\gamma \in \mathcal{S}_0$. If $\mathcal{S}_0$ is the class of *all* morphisms in $\mathcal{S}$, then we denote $\mathcal{S}^{\mathrm{cor},0}$ by $\mathcal{S}^{\mathrm{cor},G}$.

Lemma 6.2.2. *Consider the 2-category $\mathcal{S}^{\mathrm{cor},0}$ and a morphism $f : S \to T$ in $\mathcal{S}_0$ such that also $\Delta_f : S \to S \times_T S$ is in $\mathcal{S}_0$. Then the morphisms*

$$f^{\mathrm{op}} : \begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, equal] & \\ T & & S \end{tikzcd} \qquad f : \begin{tikzcd} & S \arrow[dl, equal] \arrow[dr, "f"] & \\ S & & T \end{tikzcd}$$

are adjoints in the 2-category $\mathcal{S}^{\mathrm{cor},0}$.

Proof. We give unit and counit:

$$f \circ f^{\mathrm{op}} \Rightarrow \mathrm{id} : \begin{tikzcd} & T \arrow[dl, equal] \arrow[dr, equal] & \\ T & & T \\ & S \arrow[ul, "f"] \arrow[ur, "f"'] \arrow[uu, "f"] & \end{tikzcd}$$

$$\mathrm{id} \Rightarrow f^{\mathrm{op}} \circ f : \begin{tikzcd} & S \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta_f"] & \\ S & & S \\ & S \times_T S \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] & \end{tikzcd} \tag{46}$$

One easily checks the unit/counit equations. $\square$

Proposition 6.2.3. *1. Let $\mathcal{D} \to \mathcal{S}^{\mathrm{cor},0}$ be a proper six-functor-formalism (cf. 3.1.4). If $\Delta_f \in \mathcal{S}_0$ (in many examples this is always the case) then there is a canonical natural transformation*

$$f_! \to f_*$$

which is an isomorphism if $f \in \mathcal{S}_0$.

2. Let $\mathcal{D} \to \mathcal{S}^{\mathrm{cor},0}$ be an etale six-functor-formalism (cf. 3.1.4). If $f \in \mathcal{S}_0$ then there is a canonical natural transformation

$$f^* \to f^!$$

which is an isomorphism if $\Delta_f \in \mathcal{S}_0$.

In particular, for a Wirthmüller context, we have a canonical isomorphism $f^* \cong f^!$ for all morphisms f in $\mathcal{S}$, and for a Grothendieck context, we have a canonical isomorphism $f_! \cong f_*$ for all morphisms f in $\mathcal{S}$. This justifies the naming.

Proof. We prove the first assertion, the second is shown analogously. To give a natural transformation as claimed is equivalent to give a morphism

$$f^* f_! \to \mathrm{id},$$

or equivalently

$$\mathrm{pr}_{2,!}\, \mathrm{pr}_1^* \to \mathrm{id}$$

with pr_1 and pr_2 as in (46). This natural transformation is given by means of the 2-pullback along the 2-morphism of correspondences (46). If f is in $\mathcal{S}_0$ then this is the counit of an adjunction and hence it induces a canonical isomorphism $f_! \cong f_*$ (uniqueness of adjoints up to canonical isomorphism). □

Example 6.2.4. *From the properties of 1/2 (op)fibrations of 2-multicategories one can derive many compatibilities of the morphism $f_! \to f_*$. For example in a proper six-functor-formalism $\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$ for a Cartesian square*

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d, "G"'] & T \arrow[d, "g"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

in $\mathcal{S}$ the following diagram is 2-commutative

$$\begin{tikzcd} G^* F_! \arrow[r, "\sim"] \arrow[d] & f_! g^* \arrow[d] \\ G^* F_* \arrow[r, "\mathrm{exc.}"] & f_* g^* \end{tikzcd}$$

provided that Δ_f, Δ_F are in $\mathcal{S}_0$.

Example 6.2.5. *The following diagram, which is depicted on the front cover of Lipman's book [LH09] (there a specific Grothendieck context is considered, namely quasi-coherent sheaves on a certain class of proper schemes), is commutative:*

$$\begin{tikzcd}
f_* \operatorname{Hom}(-, f^!-) \arrow[r, "\sim"] \arrow[d] & \operatorname{Hom}(f_!-, -) \\
f_* \operatorname{Hom}(f^* f_!-, f^!-) \arrow[r, "\sim"] & \operatorname{Hom}(f_!-, f_* f^!-) \arrow[u]
\end{tikzcd} \tag{47}$$

Here the horizontal morphisms are induced by the natural transformations

$$f^* f_! \to \mathrm{id}, \tag{48}$$

and

$$f_* f^! \to \mathrm{id}, \tag{49}$$

respectively, which are the natural transformations on the push-forward (resp. the pull-back) induced by the 2-morphisms of correspondences given by

$$f \circ f^{\mathrm{op}} \Rightarrow \mathrm{id} : \begin{tikzcd}
& T \arrow[dl, equal] \arrow[dr, equal] & \\
T & & T \\
& S \arrow[ul, "f"] \arrow[ur, "f"'] \arrow[uu, "f"] &
\end{tikzcd} \quad \textit{and} \quad \mathrm{id} \Rightarrow f^{\mathrm{op}} \circ f : \begin{tikzcd}
& S \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta_f"] & \\
S & & S \\
& S \times_T S \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

Note: The isomorphism $f_! \cong f_*$ of Proposition 6.2.3, 1. is constructed in such a way that the two morphisms (48) and (49) are identified with the two counits

$$f^* f_* \to \mathrm{id} \qquad \textit{and} \qquad f_! f^! \to \mathrm{id}\,.$$

Proof. Taking adjoints this is the same as to show that the diagram

$$\begin{tikzcd}
f_!(- \otimes f^*-) & (f_!-) \otimes - \arrow[l, "\sim"'] \arrow[d] \\
f_!((f^* f_!-) \otimes (f^*-)) \arrow[u] & f_! f^*((f_!-) \otimes -) \arrow[l, "\sim"']
\end{tikzcd}$$

is commutative. This is just the diagram induced on push-forwards by the following commutative diagram of 2-morphisms of multicorrespondences.

$$\begin{array}{c}
\left(\begin{tikzcd}[ampersand replacement=\&] \& \& S \arrow[dll, equal] \arrow[dl, "f"] \arrow[dr, "f"] \& \\ S \& T \& ; \& T \end{tikzcd} \right) \xrightarrow{\sim} \cdots \\
\downarrow \\
\left(\begin{tikzcd}[ampersand replacement=\&] \& \& S \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[dr, "f"] \& \\ T \& T \& ; \& T \end{tikzcd} \right) \circ_1 \left(\begin{tikzcd}[ampersand replacement=\&] \& S \arrow[dl, equal] \arrow[dr, "f"] \& \\ S \& ; \& T \end{tikzcd} \right) \xrightarrow{\sim} \cdots
\end{array}$$

$$\begin{array}{c}
\cdots \xrightarrow{\sim} \left(\begin{tikzcd}[ampersand replacement=\&] \& \& T \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] \& \\ T \& T \& ; \& T \end{tikzcd} \right) \circ_1 \left(\begin{tikzcd}[ampersand replacement=\&] \& S \arrow[dl, equal] \arrow[dr, "f"] \& \\ S \& ; \& T \end{tikzcd} \right) \\
\uparrow \\
\cdots \xrightarrow{\sim} \left(\begin{tikzcd}[ampersand replacement=\&] \& S \arrow[dl, "f"'] \arrow[dr, "f"] \& \\ T \& ; \& T \end{tikzcd} \right) \circ \left(\begin{tikzcd}[ampersand replacement=\&] \& \& T \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] \& \\ T \& T \& ; \& T \end{tikzcd} \right) \circ_1 \left(\begin{tikzcd}[ampersand replacement=\&] \& S \arrow[dl, equal] \arrow[dr, "f"] \& \\ S \& ; \& T \end{tikzcd} \right)
\end{array}$$

□

Hence, for a Grothendieck context given by Definiton 3.3.2 (as is the context considered in [LH09]) the commutativity of the diagram (47) follows from Proposition 3.3.3.

6.2.6. Analogously we can say that a 1-bifibration and 2-opfibration over $\mathrm{Dia}^{\mathrm{cor}}$ is a Wirthmüller context. Note that in $\mathrm{Dia}^{\mathrm{cor}}$ all functors supply valid 2-morphisms. This shows that to construct e.g. a monoidal derivator one does not have to start with a pre-multiderivator but could use an arbitrary 1-bifibration and and 2-opfibration over $\mathrm{Dia}^{\mathrm{cor}}$. In detail:
Let $\alpha : I \to J$ be a functor between diagrams in Dia. Recall from Lemma 5.5.6, 1. that in the category $\mathrm{Dia}^{\mathrm{cor}}$ the correspondences

$$[\alpha]' : \begin{tikzcd} & J \times_{/J} I \arrow[dl, "\alpha"'] \arrow[dr] & \\ J & & I \end{tikzcd} \qquad [\alpha] : \begin{tikzcd} & I \times_{/J} J \arrow[dl] \arrow[dr, "\alpha"] & \\ I & & J \end{tikzcd}$$

are adjoints. Using this, we can reconstruct from a strict 2-functor $\mathcal{D} \to \mathrm{Dia}^{\mathrm{cor}}$ which is 1-opfibered and 2-fibered with 1-categorical fibers a (non-strict) pre-multiderivator as follows: Consider the embedding ι from Proposition 5.2.9 and consider the pull-back (cf.

Definition 2.4.23) of $\mathcal{D}$
$$\begin{tikzcd} \iota^*\mathcal{D} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathrm{Dia}^{1-\mathrm{op}} \arrow[r, "\iota"] & \mathrm{Dia}^{\mathrm{cor}}. \end{tikzcd}$$
The 1-opfibration, and 2-fibration of 2-multicategories $\iota^*\mathcal{D} \to \mathrm{Dia}^{1-\mathrm{op}}$ with 1-categorical fibers can be seen (cf. Proposition 2.4.16) as a pseudo-functor
$$\mathbb{D} : \mathrm{Dia}^{1-\mathrm{op}} \to \mathcal{MCAT}.$$
The adjointness of $[\alpha]$ and $[\alpha]'$ shows that $\mathcal{D}$ *is determined by* $\iota^*\mathcal{D}$ and can thus be reconstructed by the construction in 5.3.7 that associates the 2-multicategory $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ with a pre-multiderivator $\mathbb{D}$. The only difference is that the $\mathbb{D}$ reconstructed from $\mathcal{D}$ might not be a strict 2-functor.

7 (Co)homological descent

7.1 Categories of $\mathbb{S}$-diagrams

Definition 7.1.1. *Let $\mathbb{S}$ be a strong right derivator with Grothendieck pre-topology (cf. Definition 4.5.2). Recall the 2-category* $\mathrm{Cat}(\mathbb{S})$ *from Definition 4.6.1.*
A **category of $\mathbb{S}$-diagrams in** $\mathrm{Cat}(\mathbb{S})$ *is a full sub-2-category $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$, satisfying the following axioms:*

(SDia1) The empty diagram $(\varnothing, -)$, the diagrams $(\cdot, S)$ for any $S \in \mathbb{S}(\cdot)$, and (Δ_1, f) for any $f \in \mathbb{S}(\Delta_1)$ are objects of $\mathcal{DIA}$.

(SDia2) $\mathcal{DIA}$ is stable under taking finite coproducts and such fibered products, where one of the morphisms is of pure diagram type.

(SDia3) For each morphism $\alpha : D_1 \to D_2$ with $D_i = (I_i, F_i)$ in $\mathcal{DIA}$ and for each object $i \in I_2$ and morphism $U \to F_2(i)$ being part of a cover in the chosen pre-topology, the slice diagram $D_1 \times_{/D_2} (i, U)$ is in $\mathcal{DIA}$, and if α is of pure diagram type then also $(i, F_2(i)) \times_{/D_2} D_1$ is in $\mathcal{DIA}$.

A **category of $\mathbb{S}$-diagrams** *$\mathcal{DIA}$ is called* **infinite**, *if it satisfies in addition:*

(SDia5) $\mathcal{DIA}$ is stable under taking arbitrary coproducts.

There is an obvious dual notion of a category of $\mathbb{S}$-diagrams in $\mathrm{Cat}^{\mathrm{op}}(\mathbb{S})$. If $\mathbb{S}$ is the trivial derivator both definitions boil down to the previous definition of a diagram category 4.1.1.

7.2 Fundamental (co)localizers

Definition 7.2.1. *A class of morphisms $\mathcal{W}$ in a category is called* **weakly saturated**, *if it satisfies the following properties:*

(WS1) Identities are in $\mathcal{W}$.

(WS2) $\mathcal{W}$ has the 2-out-of-3 property.

(WS3) If $p : Y \to X$ and $s : X \to Y$ are morphisms such that $p \circ s = \mathrm{id}_X$ and $s \circ p \in \mathcal{W}$ then $p \in \mathcal{W}$ (and hence $s \in \mathcal{W}$ by (WS2)).

Definition 7.2.2. *Let $\mathbb{S}$ be a strong right derivator with Grothendieck pre-topology (4.5.2). Let $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$ be a category of $\mathbb{S}$-diagrams (cf. 7.1.1).*
Consider a family of subclasses $\mathcal{W}_S$ of morphisms in $\mathcal{DIA} \times_{/\mathcal{DIA}} (\cdot, S)$ parametrized by all objects $S \in \mathbb{S}(\cdot)$. Such a family $\{\mathcal{W}_S\}_S$ is called a **system of relative localizers** *if the following properties are satisfied:*

(L0) For any morphism $S_1 \to S_2$ the induced functor $\mathcal{DIA} \times_{/\mathcal{DIA}} (\cdot, S_1) \to \mathcal{DIA} \times_{/\mathcal{DIA}} (\cdot, S_2)$ maps $\mathcal{W}_{S_1}$ to $\mathcal{W}_{S_2}$.

(L1) *Each $\mathcal{W}_S$ is weakly saturated.*

(L2 left) *If $D = (I, F) \in \mathcal{DIA}$, and I has a final object e, then the projection $D \to (e, F(e))$ is in $\mathcal{W}_{F(e)}$.*

(L3 left) *For any commutative diagram in $\mathcal{DIA}$ over $(\cdot, S)$*

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 = (E, F) & \end{tikzcd}$$

and for any chosen covers $\{U_{e,i} \to F(e)\}$ for all $e \in E$, the following implication holds true:

$$\forall e \in E \ \forall i \quad w \times_{/D_3} (e, U_{e,i}) \in \mathcal{W}_{U_{e,i}} \quad \Rightarrow \quad w \in \mathcal{W}_S.$$

(L4 left) *For any morphism $w : D_1 \to D_2 = (E, F)$ of pure diagram type over $(\cdot, S)$ the following implication holds true:*

$$\forall e \in E \quad (e, F(e)) \times_{/D_2} D_1 \to (e, F(e)) \in \mathcal{W}_{(e,F(e))} \quad \Rightarrow \quad w \in \mathcal{W}_S.$$

There is an obvious dual notion of a **system of colocalizers** in $\mathcal{DIA} \subset \mathrm{Cat}^{\mathrm{op}}(\mathbb{S})$ where $\mathbb{S}$ is supposed to be a strong left derivator with Grothendieck pre-cotopology.

Definition 7.2.3. *Let $\mathbb{S}$ be a strong right derivator. Assume we are given a Grothendieck pre-topology on $\mathbb{S}$ (cf. 4.5.2). Let $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$ be a category of $\mathbb{S}$-diagrams (cf. 7.1.1). A subclass $\mathcal{W}$ of morphisms in $\mathcal{DIA}$ is called an* **absolute localizer** *(or just* **localizer***) if the following properties are satisfied:*

(L1) *$\mathcal{W}$ is weakly saturated.*

(L2 left) *If $D = (I, F) \in \mathcal{DIA}$, and I has a final object e, then the projection $D \to (e, F(e))$ is in $\mathcal{W}$.*

(L3 left) *For any commutative diagram in $\mathcal{DIA}$*

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 = (E, F) & \end{tikzcd}$$

and chosen covering $\{U_{i,e} \to F_3(e)\}$ for all $e \in E$, the following implication holds true:

$$\forall e \in E \ \forall i \quad w \times_{/D_3} (e, U_i) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

(L4 left) *For any morphism $w : D_1 \to D_2 = (E, F)$ of pure diagram type, the following implication holds true:*

$$\forall e \in E \quad (e, F(e)) \times_{/D_2} D_1 \to (e, F(e)) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

There is an obvious dual notion of absolute **colocalizer** in $\mathcal{DIA} \subset \mathrm{Cat}^{\mathrm{op}}(\mathbb{S})$ where $\mathbb{S}$ is supposed to be a strong left derivator with Grothendieck pre-cotopology.
Recall the identification

$$\begin{array}{rcl} \mathrm{Cat}(\mathbb{S}) & \to & \mathrm{Cat}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}} \\ (I, F) & \mapsto & (I^{\mathrm{op}}, F^{\mathrm{op}}) \end{array}$$

By abuse of notation, we denote the image of $\mathcal{DIA}$ under this identification by $\mathcal{DIA}^{\mathrm{op}}$. Note that if $\mathbb{S}$ is a strong right derivator with Grothendieck pre-topology, then $\mathbb{S}^{\mathrm{op}}$ is a strong left derivator with Grothendieck pre-cotopology.

Remark 7.2.4.

1. *If $\mathcal{W}$ is a localizer in $\mathcal{DIA}$, then $\mathcal{W}^{\mathrm{op}}$ is a colocalizer in $\mathcal{DIA}^{\mathrm{op}}$ and vice versa. The same holds true for systems of relative localizers.*

2. *If $\mathbb{S}$ is the trivial derivator, then a system of relative localizers or a localizer are the same notion, and (L1–L3 left) are precisely the definition of fundamental localizer of Grothendieck.*

Proposition 7.2.5 (Grothendieck)**.** *If $\mathbb{S} = \{\cdot\}$ is the trivial derivator, then* $\mathrm{Cat}(\cdot) = \mathrm{Cat}^{\mathrm{op}}(\cdot)$ *as 2-categories. If $\mathcal{DIA}$ is self-dual, i.e. if $\mathcal{DIA}^{\mathrm{op}} = \mathcal{DIA}$ under this identification, then the notions of localizer, localizer without (L4 left), colocalizer, and colocalizer without (L4 right) are all equivalent.*

Proof. [Cis04, Proposition 1.2.6] □

Remark 7.2.6. *The class of localizers is obviously closed under intersection, hence there is a smallest localizer $\mathcal{W}^{\min}_{\mathcal{DIA}}$. Furthermore the smallest localizer in $\mathcal{DIA}$ and the smallest colocalizer in $\mathcal{DIA}^{\mathrm{op}}$ correspond. If $\mathbb{S}$ is the trivial derivator and $\mathcal{DIA} = \mathrm{Cat}$, Cisinski [Cis04, Théorème 2.2.11] has shown that $\mathcal{W}^{\min}_{\mathrm{Cat}}$ is precisely the class $\mathcal{W}_\infty$ of functors $\alpha : I \to J$ such that $N(\alpha)$ is a weak equivalence in the classical sense (of simplicial sets, resp. topological spaces). For a localizer in the sense of Definition 7.2.3 this implies the following:*

Theorem 7.2.7. *If $\mathcal{DIA} = \mathrm{Cat}(\mathbb{S})$ and $\mathcal{W}$ is an absolute localizer in $\mathcal{DIA}$ and $\alpha \in \mathcal{W}_\infty$, i.e. $\alpha : I \to J$ is a functor such that $N(\alpha)$ is a weak equivalence of topological spaces, the morphism $(\alpha, \mathrm{id}) : (I, p_I^* S) \to (J, p_J^* S)$ is in $\mathcal{W}$ for all $S \in \mathbb{S}(\cdot)$. The same holds analogously for a system of relative localizers.*

Proof. The class of functors $\alpha : I \to J$ in Cat such that $(\alpha, \mathrm{id}) : (I, p_I^* S) \to (J, p_J^* S)$ is in $\mathcal{W}$ obviously form a fundamental localizer in the classical sense. □

7.2.8. We will for (notational) simplicity assume that the following properties hold:

1. $\mathbb{S}$ has all **relative finite coproducts** (i.e. for each Grothendieck opfibration with finite *discrete* fibers $p : O \to I$ the functor p^* has a left adjoint $p_!$ and Kan's formula holds true for it).
2. For all finite families $(S_i)_{i \in I}$ of objects in $\mathbb{S}(\cdot)$ the collection $\{S_i \to \coprod_{j \in I} S_j\}_{i \in I}$ is a cover.

Let $\varnothing$ be the initial object of $\mathcal{S}$ (which exists by 1.). Then the map

$$\varnothing \to (\cdot, \varnothing),$$

where $\varnothing$ on the left denotes the empty diagram, is in $\mathcal{W}$ (resp. in $\mathcal{W}_\varnothing$, and hence in all $\mathcal{W}_S$) by (L3 left) applied to the empty cover.
From this and (L3 left) again it follows that for a finite collection $(S_i)_{i \in I}$ of objects of $\mathbb{S}(\cdot)$ the map

$$(I, (S_i)_{i \in I}) \to (\cdot, \coprod_{i \in I} S_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{\coprod_{i \in I} S_i}$). More generally, if we have a Grothendieck opfibration with finite *discrete* fibers $p : O \to I$ and a diagram $F \in \mathbb{S}(O)$ (over $S \in \mathbb{S}(\cdot)$), then the morphism

$$(O, F) \to (I, p_! F)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Example 7.2.9 (Mayer-Vietoris)**.** *For the simplest non-trivial example of a non-constant map in $\mathcal{W}$ consider a cover $\{U_1 \to S, U_2 \to S\}$ in $\mathbb{S}(\cdot)$ consisting of two* mono*morphisms*[27]*. Then the projection*

$$p : \left(\begin{tikzcd} \text{“}U_1 \times_S U_2\text{”} \arrow[r] \arrow[d] & U_1 \\ U_2 & \end{tikzcd} \right) \to S$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) as is easily derived from the axioms (L1–L4). See 7.5.14 for how the Mayer-Vietoris long exact sequence is related to this.

7.2.10. Let $\alpha, \beta : D_1 \to D_2$ be two morphisms in $\mathcal{DIA}$. Recall that it is the same to give a 2-morphism $\alpha \Rightarrow \beta$ or a morphism $D_1 \times \Delta_1 \to D_2$ such that for $i = 1, 2$ the compositions $D_1 \xrightarrow{e_i} D_1 \times \Delta_1 \longrightarrow D_2$ are α and β respectively. We call α and β **homotopic** if they are equivalent for the smallest equivalence relation containing by the following relation: $\alpha \sim \beta$, if there exists a 2-morphism $\alpha \Rightarrow \beta$. In other words α and β are homotopic if there is a finite set of 1-morphisms $\gamma_0, \gamma_1, \dots, \gamma_n : D_1 \to D_2$ such that $\gamma_0 = \alpha$ and $\gamma_n = \beta$ and a chain of 2-morphisms:

$$\gamma_0 \Leftarrow \gamma_1 \Rightarrow \gamma_2 \Leftarrow \cdots \Rightarrow \gamma_n.$$

[27] For an arbitrary $\mathbb{S}$ this means that the projections “$U_i \times_S U_i$” $\to U_i$ are isomorphisms.

Proposition 7.2.11. *Let $\mathcal{DIA}$ be a category of $\mathbb{S}$-diagrams (cf. 7.1.1) and let $\mathcal{W}$ be localizer in $\mathcal{DIA}$ (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers). Then $\mathcal{W}$ (resp. $\{\mathcal{W}_S\}_S$) satisfies the following properties:*

1. *The localizer $\mathcal{W}$ (resp. each $\mathcal{W}_S$) is closed under coproducts.*

2. *Let $\widetilde{s} = (s, \mathrm{id}) : D_2 = (I_2, s^*F) \to D_1 = (I_1, F)$ be a morphism in $\mathcal{DIA}$ (resp. over $(\cdot, S)$) of pure diagram type such that s has a left adjoint $p : I_1 \to I_2$. Then the obvious morphisms $\widetilde{p} : D_1 \to D_2$ and $\widetilde{s}$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).*

3. *Given a commutative diagram in $\mathcal{DIA}$ (resp. one over $(\cdot, S)$)*

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 & \end{tikzcd}$$

   where the underlying functors of the morphisms to D_3 are Grothendieck opfibrations and the underlying functor of w is a morphism of opfibrations, and coverings $\{U_{e,i} \to F_3(e)\}$ for all $e \in I_3$, then (in the relative case)

$$\forall e \in I_3 \ \forall i \quad w \times_{D_3} (e, U_{e,i}) \in \mathcal{W}_{U_{e,i}} \quad \Rightarrow \quad w \in \mathcal{W}_S$$

   or (in the absolute case)

$$\forall e \in I_3 \ \forall i \quad w \times_{D_3} (e, U_{e,i}) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}$$

4. *If $f : D_1 \to D_2$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) then also $f \times E : D_1 \times E \to D_2 \times E$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) for any $E \in \mathrm{Cat}$ such that the morphism $f \times E$ is a morphism in $\mathcal{DIA}$.*

5. *Any morphism which is homotopic (in the sense of 7.2.10) to a morphism in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).*

Proof. 1. This property follows immediately from (L3 left) applied to a diagram

$$\begin{tikzcd} \coprod_{i \in I} D_{1,i} \arrow[rr] \arrow[dr] & & \coprod_{i \in I} D_{2,i} \arrow[dl] \\ & (I, p_I^* S) & \end{tikzcd}$$

where I is considered to be a discrete category. (In the absolute case let S be the final object of $\mathbb{S}(\cdot)$.)
2. We first show that $\widetilde{p} \in \mathcal{W}$. Using (L3 left), it suffices to show that $\widetilde{p}_i : D_1 \times_{/I_2} i \to D_2 \times_{/I_2} i$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) for all $i \in I_2$, however by the adjunction we have $I_1 \times_{/I_2} i = I_1 \times_{/I_1} s(i)$

and therefore $I_1 \times_{/I_2} i$ has a final object. In the diagram

$$\begin{tikzcd} D_1 \times_{/I_2} i \arrow[r, "\widetilde{p}_i"] \arrow[d] & D_2 \times_{/I_2} i \arrow[d] \\ (\cdot, s(i)^* F) \arrow[r, equal] & (\cdot, s(i)^* F) \end{tikzcd}$$

the vertical morphisms are thus in $\mathcal{W}$ (resp. $\mathcal{W}_S$) and so is the upper horizontal morphism. That $\widetilde{s}$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) will follow from 4. because this implies that $\widetilde{s} \circ \widetilde{p}$ and $\widetilde{p} \circ \widetilde{s}$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) therefore by (L1) also $\widetilde{s}$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$). For note that the unit and the counit extend to 2-morphisms of diagrams.
3. Using (L3 left), we have to show that $D_1 \times_{/D_3} (e, U_{e,i}) \to D_2 \times_{/D_3} (e, U_{e,i})$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$). Since the underlying functors of $D_1 \to D_3$ and $D_2 \to D_3$ are Grothendieck opfibrations, we have a diagram over $(e, U_{e,i})$:

$$\begin{tikzcd} D_1 \times_{D_3} (e, U_{e,i}) \arrow[r] \arrow[d, shift left, "\iota_e"] & D_2 \times_{D_3} (e, U_{e,i}) \arrow[d, shift left, "\iota_e"] \\ D_1 \times_{/D_3} (e, U_{e,i}) \arrow[r] \arrow[u, shift left, "s_e"] & D_2 \times_{/D_3} (e, U_{e,i}) \arrow[u, shift left, "s_e"] \end{tikzcd}$$

where the underlying functor of ι_e is of diagram type and is right adjoint to s_e. Therefore s_e is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$) by 2. and hence the same holds for ι_e because $s_e \iota_e = \mathrm{id}$ (using L1). Note: we are not using the not yet proven part of 2. Since the top arrow is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$) the same holds for the bottom arrow.
4. This is a special case of 2.
5. A natural transformation $\mu : f \Rightarrow g$ for $f, g : D_1 \to D_2$ can be seen as a morphism of diagrams $\mu : \Delta_1 \times D_1 \to D_2$ such that $\mu \circ e_0 = f$ and $\mu \circ e_1 = g$. Since the projection $p : \Delta_1 \times D_1 \to D_1$ is in $\mathcal{W}$ by 3. also the morphisms $e_{0,1} : D_1 \to \Delta_1 \times D_1$ are in $\mathcal{W}$. Since $\mu \circ e_0 = f$ and $\mu \circ e_1 = g$, the morphism f is in $\mathcal{W}$ if and only if $g \in \mathcal{W}$. □

Proposition 7.2.12. *Axiom (L4 left) is, in the presence of (L1–L3 left), equivalent to the following, apparently weaker axiom:*

(L4' left) *Let $w : D_1 \to D_2$ be a morphism (resp. a morphism over $(\cdot, S)$) of pure diagram type such that the underlying functor is a Grothendieck fibration. Then (in the relative case)*

$$\forall e \in I_2 \quad (e, F_2(e)) \times_{D_2} D_1 \to (e, F_2(e)) \in \mathcal{W}_{F_2(e)} \quad \Rightarrow \quad w \in \mathcal{W}_S$$

or (in the absolute case)

$$\forall e \in I_2 \quad (e, F_2(e)) \times_{D_2} D_1 \to (e, F_2(e)) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

Proof. (L4' left) implies (L4 left): Consider the following 2-commutative diagram

$$\begin{tikzcd} (E,F)\times_{/(E,F)}(I,p^*F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & (I,p^*F)=D_1 \arrow[d] \\ (E,F) \arrow[r, equal] & (E,F)=D_2 \end{tikzcd}$$

The underlying diagram functor of the top horizontal map (which is not purely of diagram type) is a Grothendieck opfibration and hence by Proposition 7.2.11, 3. it is in $\mathcal{W}$ (resp. $\mathcal{W}_S$), provided that the morphisms of the fibers $(E\times_{/E}e, \mathrm{pr}_1^*F)\to(\cdot, F(e))$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$). However $E\times_{/E}e$ has the final object id_e whose value under pr_1^*F is $F(e)$. The morphisms of the fibers are therefore in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$) by (L2 left). The underlying diagram functor of the left vertical map is a Grothendieck fibration and pr_1^*F is constant along the fibers. Therefore the fact that all $(e,F(e))\times_{/D_2}D_1\to(e,F(e))$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$) implies that the left vertical map is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) by (L4' left). Thus also the right vertical map is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$). (This uses Proposition 7.2.11, 5. and the fact that the two compositions in the diagram are homotopic).
(L4 left) implies (L4' left): If $D_1\to D_2=(E,F)$ is a morphism whose underlying functor is a Grothendieck fibration as in Axiom (L4' left), the morphism of *constant* diagrams $(e,F(e))\times_{D_2}D_1\to(e,F(e))\times_{/D_2}D_1$ is in $\mathcal{W}$ (resp. $\mathcal{W}_{F(e)}$) (their underlying functors being part of an adjunction), therefore (L4 left) applies. □

7.3 Simplicial objects in a localizer

7.3.1. In this section, we fix a strong right derivator $\mathbb{S}$ equipped with a Grothendieck pre-topology and satisfying the assumptions of 7.2.8 and a category of $\mathbb{S}$-diagrams $\mathcal{DIA}$ (cf. 7.1.1). Assume that for all $S_\bullet\in\mathbb{S}(\Delta^{\mathrm{op}})$ the diagrams $(\Delta^{\mathrm{op}},S_\bullet)$ and also all truncations $((\Delta^{\le n})^{\mathrm{op}},S_\bullet)$ are in $\mathcal{DIA}$. Later we will assume that also $((\Delta^\circ)^{\mathrm{op}},S_\bullet)$ for all $S_\bullet\in\mathbb{S}((\Delta^\circ)^{\mathrm{op}})$ (injective simplex diagram) and all truncations $((\Delta^{\circ,\le n})^{\mathrm{op}},S_\bullet)$ are in $\mathcal{DIA}$. The reasoning in this section uses little of the explicit definition of Δ^{op}. For comparison with classical texts on cohomological descent we stick to the particular diagram Δ^{op}.
Consider the category $\mathbb{S}(\Delta^{\mathrm{op}})$. Since $\mathbb{S}$ has all (relative) finite coproducts, $\mathbb{S}(\cdot)$ is actually tensored over $\mathcal{SETF}$, hence $\mathbb{S}(\Delta^{\mathrm{op}})$ will be tensored over $\mathcal{SETF}^{\Delta^{\mathrm{op}}}$. We sketch this construction. A finite simplicial set, i.e. a functor $\xi:\Delta^{\mathrm{op}}\to\mathcal{SETF}$, can be seen as a functor with values in finite discrete categories. The corresponding Grothendieck construction yields a Grothendieck opfibration $\pi_\xi:\int\xi\to\Delta^{\mathrm{op}}$. We define for $X_\bullet\in\mathbb{S}(\Delta^{\mathrm{op}})$:

$$\xi\otimes X_\bullet := (\pi_\xi)_!(\pi_\xi)^*X_\bullet.$$

Recall that the notion '$\mathbb{S}$ has *relative finite coproducts*' means that all functors $(\pi_\xi)_!$ arising this way exist and can be computed fiber-wise.

7.3.2. Consider the full subcategory $\Delta^{\le n}$ of Δ consisting of $\Delta_0,\dots,\Delta_n$. Since $\mathbb{S}$ is assumed to be a right derivator, the restriction functor

$$\iota^*:\mathbb{S}(\Delta^{\mathrm{op}})\to\mathbb{S}((\Delta^{\le n})^{\mathrm{op}})$$

has a right adjoint ι_*, which is usually called the **coskeleton** and denoted cosk^n.
Let some simplicial object $Y_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$ and a morphism $\alpha : X_{\le n} \to \iota^* Y_\bullet$ be given. Consider the full subcategory $(\Delta^{\mathrm{op}} \times \Delta_1)^{0-\le n}$ of all objects $\Delta_i \times [1]$ for all $i \in \mathbb{N}_0$, and $\Delta_i \times [0]$ for $i \le n$. The restriction

$$\iota^* : \mathbb{S}(\Delta^{\mathrm{op}} \times \Delta_1) \to \mathbb{S}((\Delta^{\mathrm{op}} \times \Delta_1)^{0-\le n})$$

has again an adjoint ι_*. Since $\mathbb{S}$ is assumed to be strong we can consider α as an object over $(\Delta \times \Delta_1)^{0-\le n}$. The first row of $\iota_* \alpha$ is called the **relative coskeleton** $\mathrm{cosk}^n(X_{\le n}|Y_\bullet)$ of $X_{\le n}$. For $n = -1$ we understand $\mathrm{cosk}^{-1}(-|Y_\bullet) = Y_\bullet$.
These constructions work the same way with Δ replaced by Δ°. The functor 'coskeleton' and 'relative coskeleton' is in both cases even *the same functor*, i.e. these functors commute with the restriction of a simplicial to a semi-simplicial object[28]. This would not at all be true for the corresponding left adjoint, the functor 'skeleton'.

We call a diagram I in a diagram category Dia **contractible**, if $I \to \cdot$ lies in every fundamental localizer on Dia.

Lemma 7.3.3. 1. *Let Δ be the simplex category, and I a category admitting a final object i. Let $N(I)$ be the nerve of I. Then the category*

$$\int_{(\Delta^\circ)^{\mathrm{op}}} N(I)$$

is contractible.

2. *Let Δ be the simplex category, and I a category admitting a final object i. Let $N(I)$ be the nerve of I. Then the category*

$$\int_{\Delta^{\mathrm{op}}} N(I)$$

is contractible.

3. *Let Δ° be the injective simplex category and I a* directed *category admitting a final object i. Let $N^\circ(I)$ be the semi-simplicial nerve of I, defined by letting $N^\circ(I)_m$ be the set of functors $[n] \to I$ such that no non-identity morphism is mapped to an identity. Then the category*

$$\int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I)$$

is contractible.

[28] To see this, e.g., for the case of the 'coskeleton', observe that there is an adjunction:

$$\Delta_m \times_{/(\Delta^\circ)^{\mathrm{op}}} (\Delta^\circ_{\le n})^{\mathrm{op}} \rightleftarrows \Delta_m \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\le n}.$$

Proof. 1. is shown in [Cis04, Proposition 2.2.3]. 2. is the same but considering $N(I)$ as a functor from $(\Delta^\circ)^{\mathrm{op}}$ to $\mathcal{SET}$. The same proof works when $(\Delta^\circ)^{\mathrm{op}}$ is replaced by Δ^{op}. 3. is also just a small modification of [loc. cit.]. Define a functor $\xi : \int N^\circ(I) \to \int N^\circ(I)$ as follows: an object (n, x), where $x \in N^\circ(I)_n$ is mapped to (n, x) if $x(n) = i$ and to $(n+1, x')$ with

$$x'(k) = \begin{cases} x(k) & k \le n \\ i & k = n+1. \end{cases}$$

otherwise. There are natural transformations

$$\mathrm{id}_{\int N^\circ(I)} \Rightarrow \xi \qquad i \Rightarrow \xi$$

where i denotes here the constant functor with value $(0, i)$, showing that $\int N^\circ(I)$ is contractible. □

Corollary 7.3.4. *The diagrams Δ, Δ°, $\int_{\Delta^{\mathrm{op}}} \Delta_n$, $\int_{(\Delta^\circ)^{\mathrm{op}}} \Delta_n^\circ$, $\int_{\Delta^{\mathrm{op}}} \Delta_n \times \Delta_m$ and $\Delta_m \times_{/\Delta^{\mathrm{op}}} (\Delta^\circ)^{\mathrm{op}} = \int_{(\Delta^\circ)^{\mathrm{op}}} \Delta_m$ are contractible.*

Proof. The simplicial set Δ_n is just the nerve N of $[n]$. Likewise the semi-simplicial set Δ_n° is the semi-simplicial nerve N° of $[n]$. □

Note that the diagram $\int \Delta_n^\circ$ is even *finite*.

Lemma 7.3.5. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
Let $((\Delta^{\mathrm{op}})^2, F_{\bullet,\bullet}) \in \mathcal{DIA}$ be a bisimplicial diagram (resp. a bisimplicial diagram over $(\cdot, S)$) and let $\delta : \Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^2$ be the diagonal. Then the morphism

$$(\Delta^{\mathrm{op}}, \delta^* F_{\bullet,\bullet}) \to ((\Delta^{\mathrm{op}})^2, F_{\bullet,\bullet})$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Remark 7.3.6. *The statement of the Lemma is false when Δ is replaced by Δ°.*

Proof of Lemma 7.3.5. We focus on the absolute case. For the relative case the proof is identical. Since the morphism in the statement is of pure diagram type, we may check the condition of (L4 left): we have to show that the category

$$(\Delta_m \times \Delta_n) \times_{/(\Delta^{\mathrm{op}})^2} \Delta^{\mathrm{op}}$$

is contractible, say, on the diagram category of diagrams I such that $(I, F_{m,n}) \in \mathcal{DIA}$. Equivalently we may prove this for the dual category. Objects of that category are diagrams of the form:

$$\begin{tikzcd} \Delta_{m'} \arrow[d] \arrow[dr] & \\ \Delta_m & \Delta_n \end{tikzcd}$$

This is the category $\Delta/(\Delta_m \times \Delta_n)$ which is contractible by Lemma 7.3.3, 2. Note that this is the only feature of Δ^{op} used in the proof of this Lemma. □

Remark 7.3.7. *The previous lemma should be seen in the following context: the Grothendieck construction gives a way of embedding the category of simplicial sets into the category of small categories. This construction maps weak equivalences to weak equivalences and induces an equivalence between the corresponding homotopy categories. A bisimplicial set can be seen as a simplicial object in the category of simplicial sets. Its homotopy colimit is given by the diagonal simplicial set. On the other hand the homotopy colimit in the category of small categories is just given by the Grothendieck construction. From this perspective, the lemma is clear if $\mathbb{S}$ is the derivator associated with the category of sets (equipped with the discrete topology).*

Lemma 7.3.8. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
Consider a simplicial diagram $(\Delta^{\mathrm{op}}, F_\bullet) \in \mathcal{DIA}$ (resp. a simplicial diagram over $(\cdot, S)$). The morphism

$$(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_n) \to (\Delta^{\mathrm{op}}, F_\bullet)$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Proof. We focus on the absolute case. For the relative case the proof is identical. The diagram $(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_n)$ is equivalent to $(\int \Delta_n, \pi^* F_\bullet)$ by definition (see 7.3.1). We apply the criterion of (L4 left) to the resulting map

$$(\int \Delta_n, \pi^* F_\bullet) \to (\Delta^{\mathrm{op}}, F_\bullet)$$

and have to show that

$$\Delta_m \times_{/\Delta^{\mathrm{op}}} \int \Delta_n$$

is contractible. This category is again (dual to) the category of objects

$$\begin{tikzcd} \Delta_k \arrow[d] \arrow[dr] & \\ \Delta_m & \Delta_n \end{tikzcd}$$

and we have already seen in the proof of Lemma 7.3.5 that it is contractible. $\square$

Corollary 7.3.9. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
Let $f, g : (\Delta^{\mathrm{op}}, F_\bullet) \to (\Delta^{\mathrm{op}}, G_\bullet)$ be two homotopic morphisms of simplicial objects (resp. morphisms over $(\cdot, S)$). Then $f \in \mathcal{W}$ (resp. in $\mathcal{W}_S$) if and only if $g \in \mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. The statement follows by the standard argument because the projection $(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_1) \to (\Delta^{\mathrm{op}}, F_\bullet)$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) by Lemma 7.3.8. $\square$

Proposition 7.3.10 (Čech resolutions are in $\mathcal{W}$)**.** *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Let $U \to S$ be a local epimorphism in $\mathbb{S}(\cdot)$. Then the morphism

$$p : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U|S)) \to (\cdot, S)$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Proof. To simplify the exposition we focus on the case in which $\mathbb{S}$ is associated with a category $\mathcal{S}$. The reader may check however that everything goes through in the general case because the only constructions involved can be expressed as right Kan extensions. The assumption means that there is a cover $\mathcal{U} = \{U_i \to S\}$ in the given pre-topology, such that for all indices i, the induced map

$$p_i : U \times_S U_i \to U_i$$

has a section s_i. By axiom (L3 left) it suffices to show that for all i the map

$$\widetilde{p_i} : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)) \to (\cdot, U_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). Explicitly the simplicial object $\mathrm{cosk}^0(U \times_S U_i \,|\, U_i)$ is given by

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} U \times_S U \times_S U \times_S U_i \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} U \times_S U \times_S U_i \rightrightarrows U \times_S U_i$$

Since Δ^{op} is contractible (in particular the morphism $(\Delta^{\mathrm{op}}, p^*T) \to (\cdot, T)$ is in $\mathcal{W}$, resp. in $\mathcal{W}_T$, for any $T \in \mathbb{S}(\cdot)$), it suffices to show that the map

$$\widetilde{p_i} : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)) \to (\Delta^{\mathrm{op}}, p^* U_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). There is a section

$$\widetilde{s_i} : (\Delta^{\mathrm{op}}, p^* U_i) \to (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i))$$

induced by s_i such that $\widetilde{p_i} \circ \widetilde{s_i} = \mathrm{id}$. By (L1) it then suffices to check that $\widetilde{s_i} \circ \widetilde{p_i} \in \mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). We will construct a homotopy between id and $\widetilde{s_i} \circ \widetilde{p_i}$

$$(\Delta^{\mathrm{op}}, \Delta_1 \times \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)) \to (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i))$$

in the sense of simplicial objects. This will suffice by Corollary 7.3.9. Since Δ_1 is 1-coskeletal, and the cosk^0's anyway, it suffices to construct the homotopy in degrees 0 and 1:

$$\begin{array}{ccc}
\mathrm{Hom}(\Delta_1, \Delta_1) \times U \times_S U \times_S U_i & \rightrightarrows & \mathrm{Hom}(\Delta_0, \Delta_1) \times U \times_S U_i \\
\downarrow & & \downarrow \\
U \times_S U \times_S U_i & \rightrightarrows & U \times_S U_i
\end{array}$$

This can be achieved by mapping $\mathrm{id}_{\Delta_1} \times (U \times_S U_i) \times_{U_i} (U \times_S U_i)$ to $(U \times_S U_i) \times_{U_i} (U \times_S U_i)$ via $(s_i \circ \mathrm{pr}_2) \times \mathrm{id}$. □

Definition 7.3.11. *A morphism $X_\bullet \to Y_\bullet$ of simplicial objects is called a* **hypercover** *if the following two equivalent conditions hold:*

1. *In any diagram of simplicial objects*

$$\begin{tikzcd} \partial\Delta_n \otimes U \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ \Delta_n \otimes U \arrow[r] & Y_\bullet \end{tikzcd}$$

there is a cover $\mathcal{U} = \{U_i \to U\}$ such that for all i there is a lift (indicated by a dotted arrow) in the diagram

$$\begin{tikzcd} \partial\Delta_n \otimes U_i \arrow[r] \arrow[d] & \partial\Delta_n \otimes U \arrow[r] & X_\bullet \arrow[d] \\ \Delta_n \otimes U_i \arrow[r] \arrow[urr, dotted] & \Delta_n \otimes U \arrow[r] & Y_\bullet \end{tikzcd}$$

2. *For any $n \geq 0$ the morphism*

$$X_n \to \operatorname{cosk}^{n-1}(\iota^*_{\leq n-1} X_\bullet \,|\, Y_\bullet)_n$$

admits local sections in the pre-topology on $\mathbb{S}$ (i.e. it is a local epimorphism).

Remark 7.3.12.

1. *In particular the notion of hypercover depends only on the Grothendieck topology generated by the pre-topology because a morphism is a local epimorphism precisely if the sieve generated by it is a covering sieve.*

2. *The equivalent condition 1. of the definition of hypercover shows that, if $\mathbb{S}$ is the derivator associated with the category $\mathcal{SET}$ equipped with the discrete topology, then a hypercover is precisely a trivial Kan fibration.*

Definition 7.3.13. *If in condition 2. of Definition 7.3.11 the morphism is even an isomorphism for all sufficiently large n, then α is called a* **finite (or bounded) hypercover**. *Equivalently we have $X_\bullet \cong \operatorname{cosk}^n(\iota^*_{\leq n} X_\bullet \,|\, Y_\bullet)$ for some n.*

Lemma 7.3.14. *Let $\mathcal{W}$ be a localizer (resp. $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
For a finite hypercover $X_\bullet \to Y_\bullet$ (resp. one over $(\cdot, S)$) such that $X_\bullet \cong \operatorname{cosk}^{i+1}(X_\bullet \,|\, Y_\bullet)$ and $\iota^*_{\leq i-1} X_\bullet \cong \iota^*_{\leq i-1} Y_\bullet$ the morphism $(\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, Y_\bullet)$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. Again, to simplify the exposition we focus on the case in which $\mathbb{S}$ is associated with a category $\mathcal{S}$. We may assume $i \geq 1$ because otherwise we are in the situation of Lemma 7.3.10. The assumptions imply that the map $X_i \to Y_i$ is a local epimorphism. Indeed, this is the map $X_i \to Y_i = \operatorname{cosk}^{i-1}(\iota^*_{\leq i-1} X_\bullet \,|\, Y_\bullet)_i$ in this case. Therefore the morphism $X_j \to Y_j$ is actually a local epimorphism for all j.

Consider the following diagram in $\mathcal{DIA}$:

$$\begin{tikzcd}
(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, (X_\bullet \times_{Y_\bullet} X_\bullet \,|\, X_\bullet)) \arrow[r] \arrow[d] & (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, (X_\bullet \,|\, Y_\bullet)) \arrow[d] \\
(\Delta^{\mathrm{op}}, X_\bullet) \arrow[r] & (\Delta^{\mathrm{op}}, Y_\bullet)
\end{tikzcd}$$

where

$$(X_\bullet \,|\, Y_\bullet)_{m,n} := \mathrm{cosk}^0(X_n \,|\, Y_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+1 \text{ factors}} .$$

$$(X_\bullet \times_{Y_\bullet} X_\bullet \,|\, X_\bullet)_{m,n} := \mathrm{cosk}^0(X_n \times_{Y_n} X_n \,|\, X_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+2 \text{ factors}} .$$

The vertical morphisms are in $\mathcal{W}$ by Proposition 7.2.11, 3. because its columns are in $\mathcal{W}$ by Lemma 7.3.10. Again by Proposition 7.2.11, 3. it then suffices to show that the rows

$$p : (\Delta^{\mathrm{op}}, (X_\bullet \times Y_\bullet \,|\, X_\bullet)_{m,\bullet}) \longrightarrow (\Delta^{\mathrm{op}}, (X_\bullet \,|\, Y_\bullet)_{m,\bullet})$$

of the top horizontal morphism are in $\mathcal{W}$. These are again hypercovers of the form considered in this Lemma, in particular i-coskeletal, where the i-truncation is given by

$$\begin{tikzcd}
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] \arrow[d] & X_{i-1} = Y_{i-1} \cdots \arrow[d, equal] & \cdots \arrow[r, shift left] \arrow[r, shift right] & X_0 = Y_0 \arrow[d, equal] \\
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+1} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & Y_{i-1} \cdots & \cdots \arrow[r, shift left] \arrow[r, shift right] & Y_0
\end{tikzcd}$$

where the left-most vertical arrow is induced by the map $\Delta_{m+1} \to \Delta_{m+2}$, $i \mapsto i$. There is a section s, with s_i induced by the map

$$\Delta_{m+2} \to \Delta_{m+1}, \quad i \mapsto \begin{cases} i & i < m+2, \\ m+1 & i = m+2. \end{cases}$$

We will construct a homotopy $\mu : \mathrm{id} \Rightarrow s \circ p$ of truncated simplicial objects:

$$\begin{tikzcd}
\mathrm{Hom}(\Delta_i, \Delta_1) \times \underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] \arrow[d, "\mu_i"] & \mathrm{Hom}(\Delta_{i-1}, \Delta_1) \times Y_{i-1} \cdots \arrow[d, "\mu_{i-1}"] & \cdots \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(\Delta_0, \Delta_1) \times Y_0 \arrow[d, "\mu_0"] \\
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & Y_{i-1} \cdots & \cdots \arrow[r, shift left] \arrow[r, shift right] & Y_0
\end{tikzcd}$$

The morphism μ_i at the constant morphism $0 : \Delta_i \to \Delta_1$ is given by the identity, at the constant morphism $1 : \Delta_i \to \Delta_1$ given by $s_i \circ p_i$, and at the other morphisms $\Delta_i \to \Delta_1$ arbitrarily. The existence of this homotopy allows by Lemma 7.3.9 and by (L1) to conclude. □

Theorem 7.3.15. *Let $\mathcal{W}$ be a localizer (resp. $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
Any finite hypercover (resp. one over S) considered as a morphism of diagrams in $\mathcal{DIA}$

$$(\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, Y_\bullet) \tag{50}$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).
Let $\iota : (\Delta^\circ)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ be the inclusion. If the morphism (50) exists in $\mathcal{DIA}$ then also

$$((\Delta^\circ)^{\mathrm{op}}, \iota^* X_\bullet) \to ((\Delta^\circ)^{\mathrm{op}}, \iota^* Y_\bullet)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. Any finite hypercover is a finite succession of hypercovers of the form considered in Lemma 7.3.14. The additional statement is a consequence of the following Lemma. □

Lemma 7.3.16. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*
Let $\iota : (\Delta^\circ)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ be the inclusion and let $(\Delta^{\mathrm{op}}, X_\bullet)$ be a simplicial diagram in $\mathcal{DIA}$ (resp. a simplicial diagram over $(\cdot, S)$). Then the morphism

$$((\Delta^\circ)^{\mathrm{op}}, \iota^* X_\bullet) \to (\Delta^{\mathrm{op}}, X_\bullet)$$

(if in $\mathcal{DIA}$) is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. We focus on the absolute case. For the relative case the proof is identical. Since the morphism in the statement is of pure diagram type, we may check the condition of (L4 left): we have to show that the category

$$\Delta_m \times_{/\Delta^{\mathrm{op}}} (\Delta^\circ)^{\mathrm{op}}$$

is contractible, say, on the diagram category of diagrams I such that $(I, X_m) \in \mathcal{DIA}$. This is true by Lemma 7.3.3, 1. □

7.4 Cartesian and coCartesian objects

Definition 7.4.1. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator of domain* Dia. *Let $I, E \in$* Dia *be diagrams and let $\alpha : I \to E$ be a functor in* Dia. *We say that an object*

$$X \in \mathbb{D}(I)$$

*is E-**(co)Cartesian**, if for any morphism $\mu : i \to j$ in I mapping to an identity in E, the corresponding morphism $\mathbb{D}(\mu) : i^* X \to j^* X$ is (co)Cartesian.*
If E is the trivial category, we omit it from the notation, and talk about (co)Cartesian objects.

These notions define full subcategories $\mathbb{D}(I)^{E\text{-cart}}$ (resp. $\mathbb{D}(I)^{E\text{-cocart}}$) of $\mathbb{D}(I)$, and $\mathbb{D}(I)_F^{E\text{-cart}}$ (resp. $\mathbb{D}(I)_F^{E\text{-cocart}}$) of $\mathbb{D}(I)_F$ for any $F \in \mathbb{S}(I)$.

Lemma 7.4.2. *The functor α^* w.r.t. a morphism $\alpha : D_1 \to D_2$ in* $\mathrm{Dia}(\mathbb{S})$ *maps Cartesian objects to Cartesian objects. The functor α^* for a morphism $\alpha : D_1 \to D_2$ in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ *maps coCartesian objects to coCartesian objects.*

Remark 7.4.3. *The categories of coCartesian objects are a generalization of the* **equivariant derived categories** *of Bernstein and Lunts [BL94]. For this let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be the stable fibered derivator of sheaves of abelian groups on (nice) topological spaces, where $\mathbb{S}$ is the pre-derivator associated with the category of (nice) topological spaces. Let G be a topological group acting on a space X. Then we may form the following simplicial space which is an object of $\mathbb{S}(\Delta^{\mathrm{op}})$:*

$$[G\backslash X]_\bullet : \ \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} G \times G \times X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} G \times X \substack{\longrightarrow\\\longrightarrow} X,$$

cf. [BL94, B1]. Then the category

$$\mathbb{D}(\Delta)^{\mathrm{cocart}}_{[G\backslash X]_\bullet}$$

is equivalent to the (unbounded) equivariant derived category, cf. [BL94, Proposition B4]. Note that all pull-back functors are exact in this context.

Definition 7.4.4. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator of domain* Dia. *We say that $\mathbb{D} \to \mathbb{S}$* **admits left Cartesian projections** *if for all functors $\alpha : I \to E$ in* Dia *and $S \in \mathbb{S}(\cdot)$, the fully-faithful inclusion*

$$\mathbb{D}(I)_F^{E-\mathrm{cart}} \to \mathbb{D}(I)_F$$

has a left adjoint $\square_!^E$. More generally we have four notions with the following notations:

$\square_!^E$	*left adjoint*	*left Cartesian projection*
$\blacksquare_!^E$	*right adjoint*	*right Cartesian projection*
$\blacksquare_*^E$	*left adjoint*	*left coCartesian projection*
$\square_*^E$	*right adjoint*	*right coCartesian projection*

We will, in general, only use left Cartesian and right coCartesian projection, the others being somewhat unnatural. In 8.3.1 we will show (using Brown representability) that for an infinite fibered derivator whose fibers are stable and well-generated (cf. Definitions 8.1.1, 8.1.7) a right coCartesian projection exists. Similarly if, in addition, Brown representability for the dual holds, e.g. if the fibers are compactly generated, then a left Cartesian projection exists (see 8.3.2) in many cases. Note that for a usual (non fibered) derivator, the notions 'Cartesian' and 'coCartesian' are equivalent. If for a fibered derivator with stable fibers both left and right Cartesian projections exist, then there is actually a *recollement* [Kra10, Proposition 4.13.1]:

$$\begin{tikzcd}
\mathbb{D}(I)_F^{E-\mathrm{cart}} \arrow[r, "\mathrm{incl.}"] & \mathbb{D}(I)_F \arrow[l, bend right, "\blacksquare_*"'] \arrow[l, bend left, "\square_!"] \arrow[r] & \mathbb{D}(I)_F/\mathbb{D}(I)_F^{E-\mathrm{cart}} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

Example 7.4.5. *The projections are difficult to describe explicitly, except in very special situations. Here a rather trivial example where this is possible. Let $\mathbb{D}$ be a stable derivator and consider $I = \Delta_1$, the projection $p : \Delta_1 \to \cdot$ and the inclusions $e_0, e_1 : \cdot \to \Delta_1$. Then a left and a right Cartesian projection exist and the recollement above is explicitly given by:*

$$\begin{tikzcd} \mathbb{D}(\Delta_1)^{\mathrm{cart}} \cong \mathbb{D}(\cdot) \arrow[r, "p^*"] & \mathbb{D}(\Delta_1) \arrow[l, shift right=2ex, "e_0^*"'] \arrow[l, shift left=2ex, "e_1^*"] \arrow[r, "\mathrm{C}"] & \mathbb{D}(\cdot) \arrow[l, shift right=2ex, "e_{1!}"'] \arrow[l, shift left=2ex, "{[-1]\circ e_{0,*}}"] \end{tikzcd}$$

Note that the functor C *(Cone) may be described as either $[1] \circ e_0^!$ or $e_1^?$ (cf. [Gro13, §3]) and that the essential image of p^* is precisely the kernel of* C*, which also coincides with the full subcategory of Cartesian=coCartesian objects.*

7.5 Weak and strong $\mathbb{D}$-equivalences

Definition 7.5.1 (left)**.** *Let* Dia *be a diagram category and let $\mathbb{S}$ be a strong right derivator with domain* Dia *equipped with a Grothendieck pre-topology. Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying (FDer0 right) and let $S \in \mathbb{S}(\cdot)$. A morphism $f : D_1 \to D_2$ in* $\mathrm{Dia}(\mathbb{S})/(\cdot, S)$ *is called a* **weak $\mathbb{D}$-equivalence** *relative to S if the natural transformation*

$$p_{1!}p_1^* \to p_{2!}p_2^*$$

is an isomorphism of functors. A morphism $f \in \mathrm{Dia}(\mathbb{S})$ is called a **strong $\mathbb{D}$-equivalence** *if the functor f^* induces an equivalence of categories*

$$f^* : \mathbb{D}(D_2)^{\mathrm{cart}} \to \mathbb{D}(D_1)^{\mathrm{cart}}.$$

Note that *weak* is a relative notion whereas *strong* is absolute.

Definition 7.5.2 (right)**.** *Let* Dia *be a diagram category, and let $\mathbb{S}$ be a strong left derivator with domain* Dia *equipped with a Grothendieck pre-cotopology. Let $\mathbb{D} \to \mathbb{S}$ be a right fibered derivator satisfying (FDer0 left) and $S \in \mathbb{S}(\cdot)$. A morphism $f : D_1 \to D_2$ in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})/(\cdot, S)$ is called a* **weak $\mathbb{D}$-equivalence** *relative to S, if the natural transformation*

$$p_{2*}p_2^* \to p_{1*}p_1^*$$

is an isomorphism of functors. A morphism $f \in \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ is called a **strong $\mathbb{D}$-equivalence** *if the functor f^* induces an equivalence of categories*

$$f^* : \mathbb{D}(D_2)^{\mathrm{cocart}} \to \mathbb{D}(D_1)^{\mathrm{cocart}}.$$

For a (left and right) derivator, i.e. for $\mathbb{S} = \cdot$, there is no difference between $\mathrm{Dia}(\mathbb{S})$ and $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ and then also the two different definitions of weak, resp. strong $\mathbb{D}$-equivalence coincide (for the case of weak $\mathbb{D}$-equivalences, note that the two conditions become adjoint to each other). These notions of $\mathbb{D}$-*equivalence* (right version) should be compared to the classical notions of *cohomological descent*, see [SGA72b, Exposé V$^{\mathrm{bis}}$].

Lemma 7.5.3 (left)**.** *Let $f : D_1 \to D_2$ be a morphism in* $\mathrm{Dia}(\mathbb{S})/(\cdot, S)$. *Then the following implication holds:*

$$f \text{ strong } \mathbb{D}\text{-equivalence} \quad \Rightarrow \quad f \text{ weak } \mathbb{D}\text{-equivalence relative to } S.$$

Proof. The morphism in the definition of weak $\mathbb{D}$-equivalence is induced by the counit w.r.t. the adjunction $f^*, f_!$:

$$p_{1!}p_1^* \cong p_{2!}f_!f^*p_2^* \to p_{2!}p_2^*$$

Now let $f_{!\square}$ be an inverse to f^*, as required by the definition of strong $\mathbb{D}$-equivalence. From $f^*p_2^* \cong p_1^*$ follows $p_{2!}f_{!\square} \cong p_{1!}$ and moreover the diagram

$$\begin{tikzcd}
p_{2!}f_!f^*p_2^* \arrow[r] \arrow[d, "\sim"] & p_{2!}p_2^* \\
p_{2!}f_{!\square}f^*p_2^* \arrow[ur] &
\end{tikzcd}$$

is commutative. Since the diagonal morphism is a natural isomorphism the statement follows. □

Of course there is an analogous right version of this lemma. The goal of this section is to prove the following two theorems:

Main Theorem 7.5.4 (right)**.** *Let* Dia *be a diagram category and let* $\mathbb{S}$ *be a strong left derivator with domain* Dia *equipped with a Grothendieck pre-cotopology.*

1. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator with domain* Dia *which is colocal in the sense of Definition 4.5.6 for the Grothendieck pre-cotopology on $\mathbb{S}$. Then the collection of classes $\{\mathcal{W}_{\mathbb{D},S}\}_S$, where $\mathcal{W}_{\mathbb{D},S}$ consists of those morphisms $f : D_1 \to D_2$ in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ *which are* weak $\mathbb{D}$-equivalences *relative to $S \in \mathbb{S}(\cdot)$, forms a system of relative colocalizers.*

2. *Let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator with domain* Dia *which is colocal in the sense of Definition 4.5.6 for the Grothendieck pre-cotopology on $\mathbb{S}$, with stable, compactly generated fibers. The class $\mathcal{W}_{\mathbb{D}}$ consisting of those morphisms $f : D_1 \to D_2$ in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ *which are* strong $\mathbb{D}$-equivalences *forms an absolute colocalizer.*

Main Theorem 7.5.5 (left)**.** *Let* Dia *be a diagram category and let* $\mathbb{S}$ *be a strong right derivator with domain* Dia *equipped with a Grothendieck pre-topology.*

1. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator with domain* Dia*, which is local in the sense of Definition 4.5.4 for the Grothendieck pre-topology on $\mathbb{S}$. Then the collection of classes $\{\mathcal{W}_{\mathbb{D},S}\}_S$, where $\mathcal{W}_{\mathbb{D},S}$ consists of those morphisms $f : D_1 \to D_2$ in* $\mathrm{Dia}(\mathbb{S})$ *which are* weak $\mathbb{D}$-equivalences *relative to $S \in \mathbb{S}(\cdot)$, forms a system of relative localizers.*

2. *Let* $\mathrm{Dia}'(\mathbb{S}) \subset \mathrm{Dia}(\mathbb{S})$ *be the full subcategory of the diagrams which consist of universally* $\mathbb{D}$*-local morphisms.*

   Let $\mathbb{D} \to \mathbb{S}$ *be an infinite fibered derivator with domain* Dia*, which is local in the sense of Definition 4.5.4 for the Grothendieck pre-topology on* $\mathbb{S}$*, with stable, compactly generated fibers. The class* $\mathcal{W}_{\mathbb{D}}$ *consisting of those morphisms* $f : D_1 \to D_2$ *in* $\mathrm{Dia}'(\mathbb{S})$ *which are* strong $\mathbb{D}$-equivalences *forms an absolute localizer in* $\mathrm{Dia}'(\mathbb{S})$*.*

Remark 7.5.6. *The restriction onto* $\mathrm{Dia}'(\mathbb{S})$ *in the left-variant of the theorem is needed because otherwise we do not know whether left Cartesian projections exist (cf. Theorem 8.3.2).*

The weak $\mathbb{D}$-equivalences for the case of usual derivators (i.e. for $\mathbb{S} = \{\cdot\}$) were called just '$\mathbb{D}$-equivalences' by Cisinski [Cis08] and it is rather straight-forward to see from the definition of derivator that they from a fundamental localizer in the classical sense (= absolute localizer for $\mathbb{S} = \{\cdot\}$, = system of relative localizers for $\mathbb{S} = \{\cdot\}$).
We will only prove the left-variant of the theorem. The other follows by logical duality and the restriction to $\mathrm{Dia}'(\mathbb{S})$ is not necessary because Lemma 7.5.11 is used instead of Lemma 7.5.10. Before proving the theorem we need a couple of lemmas. We assume for the rest of this section that Dia is a diagram category and that $\mathbb{S}$ is a strong right derivator with domain Dia equipped with a Grothendieck pre-topology.

Definition 7.5.7. *Two morphisms (in* $\mathrm{Dia}(\mathbb{S})$ *or in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$*)*

$$D_1 \underset{s}{\overset{p}{\rightleftarrows}} D_2$$

such that chains of 2-morphisms

$$p \circ s \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \mathrm{id}_{D_1} \qquad s \circ p \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \mathrm{id}_{D_2}$$

exist are called a **homotopy equivalence** *(or* p *is called as such if an* s *with this property exists).*

Lemma 7.5.8 (left)**.** *Let* $\mathbb{D}$ *be a left fibered derivator satisfying (FDer0 right) and let* $D_1, D_2 \in \mathrm{Dia}(\mathbb{S})$*. Given any homotopy equivalence* (p, s)*, then the functors* p^* *and* s^* *induce an equivalence*

$$\mathbb{D}(D_2)^{\mathrm{cart}} \underset{s^*}{\overset{p^*}{\rightleftarrows}} \mathbb{D}(D_1)^{\mathrm{cart}}$$

Proof. The 2-morphisms $\mu : (\alpha, f) \Rightarrow (\beta, g)$ in Definition 7.5.7 induce morphisms between the pull-back functors

$$(\alpha, f)^*\mathcal{E} \to (\beta, g)^*\mathcal{E}$$

which are isomorphisms on Cartesian objects. □

Example 7.5.9 (cf. also Proposition 7.2.11, 2.). *Let I_1, I_2 be diagrams in* Dia. *If*

$$\begin{tikzcd} I_1 \arrow[r, shift left, "p"] & I_2 \arrow[l, shift left, "s"] \end{tikzcd}$$

is an adjunction where p is left adjoint to s, and if $F \in \mathbb{S}(I_1)$ then we get an equivalence

$$\begin{tikzcd} \mathbb{D}(D_2)^{\mathrm{cart}} \arrow[r, shift left, "p^*"] & \mathbb{D}(D_1)^{\mathrm{cart}} \arrow[l, shift left, "s^*"] \end{tikzcd}$$

where $D_1 = (I_1, F)$ and $D_2 = (I_2, s^*F)$.

Lemma 7.5.10 (left). *Let* Dia *be an infinite diagram category and let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator with domain* Dia *with stable, compactly generated fibers. Consider a morphism $D = (I, F) \to (\cdot, S)$ such that F is a diagram of universally $\mathbb{D}$-local morphisms. Let $U \to S$ be a universally $\mathbb{D}$-local morphism. Write $D_U := D \times_{(\cdot,S)} (\cdot, U)$ in* $\mathrm{Dia}(\mathbb{S})$. *Then the following diagram is 2-commutative (via the exchange natural isomorphism):*

$$\begin{tikzcd} \mathbb{D}(D) \arrow[r, "\Box_!"] \arrow[d, "\mathrm{pr}_1^*"'] & \mathbb{D}(D)^{\mathrm{cart}} \arrow[d, "\mathrm{pr}_1^*"] \\ \mathbb{D}(D_U) \arrow[r, "\Box_!"] & \mathbb{D}(D_U)^{\mathrm{cart}} \end{tikzcd}$$

Note that left Cartesian projectors exist for D and D_U by Theorem 8.3.2.

Proof. The functor pr_1^* has a right adjoint pr_{1*} by (Dloc2 left) and then Brown representability theorem. (Dloc1 left) says that pr_1^* preserves coCartesian morphisms, hence pr_{1*} preserves Cartesian morphisms. Therefore the right adjoint of the given diagram is the following *commutative* diagram:

$$\begin{tikzcd} \mathbb{D}(D) & \mathbb{D}(D)^{\mathrm{cart}} \arrow[l] \\ \mathbb{D}(D_U) \arrow[u, "\mathrm{pr}_{1*}"] & \mathbb{D}(D_U)^{\mathrm{cart}} \arrow[l] \arrow[u, "\mathrm{pr}_{1*}"'] \end{tikzcd}$$

Consequently the exchange morphism of the diagram in the statement is also a natural isomorphism. □

Lemma 7.5.11 (right). *Let* Dia *be an infinite diagram category and let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator with domain* Dia *with stable, compactly generated fibers. Consider a morphism $D = (I, F) \to (\cdot, S)$. Let $S \to U$ be a universally $\mathbb{D}$-colocal morphism. Write $D_U := D \times_{(\cdot,S)} (\cdot, U)$ in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$. *Then the following diagram is 2-commutative (via the exchange natural isomorphism):*

$$\begin{tikzcd} \mathbb{D}(D) \arrow[r, "\Box_*"] \arrow[d, "\mathrm{pr}_1^*"'] & \mathbb{D}(D)^{\mathrm{cocart}} \arrow[d, "\mathrm{pr}_1^*"] \\ \mathbb{D}(D_U) \arrow[r, "\Box_*"] & \mathbb{D}(D_U)^{\mathrm{cocart}} \end{tikzcd}$$

Note that right Cartesian projectors exist for D and D_U by Theorem 8.3.1.

Proof. The functor pr_1^* has a left adjoint $\mathrm{pr}_{1!}$ by (Dloc2 right) and by the Brown representability theorem for the dual. (Dloc1 right) says that pr_1^* preserves Cartesian morphisms, hence $\mathrm{pr}_{1!}$ preserves coCartesian morphisms. Therefore the right adjoint of the given diagram is the following *commutative* diagram:

$$\begin{tikzcd} \mathbb{D}(D) & \mathbb{D}(D)^{\mathrm{cocart}} \arrow[l] \\ \mathbb{D}(D_U) \arrow[u, "\mathrm{pr}_{1!}"] & \mathbb{D}(D_U)^{\mathrm{cocart}} \arrow[l] \arrow[u, "\mathrm{pr}_{1!}"'] \end{tikzcd}$$

Consequently the exchange morphism of the diagram in the statement is also a natural isomorphism. □

Lemma 7.5.12 (left). *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator with domain* Dia *admitting a left Cartesian projection (cf. 7.4.4). For any Grothendieck opfibration*

$$\begin{tikzcd} I \arrow[d, "\pi"] \\ E \end{tikzcd}$$

in Dia*, for any diagram in $F \in \mathbb{S}(I)$, and for each element $e \in E$, the following diagram is 2-commutative:*

$$\begin{tikzcd} \mathbb{D}(I)_F \arrow[r, "\Box_!^E"] \arrow[d, "\iota^*"'] & \mathbb{D}(I)_F^{E-\mathrm{cart}} \arrow[d, "\iota^*"] \\ \mathbb{D}(I_e)_{F_e} \arrow[r, "\Box_!"] & \mathbb{D}(I_e)_{F_e}^{\mathrm{cart}} \end{tikzcd}$$

where $\iota : I_e \to I$ is the inclusion of the fiber.

Lemma 7.5.13 (right). *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator with domain* Dia *admitting a right coCartesian projection (cf. 7.4.4). For a Grothendieck fibration*

$$\begin{tikzcd} I \arrow[d, "\pi"] \\ E \end{tikzcd}$$

in Dia*, for any diagram in $F \in \mathbb{S}(I)$, and for each element $e \in E$, the following diagram is 2-commutative:*

$$\begin{tikzcd} \mathbb{D}(I)_F \arrow[r, "\Box_*^E"] \arrow[d, "\iota^*"'] & \mathbb{D}(I)_F^{E-\mathrm{cocart}} \arrow[d, "\iota^*"] \\ \mathbb{D}(I_e)_{F_e} \arrow[r, "\Box_*"] & \mathbb{D}(I_e)_{F_e}^{\mathrm{cocart}} \end{tikzcd}$$

where $\iota : I_e \to I$ is the inclusion of the fiber.

Proof. We restrict to the right-variant, the other being dual. We will show that the functor $\iota_!$ maps coCartesian objects to E-coCartesian ones. Then the left adjoint of the given diagram is the diagram

$$\begin{tikzcd} \mathbb{D}(I)_F & \mathbb{D}(I)_F^{E\text{-cocart}} \arrow[l] \\ \mathbb{D}(I_e)_{F_e} \arrow[u, "\iota_!"] & \mathbb{D}(I_e)_{F_e}^{\text{cocart}} \arrow[l] \arrow[u, "\iota_!"'] \end{tikzcd}$$

which is commutative. Consequently also the diagram of the statement is 2-commutative via the natural exchange morphism.
Let f in E be an object and let $\nu : i_1 \to i_2$ be a morphism in I mapping to id_f. Let α_k be the inclusions of $\cdot$ into I with image i_k. The morphism ν yields a natural transformation

$$\nu : \alpha_1 \Rightarrow \alpha_2.$$

Consider the diagram

$$\begin{tikzcd} e \times_{/E} f \arrow[r, shift left, "c_k"] \arrow[d, "p'"] & I_e \times_{/I} i_k \arrow[l, shift left, "\pi"] \arrow[r, "A_k"] \arrow[d, "p_k"'] \arrow[dr, phantom, "\swarrow \mu_k"] & I_e \arrow[d, "\iota"] \\ \cdot \arrow[r, equal] & \cdot \arrow[r, "\alpha_k"] & I \end{tikzcd}$$

where c_k is given on a morphism $\beta : e \to f$ in E by the choice of a Cartesian arrow $i_k' \to i_k$. It is right adjoint to π by the definition of Cartesian arrow.
There is a functor (composition with ν):

$$\widetilde{\nu} : I_e \times_{/I} i_1 \to I_e \times_{/I} i_2$$

such that $A_2 \widetilde{\nu} = A_1$ and $p_2 \widetilde{\nu} = p_1$. We have therefore a natural (point-wise) coCartesian morphism $\mathbb{S}(\mu_1)_\bullet \widetilde{\nu}^* \to \widetilde{\nu}^* \mathbb{S}(\mu_2)_\bullet$ of functors $\mathbb{D}(I_e \times_{/I} i_2)_{A_2^* F_e} \to \mathbb{D}(I_e \times_{/I} i_1)$.
We have also a natural transformation $\rho : \widetilde{\nu} c_1 \to c_2$ defined for a morphism $\beta : e \to f$ in E as the unique arrow $\rho(\beta)$ over id_e making the following diagram commutative:

$$\begin{tikzcd} i_1' \arrow[r, "c_1(\beta)"] \arrow[d, "\rho(\beta)"'] & i_1 \arrow[d, "\nu"] \\ i_2' \arrow[r, "c_2(\beta)"] & i_2 \end{tikzcd}$$

The resulting morphism $\mathbb{D}(\rho) : c_1^* \widetilde{\nu}^* \to c_2^*$ is point-wise coCartesian on coCartesian objects.

We get a commutative diagram of natural transformations

$$\begin{tikzcd}
\mathbb{S}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d, "\sim"] & \mathbb{S}(\mu_1)_\bullet A_1^* \iota^* \iota_! \arrow[r, "\mathbb{D}(\mu_1)'"] \arrow[d, "\sim"] & p_1^* \alpha_1^* \iota_! \arrow[d, "\sim"] \\
\mathbb{S}(\mu_1)_\bullet \widetilde{\nu}^* A_2^* \arrow[r] \arrow[d] & \mathbb{S}(\mu_1)_\bullet \widetilde{\nu}^* A_2^* \iota^* \iota_! \arrow[d] & \widetilde{\nu}^* p_2^* \alpha_1^* \iota_! \arrow[d, "\widetilde{\nu}^* p_2^*(\mathbb{D}(\nu))"] \\
\widetilde{\nu}^* \mathbb{S}(\mu_2)_\bullet A_2^* \arrow[r] & \widetilde{\nu}^* \mathbb{S}(\mu_2)_\bullet A_2^* \iota^* \iota_! \arrow[r, "\widetilde{\nu}^*(\mathbb{D}(\mu_2)')"] & \widetilde{\nu}^* p_2^* \alpha_2^* \iota_!
\end{tikzcd}$$

where the first two top vertical morphisms are the natural isomorphisms induced by $A_2\widetilde{\nu} = A_1$, the third top vertical morphism is the natural isomorphism induced by $p_2\widetilde{\nu} = p_1$, and the first two lower vertical morphisms are point-wise coCartesian. Here we use the notation $\mathbb{D}(\mu_1)'$ for the morphism $\mathbb{S}(\mu_1)_\bullet X \to Y$ induced by a morphism $\mathbb{D}(\mu_1) : X \to Y$.
Now we apply $p_{1!}$ to the outer square:

$$\begin{tikzcd}
p_{1!}\mathbb{S}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d] & p_{1!} p_1^* \alpha_1^* \iota_! \arrow[d] \\
p_{1!}\widetilde{\nu}^* \mathbb{S}(\mu_2)_\bullet A_2^* \arrow[r] & p_{1!}\widetilde{\nu}^* p_2^* \alpha_2^* \iota_!
\end{tikzcd}$$

The left vertical map is still coCartesian (homotopy colimits preserve coCartesian morphisms).
There is a canonical isomorphism $p'_! c_i^* \to p_{i!}$ [Gro13, Prop. 1.23] and the natural transformation $\mathbb{D}(\rho) : c_1^* \widetilde{\nu}^* \to c_2^*$ is an isomorphism on coCartesian objects over constant diagrams. Consider the commutative diagram:

$$\begin{tikzcd}
p'_! c_1^* \widetilde{\nu}^* \arrow[r, "\sim"] \arrow[d, equal] & p_{2!}\widetilde{\nu}_! c_{1!} c_1^* \widetilde{\nu}^* \arrow[r] & p_{2!} \arrow[dd, equal] \\
p'_! c_1^* \widetilde{\nu}^* \arrow[r, "\sim"] & p_{2!} c_{2!} c_1^* \widetilde{\nu}^* \arrow[u, "\mathbb{D}(\rho)^{ad}"] \arrow[d, "\mathbb{D}(\rho)"] & \\
 & p_{2!} c_{2!} c_2^* \arrow[r, "\sim"] & p_{2!}
\end{tikzcd}$$

where the rightmost horizontal morphisms are the respective counits. Since $\mathbb{D}(\rho)$ is an isomorphism on coCartesian objects over constant diagrams, so is the morphism $p'_! c_1^* \widetilde{\nu}^* \to p_{2!}$. Now we have the commutative diagram

$$\begin{tikzcd}
p'_! c_1^* \widetilde{\nu}^* \arrow[r] \arrow[d, "\sim"] & p_{2!} \\
p_{1!}\widetilde{\nu}^* \arrow[r, "\sim"] & p_{2!}\widetilde{\nu}_! \widetilde{\nu}^* \arrow[u]
\end{tikzcd}$$

which shows that also the natural map $p_{1!}\widetilde{\nu}^* \to p_{2!}$ is an isomorphism on coCartesian objects over constant diagrams.
We get a commutative diagram

$$\begin{tikzcd}
p_{1!}\mathbb{S}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d] & p_{1!}p_1^*\alpha_1^*\iota_! \arrow[r] \arrow[d] & \alpha_1^*\iota_! \arrow[dd, "\mathbb{D}(\nu)"] \\
p_{1!}\widetilde{\nu}^*\mathbb{S}(\mu_2)_\bullet A_2^* \arrow[r] \arrow[d] & p_{1!}\widetilde{\nu}^* p_2^*\alpha_2^*\iota_! \arrow[d] & \\
p_{2!}\mathbb{S}(\mu_2)_\bullet A_2^* \arrow[r] & p_{2!}p_2^*\alpha_2^*\iota_! \arrow[r] & \alpha_2^*\iota_!
\end{tikzcd}$$

where the composition of the left vertical morphisms is coCartesian on coCartesian objects because the functor $\mathbb{S}(\mu_2)_\bullet A_2^*$ maps coCartesian objects to coCartesian objects over constant diagrams. The composition of the horizontal morphisms in the top and bottom rows are isomorphisms by (FDer4 left). Hence the rightmost vertical map is coCartesian as well. □

Proof of Main Theorem 7.5.5, 1. This is the case of weak $\mathbb{D}$-equivalences.
(L0) and (L1) are clear.
For (L2 left), let $D_1 = (I, F)$ and $D_2 = (\{e\}, F(e))$. The projection p and the inclusion i of the final object induce morphisms:

$$\begin{tikzcd} D_1 \arrow[r, shift left, "p"] & D_2 \arrow[l, shift left, "i"] \end{tikzcd}$$

We have $p \circ i = \mathrm{id}$ and there is a 2-morphism $\beta : \mathrm{id} \Rightarrow i \circ p$. Therefore the statement is clear for weak $\mathbb{D}$-equivalences over any base S.
(L3 left): Let

$$\begin{tikzcd}
D_1 \arrow[rr, "w"] \arrow[dr, "p_1'"] \arrow[ddr, bend right, "p_1"'] & & D_2 \arrow[dl, "p_2'"'] \arrow[ddl, bend left, "p_2"] \\
& D_3 = (E, F) \arrow[d, "p"] & \\
& (\cdot, S) &
\end{tikzcd}$$

be a morphism as in (L3 left) over a base $S \in \mathbb{S}(\cdot)$. We have to show that

$$p_{1!}\, p_1^* \to p_{2!}\, p_2^*$$

is an isomorphism and it suffices to show that the morphism

$$p'_{1!}\, (p_1)^* \to p'_{2!}\, (p_2)^*$$

is an isomorphism. This may be checked point-wise by (Der2) and after pull-back to an open cover by condition 2. of 'local' for a fibered derivator (see Definition 4.5.4), so fix $e \in E$ and consider the 2-commutative diagrams

$$\begin{tikzcd} D_i \times_{/D_3} (e, U_j) \arrow[r, "\iota_{i,e,j}"] \arrow[d, "p'_{i,e,j}"'] & D_i \arrow[d, "p'_i"] \\ (e, U_j) \arrow[r, "\epsilon_{e,j}"] & D_3 \end{tikzcd}$$

and let $p_{i,e} : D_i \times_{/D_3} (e, U_i) \to D_i$ be the projection. Applying the functor $\epsilon_{e,j}^*$, we get

$$\epsilon_{e,j}^* p'_{1!}(p_1)^* \to \epsilon_{e,j}^* p'_{2!}(p_2)^*$$

which is, using Proposition 4.6.9 (note that $\iota_{e,j}$ is $\mathbb{D}$-local by assumption), the same as

$$(p'_{1,e,j})_!(\iota_{i,e,j})^*(p_1)^* \to (p'_{2,e,j})_!(\iota_{i,e,j})^*(p_2)^*.$$

Now $p_i \circ \iota_{i,e,j} = \pi_j \circ p'_{i,e,j}$, where $\pi : (\cdot, U_j) \to (\cdot, S)$ is the structural morphism. Therefore we get:

$$(p'_{1,e,j})_!(p'_{1,e,j})^* \pi_j^* \to (p'_{2,e,j})_!(p'_{2,e,j})^* \pi_j^*.$$

By Lemma 4.6.7 this is induced by the canonical natural transformation which is an isomorphism by assumption.

(L4 left): By Lemma 7.2.12 we may prove axiom (L4' left) instead. Consider a morphism $p : D_1 \to (E, F) = D_2$ in $\mathrm{Dia}(\mathbb{S})$ of pure diagram type, where the underlying functor of p is a Grothendieck fibration. It suffices to show that the counit

$$p_! \, p^* \to \mathrm{id}$$

is an isomorphism. This is the same as showing that the unit

$$\mathrm{id} \to p_* \, p^*$$

is an isomorphism. Note that p_* exists because this is a morphism of diagram type and $\mathbb{D} \to \mathbb{S}$ is assumed to be a right fibered derivator as well (this is the only place, where this assumption is used for the case of weak $\mathbb{D}$-equivalences). Now, since p is a Grothendieck fibration, p_* can be computed fiber-wise. So we have to show that

$$\mathrm{id} \to p_{e,*} \, p_e^*$$

is an isomorphism or, equivalently, that

$$p_{e,!} \, p_e^* \to \mathrm{id}$$

is an isomorphism. This holds true because by assumption the map of fibers $I_e \to e$ is in $\mathcal{W}_{F(e)}$. □

We proceed to state some consequences of the fact that *weak* $\mathbb{D}$-equivalences form a fundamental localizer.

Example 7.5.14 (Mayer-Vietoris). *Let $\mathbb{S}$ be a strong right derivator (e.g. associated with a category with limits) with a Grothendieck pre-topology. We saw in Example 7.2.9 that for a cover $\{U_1 \to S, U_2 \to S\}$ consisting of 2* mono*morphisms, the projection*

$$p : \left(\begin{tikzcd} \text{“}U_1 \times_S U_2\text{”} \arrow[r] \arrow[d] & U_1 \\ U_2 & \end{tikzcd} \right) \to S$$

belongs to any fundamental localizer. If $\mathbb{D} \to \mathbb{S}$ is a fibered derivator which is local w.r.t. the pre-topology on $\mathbb{S}$, Theorem 7.5.5 implies therefore that p is a weak $\mathbb{D}$-equivalence in $\mathrm{Dia}(\mathbb{S})/(\cdot, S)$*, i.e. for $A \in \mathbb{D}(\cdot)_S$ we have*

$$p_! \, p^* A \cong A,$$

i.e. the homotopy colimit of

$$\begin{tikzcd} i_{1,2,\bullet} \, i_{1,2}^\bullet A \arrow[r] \arrow[d] & i_{1,\bullet} \, i_1^\bullet A \\ i_{2,\bullet} \, i_2^\bullet A & \end{tikzcd}$$

is isomorphic to A. If $\mathbb{D}$ has stable fibers, this translates to the usual distinguished triangle

$$i_{1,2,\bullet} \, i_{1,2}^\bullet A \to i_{1,\bullet} \, i_1^\bullet A \oplus i_{2,\bullet} \, i_2^\bullet A \to A \to i_{1,2,\bullet} \, i_{1,2}^\bullet A[1]$$

in the language of triangulated categories.
Dually, if $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is a fibered derivator which is colocal w.r.t. the pre-cotopology on $\mathbb{S}^{\mathrm{op}}$, Theorem 7.5.4 implies that p^{op} is a weak $\mathbb{D}$-equivalence in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})/(\cdot, S)$*, i.e. for $A \in \mathbb{D}(\cdot)_S$ we have*

$$A \cong p_* p^* A.$$

This means that the homotopy limit of

$$\begin{tikzcd} & i_1^\bullet \, i_{1,\bullet} A \arrow[d] \\ i_2^\bullet \, i_{2,\bullet} A \arrow[r] & i_{1,2}^\bullet \, i_{1,2,\bullet} A \end{tikzcd}$$

is isomorphic to A. If $\mathbb{D}$ has stable fibers, this translates to the usual distinguished triangle

$$A \to i_1^\bullet \, i_{1,\bullet} A \oplus i_2^\bullet \, i_{2,\bullet} A \to i_{1,2}^\bullet \, i_{1,2,\bullet} A \to A[1]$$

in the language of triangulated categories. Note that $i_\bullet$ denotes a left adjoint push-forward along a morphism in $\mathbb{S}^{\mathrm{op}}$*, i.e. a left adjoint pull-back along a morphism in $\mathbb{S}$.*

Example 7.5.15 ((Co)homological descent)**.** *Let $\mathbb{S}$ be a strong right derivator with a Grothendieck pre-topology and let $X_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$ be a simplicial diagram over $S \in \mathbb{S}(\cdot)$ with underlying diagram*

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_1 \substack{\longrightarrow\\\longrightarrow} X_0$$

such that $(\mathrm{id}, p) : (\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, \pi^* S)$ *is a finite hypercover. Here* $\pi : \Delta^{\mathrm{op}} \to \cdot$ *denotes the projection. If* $\mathbb{D} \to \mathbb{S}$ *is a fibered derivator which is local w.r.t. the pre-topology on* $\mathbb{S}$*, Theorem 7.5.5 implies that* (π, p) *is a weak* $\mathbb{D}$*-equivalence in* $\mathbb{S}(\cdot)/(\cdot, S)$*, i.e. for* $A \in \mathbb{D}(\cdot)_S$ *we have*

$$A \cong \pi_! \, p_\bullet \, p^\bullet \, \pi^* A.$$

This means that the homotopy colimit of $p_\bullet \, p^\bullet \, \pi^* A$ *is equal to* A*. If the fibers of* $\mathbb{D} \to \mathbb{S}$ *are in fact* derived categories*, this yields a spectral sequence of homological descent because the homotopy colimit over a simplicial complex is the total complex of the associated double complex (a well-known fact). This double complex looks like*

$$\cdots \longrightarrow p_{2,\bullet} \, p_2^\bullet A \longrightarrow p_{1,\bullet} \, p_1^\bullet A \longrightarrow p_{0,\bullet} \, p_0^\bullet A.$$

The point is that we get a coherent *double complex. Knowing the individual morphisms* $p_{i,\bullet} \, p_i^\bullet A \to p_{i-1,\bullet} \, p_{i-1}^\bullet A$ *as morphisms in the derived category* $\mathbb{D}(\cdot)_S$ *would not be sufficient! Dually (applying everything to a fibered derivator* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$*, and working in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$*), one obtains the more classical spectral sequence of* cohomological descent*.*

Proof of Main Theorem 7.5.5, 2. This is the case of strong $\mathbb{D}$-equivalences.
(L1) is clear.
For (L2 left), let $D_1 = (I, F)$ and $D_2 = (e, F(e))$. The projection p and the inclusion i of the final object induce morphisms:

$$D_1 \underset{i}{\overset{p}{\rightleftarrows}} D_2$$

We have $p \circ i = \mathrm{id}$ and there is a 2-morphism $\beta : \mathrm{id} \Rightarrow i \circ p$. Therefore the statement follows from Lemma 7.5.8. (Actually $(i \circ p)^*$ is left adjoint to the inclusion $\mathbb{D}(D_1)^{\mathrm{cart}} \to \mathbb{D}(D_1)$.)
(L3 left): It suffices to prove the following two statements:

1. Consider a morphism of diagrams $w = (\alpha, f) : D_1 = (I_1, F_1) \to D_2 = (I_2, F_2)$ such that we have a commutative diagram

$$\begin{tikzcd} I_1 \arrow[rr, "\alpha"] \arrow[dr, "p_1"'] & & I_2 \arrow[dl, "p_2"] \\ & E & \end{tikzcd}$$

   and such that $w \times_{/E} e$ is a strong $\mathbb{D}$-equivalence for all objects e in E. Then w is a strong $\mathbb{D}$-equivalence.

2. Consider a morphism of diagrams $w : D_1 = (I_1, F_1) \to D_2 = (I_2, F_2)$ over $(\cdot, S)$ and let $\{U_i \to S\}$ be a covering. If $w \times_{(\cdot,S)} (\cdot, U_i)$ is a strong $\mathbb{D}$-equivalence for all i then w is a strong $\mathbb{D}$-equivalence.

We proceed by showing statement 1. Consider the following diagram over E

$$\begin{tikzcd} D_1 \arrow[r, "w"] \arrow[d] & D_2 \arrow[d] \\ D_1 \times_{/E} E \arrow[r, "w'"] & D_2 \times_{/E} E \end{tikzcd}$$

where the vertical morphisms are of pure diagram type. We have an adjunction

$$\begin{tikzcd} I_i \arrow[r, shift left, "\kappa_i"] & I_i \times_{/E} E \arrow[l, shift left, "\iota_i"] \end{tikzcd}$$

where κ_i maps an object i to $(i, \mathrm{id}_{p(i)})$. We have a natural transformation $\kappa_i \circ \iota_i \Rightarrow \mathrm{id}_{I_i \times_{/E} E}$ and moreover $\iota_i \circ \kappa_i = \mathrm{id}_E$ holds. Actually this defines an adjunction with κ_i left-adjoint to ι_i. Furthermore, we get lifts to diagrams

$$\begin{tikzcd} D_i \arrow[r, shift left, "\widetilde{\kappa}_i"] & (I_i \times_{/E} E, \iota_i \circ F) = D_1 \times_{/E} E, \arrow[l, shift left, "\widetilde{\iota}_i"] \end{tikzcd}$$

and a 2-morphism $\widetilde{\kappa}_i \circ \widetilde{\iota}_i \Rightarrow \mathrm{id}_{D_1 \times_{/E} E}$, and we have $\widetilde{\iota}_i \circ \widetilde{\kappa}_i = \mathrm{id}_{D_1}$.
Hence, by Lemma 7.5.8, the pull-backs along $\widetilde{\iota}_1$ and $\widetilde{\iota}_2$ induce equivalences on Cartesian objects, so we are reduced to showing that the pull-back along w' induces an equivalence on Cartesian objects. The underlying diagrams $I_k \times_{/E} E$ are Grothendieck opfibratons over E and the functor underlying w' is a map of Grothendieck opfibrations (the push-forward along a map $\mu : e \to f$ in E being given by mapping $(i, \nu : p(i) \to e)$ to $(i, \nu \circ \mu)$). Hence w.l.o.g. we may assume that $I_1 \to E$ is a Grothendieck opfibration and the morphism $I_1 \to I_2$ underlying f is a morphism of Grothendieck opfibrations.
We keep the notation $w : D_1 \to D_2$, however, and the assumption translates to the statement that the composition

$$\mathbb{D}(D_{2,e})^{\mathrm{cart}} \xrightarrow{w_e^*} \mathbb{D}(D_{1,e})^{\mathrm{cart}}$$

for the fibers is an equivalence with inverse $\Box_! w_{e,!}$.
Consider the two functors:

$$\mathbb{D}(D_2)^{E-\mathrm{cart}} \xrightarrow{incl.} \mathbb{D}(D_2) \xrightarrow{w^*} \mathbb{D}(D_1).$$

We first show that the counit

$$\Box_!^E w_! w^* \mathcal{E} \to \mathcal{E}$$

is an isomorphism for every E-Cartesian $\mathcal{E}$.

This can be checked after pulling back to the fibers. Let $\iota_k : I_{k,e} \to I_k$ be the inclusion of the fiber over some $e \in E$.
We have the isomorphisms

$$\iota_2^* \, \Box_!^E \, w_! w^* \mathcal{E} \cong \Box_! w_{e,!} \iota_1^* w^* \mathcal{E} \cong \Box_! w_{e,!} w^{e,*} \iota_2^* \mathcal{E} \cong \iota_2^* \mathcal{E},$$

where we used the isomorphism $\iota_2^* \Box_!^E \cong \Box_! \iota_2^*$ (Lemma 7.5.12) and the isomorphism $\iota_2^* w_! \cong w_{e,!} \iota_1^*$ (exists for morphisms of pure diagram type because we have a morphism of Grothendieck opfibrations, see Proposition 4.3.23, 3. and for morphisms of fixed shape by axiom (FDer0 left)). The morphism $\Box_! w_{e,!} w^{e,*} \mathcal{E} \to \mathcal{E}$ is an isomorphism for Cartesian $\mathcal{E}$ by assumption.
We now show that the unit

$$\mathcal{E} \to w^* \, \Box_!^E \, w_! \mathcal{E}$$

is an isomorphism for every E-Cartesian $\mathcal{E}$. This can be checked again on the fibers:

$$\iota_1^* w^* \, \Box_!^E \, w_! \mathcal{E} \cong w_e^* \iota_2^* \, \Box_!^E \, w_! \mathcal{E} \cong w_e^* \, \Box_! \, w_{e,!} \iota_1^* \mathcal{E} \cong \iota_1^* \mathcal{E}.$$

Therefore we have already proven that the functors

$$\begin{tikzcd} \mathbb{D}(D_2)^{E-\mathrm{cart}} \arrow[r, shift left, "w^*"] & \mathbb{D}(D_1)^{E-\mathrm{cart}} \arrow[l, shift left, "\Box_!^E w_!"] \end{tikzcd}$$

form an equivalence.
We conclude by showing that $\Box_!^E w_!$ maps Cartesian objects to Cartesian objects: Let $\nu : e \to f$ be a morphisms of E. It induces a morphism (choice of push-forward for $I_k \to E$)

$$\widetilde{\nu}_k : D_{k,e} \to D_{k,f}$$

(not of diagram type!) and a 2-morphism: $\iota_{k,e} \to \iota_{k,f} \circ \widetilde{\nu}_k$.

- *Claim:* It suffices to show that for all $\nu : e \to f$ the induced morphism

  $$\iota_{2,e}^* \, \Box_!^E \, w_! \mathcal{E} \to \widetilde{\nu}_2^* \iota_{2,f}^* \, \Box_!^E \, w_! \mathcal{E}$$

  is an isomorphism for every Cartesian $\mathcal{E}$.

Proof of the claim: Every morphism $\mu : i \to i''$ in I with $p(\mu) = \nu$, say, is the composition of a coCartesian μ' and some morphism μ'' with $p(\mu'') = \mathrm{id}_f$. Since $\mathcal{E}$ is E-Cartesian, the morphism $\mathcal{E}(\mu'')$ is Cartesian. Hence to show that $\mathcal{E}(\mu)$ is Cartesian it suffices to see that $\mathcal{E}(\mu')$ is Cartesian. A reformulation is, however, that the morphism of the claim be an isomorphism. □
Using the same argument as in the first part of the proof, we have to show that

$$\Box_! w_{e,!} \iota_{1,e}^* \mathcal{E} \to \widetilde{\nu}_2^* \, \Box_! \, w_{f,!} \iota_{1,f}^* \mathcal{E}$$

is an isomorphism for every Cartesian $\mathcal{E}$. Since both sides are Cartesian objects, this can be checked after applying w_e^* which is an equivalence on Cartesian objects:

$$w_e^* \,\Box_!\, w_{e,!}\iota_{1,e}^*\mathcal{E} \to w_e^*\widetilde{\nu}_2^* \,\Box_!\, w_{f,!}\iota_{1,f}^*\mathcal{E}.$$

We have $w_e^*\widetilde{\nu}_2^* = \widetilde{\nu}_1^* w_f^*$ because the map of diagrams underlying w is a morphism of Grothendieck opfibrations. Hence, after applying w_e^*, we get

$$w_{e^*} \,\Box_!\, w_{e,!}\iota_{1,e}^*\mathcal{E} \to \widetilde{\nu}_1^* w_{f^*} \,\Box_!\, w_{f,!}\iota_{1,f}^*\mathcal{E}.$$

Since $w_{e^*} \,\Box_!\, w_{e,!}$ and $w_{f^*} \,\Box_!\, w_{f,!}$ are equivalences on Cartesian objects, we get

$$\iota_{1,e}^*\mathcal{E} \to \widetilde{\nu}_1^*\iota_{1,f}^*\mathcal{E}.$$

A slightly tedious check shows that this is again the morphism induced by the 2-morphism $\iota_{1,e} \to \iota_{1,f} \circ \widetilde{\nu}_1$. It is an isomorphism because $\mathcal{E}$ is Cartesian.
We will now show statement 2. Consider a diagram

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr, "p_1"'] & & D_2 \arrow[dl, "p_2"] \\ & (\cdot, S) & \end{tikzcd}$$

For any i (index of the cover in L4 left) we have the following commutative diagrams of objects in $\mathrm{Dia}(\mathbb{S})$:

$$\begin{tikzcd} D_1 \times_S U_i \arrow[r, "w_i"] \arrow[d, "\mathrm{pr}_1^{(i)}"'] & D_2 \times_S U_i \arrow[d, "\mathrm{pr}_1^{(i)}"] \\ D_1 \arrow[r, "w"] & D_2 \end{tikzcd}$$

The morphisms $\mathrm{pr}_1^{(i)}$ are of fixed shape. We first show that the unit is an isomorphism

$$\mathcal{E} \to w^* \,\Box_!\, w_!\,\mathcal{E}$$

for any Cartesian $\mathcal{E}$. Note that by the stability axiom of a Grothendieck pre-topology also the collections $(D_1 \times_S U_i)_j \to D_{1,j}$ are covers for any $j \in I_1$, where I_1 is the underlying diagram of D_1. Since $\mathbb{D}$ is local w.r.t. the Grothendieck pre-topology (and by axiom Der2), the family $(\mathrm{pr}_1^{(i)})^*$ is jointly conservative. Therefore it suffices to show that the unit is an isomorphism after applying $(\mathrm{pr}_1^{(i)})^*$. We get

$$(\mathrm{pr}_1^{(i)})^*\mathcal{E} \to (\mathrm{pr}_1^{(i)})^*\, w^* \,\Box_!\, w_!\,\mathcal{E}$$

which is the same as

$$(\mathrm{pr}_1^{(i)})^*\mathcal{E} \to w_i^*\,(\mathrm{pr}_1^{(i)})^* \,\Box_!\, w_!\,\mathcal{E}.$$

Since $(\mathrm{pr}_1^{(i)})^*$ commutes with $\Box_!$ (Lemma 7.5.10) and with $w_!$ (Proposition 4.6.9, 2.), we get

$$(\mathrm{pr}_1^{(i)})^* \mathcal{E} \to w_i^* \,\Box_!\, w_{i,!}\, (\mathrm{pr}_1^{(i)})^* \mathcal{E}.$$

This morphism is an isomorphism by assumption. In the same way one shows that the counit is an isomorphism.

(L4 left): By Lemma 7.2.12 we may prove axiom (L4' left) instead. We have shown during the proof for (L4' left) for the case of weak $\mathbb{D}$-equivalences that

$$p_!\, p^* \to \mathrm{id}$$

is an isomorphism, hence on Cartesian objects the same holds for the natural transformation

$$\Box_!\, p_!\, p^* \to \mathrm{id}\,.$$

We have to show that also the counit

$$\mathrm{id} \to p^* \,\Box_!\, p_! \tag{51}$$

is an isomorphism on Cartesian objects. First note that p_* also is a right adjoint of p^* when restricted to the full subcategories of Cartesian objects because p_* preserves Cartesian objects. Indeed, p_* can be computed fiber-wise because p is a Grothendieck fibration. The fibers being contractible in the sense of any localizer on Dia implies that the functors $p_e^*, p_{e,*}$ induce an equivalence $\mathbb{D}(D_e)^{\mathrm{cart}} \cong \mathbb{D}(\cdot)_{F(e)}$. Note: This uses that (L1–L3 left) hold for the class of strong $\mathbb{D}$-equivalences on the fiber $\mathbb{D}_{F(e)}$, a fact which has been proven already. Therefore we pass to the right adjoints of the functors in (51) and have to show that the counit

$$p^*\, p_* \to \mathrm{id}$$

is an isomorphism on Cartesian objects. Again this can be checked fiber-wise, i.e. we have to show that the counit

$$p_e^*\, p_{e,*} \to \mathrm{id}$$

is an isomorphism on Cartesian objects. But the pair of functors is an equivalence as we have seen, and the claim follows. □

We proceed to state some consequences of the fact that *strong* $\mathbb{D}$-equivalences form a fundamental localizer.

Corollary 7.5.16 (left)**.** *Let $\mathbb{S}$ be a strong right derivator. If $\mathbb{D} \to \mathbb{S}$ is an infinite fibered derivator which is local w.r.t. the pre-topology on $\mathbb{S}$ (cf. 4.5.2) with stable, compactly generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered in* $\mathrm{Dia}(\mathbb{S})'$ *the functor f^* induces an equivalence*

$$\mathbb{D}(Y_\bullet)^{\mathrm{cart}} \to \mathbb{D}(X_\bullet)^{\mathrm{cart}}.$$

Here $\mathrm{Dia}(\mathbb{S})'$ *is the full subcategory of diagrams with universally $\mathbb{D}$-local morphisms.*

Corollary 7.5.17 (right)**.** *Let $\mathbb{S}$ be a strong right derivator. If $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is an infinite fibered derivator which is colocal w.r.t. the pre-cotopology on $\mathbb{S}^{\mathrm{op}}$ (cf. 4.5.2) with stable, compactly generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$ *the functor f^* induces an equivalence*

$$\mathbb{D}(Y_\bullet)^{\mathrm{cocart}} \to \mathbb{D}(X_\bullet)^{\mathrm{cocart}}.$$

Corollary 7.5.18. *If $\mathbb{D}$ is an infinite derivator (not fibered) with domain* Cat *which is stable and well-generated, then for each homotopy type I, we get a category $\mathbb{D}(I)^{\mathrm{cart}}$ well-defined up to equivalence of categories. Moreover each morphism $I \to J$ of homotopy types gives rise to a corresponding functor $\alpha^* : \mathbb{D}(J)^{\mathrm{cart}} \to \mathbb{D}(I)^{\mathrm{cart}}$. It is, however, not possible to arrange those as a pseudo functor $\mathcal{HOT} \to \mathcal{CAT}$, but it is possible to arrange them as a pseudo-functor $\mathcal{HOT}^{(2)} \to \mathcal{CAT}$ where $\mathcal{HOT}^{(2)}$ is the homotopy 2-category (2-truncation) of any model for the homotopy theory of spaces (cf. also 2.4).*

8 Representability

In this chapter we exploit the consequences that Brown representability type results have for fibered derivators. In particular these results are useful to see that under certain circumstances a left fibered (multi-)derivator is already a right fibered (multi-)derivator, provided that its fibers are nice (i.e. stable and well-generated derivators). Furthermore they provide us with (co)Cartesian projectors that are needed for the strong form of (co)homological descent. In contrast to the rest of the article the results are stated in a rather unsymmetric form. This is due to the fact that in applications the stable derivators will rather be well-generated or compactly generated whereas their duals will rather not satisfy this condition. All the auxiliary results are taken from [Kra10] and [Nee01].

8.1 Well-generated triangulated categories and Brown representability

Definition 8.1.1 (cf. [Kra10, 5.1, 6.3])**.** *Let $\mathcal{D}$ be a category with zero object and small coproducts. We call $\mathcal{D}$* **perfectly generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that the following conditions hold:*

1. *An object $X \in \mathbb{D}(\cdot)$ is zero if and only if* $\mathrm{Hom}(T, X) = 0$ *for all $T \in \mathcal{T}$.*
2. *If $\{X_i \to Y_i\}$ is any set of maps, and* $\mathrm{Hom}(T, X_i) \to \mathrm{Hom}(T, Y_i)$ *is surjective for all i, then* $\mathrm{Hom}(T, \coprod_i X_i) \to \mathrm{Hom}(T, \coprod_i Y_i)$ *is also surjective.*

The category $\mathcal{D}$ is called **well-generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that in addition to 1., 2. there is a regular cardinal α such that the following condition holds:*

3. *All objects $T \in \mathcal{T}$ are α-small, cf. [Kra10, 6.3].*

The category $\mathcal{D}$ is called **compactly generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that in addition to 1., 2. the following two equivalent conditions hold:*

4. *All $T \in \mathcal{T}$ are $\aleph_0$-small.*

4'. *All $T \in \mathcal{T}$ are compact, i.e. for each morphism $\gamma : T \to \coprod_{i \in I} X_i$ there is a finite subset $J \subseteq I$ such that γ factors through $\coprod_{i \in J} X_i$.*

Definition 8.1.2. *A pre-derivator $\mathbb{D}$ whose domain* Dia *is infinite (i.e. closed under infinite disjoint unions) is called* **infinite** *if the restriction-to-I_j functors induce an equivalence*

$$\mathbb{D}(\coprod_{j \in J} I_j) \cong \prod_{j \in J} \mathbb{D}(I_j)$$

for all sets J.

Recall (cf. [Kra10, 4.4]) that a functor from a triangulated category $\mathcal{D}$ to an abelian category is called **cohomological** if it sends distinguished triangles to exact sequences. We recall the following theorem:

Theorem 8.1.3 (right Brown representability)**.** *Let $\mathcal{D}$ be a perfectly generated triangulated category with small coproducts. Then a functor $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{AB}$ is cohomological and sends coproducts to products if and only if it is representable. An exact functor $\mathcal{D} \to \mathcal{E}$ between triangulated categories commutes with coproducts if and only if it has a right adjoint.*

Proof. See [Kra10, Theorem 5.1.1]. □

It can be shown that for a compactly generated triangulated category $\mathcal{D}$ with small coproducts, $\mathcal{D}^{\mathrm{op}}$ is perfectly generated and has small coproducts. Therefore the dual version of the previous theorem holds in this case:

Theorem 8.1.4 (left Brown representability)**.** *Let $\mathcal{D}$ be a compactly generated triangulated category with small coproducts. Then a functor $F : \mathcal{D} \to \mathcal{AB}$ is homological and sends products to products if and only if F is representable. An exact functor $\mathcal{D} \to \mathcal{E}$ between triangulated categories commutes with products if and only if it has a left adjoint.*

Theorem 8.1.5. *Let $\mathcal{D}$ be a well-generated triangulated category with small coproducts. Consider a functor $F : \mathcal{D} \to \mathcal{AB}$ which is cohomological and commutes with coproducts. Then there exists a right adjoint to the inclusion of the full subcategory of objects X such that $F(X[n]) = 0$ for all $n \in \mathbb{Z}$ (i.e. this subcategory is coreflective).*

Proof. See [Kra10, Theorem 7.1.1]. □

Lemma 8.1.6. *Let* Dia *be an infinite diagram category (4.1.1). Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered derivator with domain* Dia*. If $\mathbb{D}(\cdot)_S$ for all $S \in \mathbb{S}(\cdot)$ is perfectly generated (resp. well-generated, resp. compactly generated), then the same holds for $\mathbb{D}(I)_{S'}$ for all $I \in$* Dia *and for all $S' \in \mathbb{S}(I)$. Furthermore the categories $\mathbb{D}(I)_{S'}$ all have small coproducts.*

Proof. A set of generators as requested is given by the set $\mathcal{T}_I := \{i_! T\}_{i \in I, T \in \mathcal{T}}$. Indeed, an object $X \in \mathbb{D}(I)$ is zero if $i^* X$ is zero for all $i \in X$ by (Der2). Therefore X is zero if $\mathrm{Hom}(i_! T, X) = \mathrm{Hom}(T, i^* X) = 0$ for all $i \in I$ and for every $T \in \mathcal{T}$. We have to show that $\mathrm{Hom}(i_! T, \coprod_i X_i) \to \mathrm{Hom}(i_! T, \coprod_i Y_i)$ is an isomorphism for a family $\{X_i \to Y_i\}_{i \in O}$ of morphisms as in Definition 8.1.1, 2. We have $\mathrm{Hom}(i_! T, \coprod_i X_i) = \mathrm{Hom}(T, i^* \coprod_i X_i) = \mathrm{Hom}(T, \coprod_i i^* X_i)$, where we used that i^* commutes with coproducts. This follows because the Cartesian diagram

$$\begin{tikzcd} O \arrow[r] \arrow[d] & O \times I \arrow[d] \\ \cdot \arrow[r] & I \end{tikzcd}$$

is homotopy exact. Note that, since $\mathbb{D}$ is infinite, coproducts exist and are equal to the corresponding homotopy coproducts. The map $\mathrm{Hom}(T, \coprod_i i^* X_i) \to \mathrm{Hom}(T, \coprod_i i^* Y_i)$ is surjective by assumption.

We have to show that a morphism

$$i_! T \to \coprod_{i \in I} Y_i$$

in $\mathbb{D}(I)_{S'}$ factors through $\coprod_{i \in J} Y_i$ for some subset $J \subset I$ of cardinality less than α. By the same reasoning as above, we get a morphism

$$T \to \coprod_{i \in I} i^* Y_i$$

Hence, there is some subset $J \subset I$, as required, such that this morphism factors through it. The same then holds for the original morphism. Since there is no need to enlarge J, the same statement holds for finite subsets.

The categories $\mathbb{D}(I)_{S'}$ have small coproducts because $\mathbb{D} \to \mathbb{S}$ is infinite and left fibered. □

Definition 8.1.7. *Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered derivator with domain* Dia. *We will say that $\mathbb{D} \to \mathbb{S}$ has* **perfectly-generated** *(resp.* **well-generated**, *resp.* **compactly-generated***)* **fibers**, *if all categories $\mathbb{D}(\cdot)_S$ are perfectly-generated (resp. well-generated, resp. compactly-generated) for all $S \in \mathbb{S}(\cdot)$. It follows from the previous Lemma that, in this case, for all $I \in$* Dia *and for all $S' \in \mathbb{S}(I)$ the category $\mathbb{D}(I)_{S'}$ is also perfectly-generated (resp. well-generated, resp. compactly-generated).*

8.2 Left and right

Theorem 8.2.1 (left)**.** *Let* Dia *be an infinite diagram category (cf. 4.1.1). Let $\mathbb{D}$ and $\mathbb{E}$ be infinite left derivators with domain* Dia *such that for all $I \in$* Dia *the pre-derivators $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable (left and right) derivators with domain* Posf*. Assume that $\mathbb{D}$ is perfectly generated. Then a morphism of derivators $F : \mathbb{D} \to \mathbb{E}$ commutes with all homotopy colimits w.r.t.* Dia *if and only if it has a right adjoint.*

Proof. Let I be in Dia. Since $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable, $\mathbb{D}(I)$ is canonically triangulated, and we may use Theorem 8.1.3 of right Brown representability. It follows that the functor $F(I) : \mathbb{D}(I) \to \mathbb{E}(I)$ has a right adjoint $G(I)$, because it is triangulated, commutes with small coproducts and $\mathbb{D}(I)$ is perfectly generated. To construct a morphism of derivators out of this collection, for any $\alpha : I \to J$, we have to give an isomorphism: $G(J)\alpha^* \to \alpha^* G(I)$. We may take the adjoint of the isomorphism $\alpha_! F(J) \to F(I)\alpha_!$ expressing that F commutes with all homotopy colimits (see [Gro13, Lemma 2.1] for details). □

Analogously, using Theorem 8.1.4 of left Brown representability, we obtain:

Theorem 8.2.2 (right)**.** *Let* Dia *be an infinite diagram category (cf. 4.1.1). Let $\mathbb{D}$ and $\mathbb{E}$ be infinite right derivators with domain* Dia *such that for all $I \in$* Dia*, the pre-derivators $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable (left and right) derivators with domain* Posf*. Assume that $\mathbb{D}$ is compactly generated. Then a morphism of derivators $F : \mathbb{D} \to \mathbb{E}$ commutes with all homotopy limits w.r.t.* Dia *if and only if it has a left adjoint.*

Theorem 8.2.3 (left)**.** *Let* Dia *be an infinite diagram category (cf. 4.1.1). Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered (multi)derivator with domain* Dia *whose* fibers *$\mathbb{D}_S$ for every $I \in$* Dia *and all $S \in \mathbb{S}(I)$ are stable (left and right) derivators with domain* Posf*. Assume that $\mathbb{D}$ has perfectly generated fibers. Then $\mathbb{D}$ is a* right *fibered (multi)derivator as well.*

Proof. Let $I \in$ Dia and let $f \in \mathrm{Hom}_{\mathbb{S}(I)}(S_1, \dots, S_n; T)$ be a multimorphism. By Lemma 4.3.13, fixing $\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n$, the association

$$\begin{array}{rcl} \mathbb{D}(I \times J)_{p^* S_i} & \to & \mathbb{D}(I)_{p^* T} \\ \mathcal{E}_i & \mapsto & (p^* f)_\bullet (p^* \mathcal{E}_1, \dots, \ \mathcal{E}_i, \ \dots, p^* \mathcal{E}_n) \end{array}$$

defines a morphism of derivators

$$\mathbb{D}_{S_i} \to \mathbb{D}_T$$

which is left continuous. Hence by Theorem 8.2.1 it has a right adjoint. This shows the first part of (FDer0 right), i.e. the functor $\mathbb{D}(I) \to \mathbb{S}(I)$ is an opfibration as well, for every $I \in$ Dia. Then axiom (FDer5 left) implies the remaining assertion of (FDer0 right) while (FDer0 left) implies (FDer5 right), see Lemma 4.3.8.
Similarly a morphism $\alpha : I \to J$ in Dia induces a morphism of derivators

$$\alpha^* : \mathbb{D}_S \to \mathbb{D}_{\alpha^* S}.$$

It commutes with homotopy colimits by Proposition 4.3.23, 2. Therefore α^* has a right adjoint α_* by the previous theorem, i.e. (FDer3 right) holds. (FDer4 right) is then a consequence of Lemma 4.3.23, 1. □

Analogously, using Theorem 8.1.4 of left Brown representability, we obtain:

Theorem 8.2.4 (right)**.** *Let* Dia *be an infinite diagram category (cf. 4.1.1). Let $\mathbb{D} \to \mathbb{S}$ be an infinite right fibered (multi)derivator with domain* Dia*, whose* fibers *$\mathbb{D}_S$ for every $I \in$* Dia *and for all $S \in \mathbb{S}(I)$ are stable (left and right) derivators with domain* Posf*. Assume that $\mathbb{D}$ has compactly generated fibers. Then $\mathbb{D}$ is a* left *fibered (multi)derivator as well.*

8.3 (Co)Cartesian projectors

Theorem 8.3.1 (right)**.** *Let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered left derivator (w.r.t.* Dia*) whose fibers are stable derivators w.r.t.* Posf*. Assume that $\mathbb{D}(\cdot)_S$ is well-generated for every $S \in \mathbb{S}(\cdot)$. Then for all $I \in$* Dia*, for all $F \in \mathbb{S}(I)$, and for all functors $I \to E$ in* Dia *the fully-faithful inclusion*

$$\mathbb{D}(I)_F^{E-\mathrm{cocart}} \to \mathbb{D}(I)_F$$

has a right adjoint $\Box_*^E$.
If $\mathbb{D} \to \mathbb{S}$ also satisfies (FDer0 right) and if F is such that $F(\mu)$ satisfies (Dloc2 left) for every μ mapping to an identity in E, then the fully-faithful inclusion

$$\mathbb{D}(I)_F^{E-\mathrm{cart}} \to \mathbb{D}(I)_F$$

has a right adjoint $\blacksquare_*^E$.

Proof. Consider the set O of morphisms $\mu : i \to j$ which map to an identity in E, and for each morphism $\mu \in O$ the composition D_μ:

$$\mathbb{D}(I)_F \xrightarrow{\mu^*} \mathbb{D}(\to)_{\mu^* F} \xrightarrow{F(\mu)_\bullet} \mathbb{D}(\to)_{i^* F} \xrightarrow{\mathrm{Cone}} \mathbb{D}(\cdot)_{i^* F}$$

We define a functor D by

$$\prod_{\mu \in O} D_\mu : \mathbb{D}(I)_F \to \prod_{\mu \in O} \mathbb{D}(\cdot)_{i^* F} = \mathbb{D}(O)_{\iota^* F},$$

where $\iota : O \to I$ is the map "source". D commutes with coproducts, as all functors in the succession do, and it is exact. Therefore by [Kra10, Theorem 7.4.1] the triangulated subcategory $\mathbb{D}(I)_F^{E-\mathrm{cocart}} = \ker D$ is well-generated and hence the inclusion in the statement has a right adjoint. In the Cartesian case, $F(\mu)^\bullet$ commutes with coproducts only if $F(\mu)$ satisfies (Dloc2 left). □

Theorem 8.3.2 (left)**.** *Let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator (with domain* Dia*) whose fibers are stable. Assume that all $\mathbb{D}(\cdot)_S$ for $S \in \mathbb{S}(\cdot)$ are compactly generated.*
Let $I \to E$ be a functor in Dia *and let $F \in \mathbb{S}(I)$. Suppose that $F(\mu)$ satisfies (Dloc2 left) for every morphism μ in I that maps to an identity in E. Then the fully-faithful inclusion*

$$\mathbb{D}(I)_F^{E-\mathrm{cart}} \to \mathbb{D}(I)_F$$

has a left adjoint $\Box_!^E$.

Proof. As in the proof of the previous theorem we have an exact functor

$$F^{\mathrm{cart}} : \mathbb{D}(I)_F \to \mathcal{T}$$

into another triangulated category which commutes with coproducts and such that the subcategory of E-Cartesian objects is precisely its kernel. Lemma 8.1.6 implies that

$\mathbb{D}(I)_F$ is compactly generated, and hence $\mathbb{D}(I)_F^{\mathrm{op}}$ is perfectly generated. Furthermore, Theorem 8.3.1 implies that the categories $\mathbb{D}(I)_F/\mathbb{D}(I)_F^{E\text{-cart}}$ are locally small. Note that

$$\mathbb{D}(I)_F^{\mathrm{op}}/(\mathbb{D}(I)_F^{E\text{-cart}})^{\mathrm{op}} = (\mathbb{D}(I)_F/\mathbb{D}(I)_F^{E\text{-cart}})^{\mathrm{op}}.$$

Therefore [Kra10, Proposition 5.2.1] implies that a right adjoint to the inclusion $(\mathbb{D}(I)_F^{E\text{-cart}})^{\mathrm{op}} \to (\mathbb{D}(I)_F)^{\mathrm{op}}$ exists. So a left adjoint to the inclusion

$$\mathbb{D}(I)_F^{E\text{-cart}} \to \mathbb{D}(I)_F$$

exists. $\square$

In the compactly generated case, $\Box_!^E$ should exist unconditionally, but we were not able to prove this.

9 Derivator six-functor-formalisms

9.1 Definitions

Our main purpose for introducing the more general notion of fibered multiderivator over 2-pre-multiderivators (as opposed to those over usual pre-multiderivators) is that it provides the right framework to think about any kind of *derived* six-functor-formalism:

Definition 9.1.1. *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks*[29]*. Recall from Section 3.1 the definition of the (symmetric) 2-multicategory $\mathcal{S}^{cor}$ (resp. $\mathcal{S}^{cor,0}$ with choice of classes of proper or etale morphisms). Denote its associated represented 2-pre-multiderivator by $\mathbb{S}^{\mathrm{cor}}$, $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$, and $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$, respectively (cf. 5.1.5).*

1. *We define a* **(symmetric) derivator six-functor-formalism** *as a left and right fibered (symmetric) multiderivator*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}.$$

2. *We define a* **(symmetric) proper derivator six-functor-formalism** *as before which has an extension as oplax left fibered (symmetric) multiderivator*

$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}},$$

and an extension as lax right fibered (symmetric) multiderivator

$$\mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}.$$

3. *We define a* **(symmetric) etale derivator six-functor-formalism** *as before which has an extension as lax left fibered (symmetric) multiderivator*

$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}},$$

and an extension as oplax right fibered (symmetric) multiderivator

$$\mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}.$$

In particular, and in view of Section 6.2, if $\mathcal{S}^{cor,0} = \mathcal{S}^{cor,G}$ is formed w.r.t. the choice of *all morphisms*, we call a proper derivator six-functor-formalism a **derivator Grothendieck context** and an etale derivator six-functor-formalism a **derivator Wirthmüller context**.

9.1.2. As mentioned, if $\mathbb{S}$ is really a 2-pre-multiderivator, as opposed to a usual pre-multiderivator, the functor

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathrm{Dia}^{\mathrm{cor}},$$

[29] e.g. a category $\mathcal{S}$ with fiber products made into a symmetric opmulticategory like in (20)

has hardly ever any fibration properties, because of the truncation involved in the definition of the categories of 1-morphisms. Nevertheless the composition

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \{\cdot\}$$

is often 1-bifibered, i.e. there exists an absolute monoidal product on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ extending the one on $\mathrm{Dia}^{\mathrm{cor}}$. For example, if $\mathcal{S}$ is a usual 1-category with fiber products and final object equipped with the opmulticategory structure (20) then for the 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$ represented by $\mathcal{S}^{\mathrm{cor}}$, we have on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ the monoidal product

$$(I,F) \boxtimes (J,G) = (I \times J, F \times G)$$

where $F \times G$ is the diagram of correspondences in $\mathcal{S}$ formed by applying $\times$ point-wise. Similarly we have

$$\mathbf{HOM}\left((I,F),(J,G)\right) = (I^{\mathrm{op}} \times J, F^{\mathrm{op}} \times G)$$

where in F^{op} all correspondences are flipped. In particular any object (I,F) in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ is dualizable with duality explicitly given by

$$\mathbf{HOM}\left((I,F),(\cdot,\cdot)\right) = (I^{\mathrm{op}}, F^{\mathrm{op}}).$$

Given a derivator six-functor-formalism $\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}$ we get an external monoidal product even on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ which prolongs the one on diagrams of correspondences, and in many concrete situations all objects will be dualizable.

9.2 Construction of derivator Grothendieck contexts

In this section we formally *construct* a (symmetric) derivator six-functor-formalism in which $f_! = f_*$, i.e. a derivator Grothendieck context, starting from a (symmetric) fibered multiderivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$. The precise statement is as follows:

Main Theorem 9.2.1. *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}^{\mathrm{op}}$ be the (symmetric) pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (symmetric) left and right fibered multiderivator such that the following holds:*

1. *The pullback along 1-ary morphisms (i.e. pushforward along 1-ary morphisms in $\mathcal{S}$) commutes also with homotopy* colimits *(of shape in* Dia*).*
2. *In the underlying bifibration $\mathbb{D}(\cdot) \to \mathbb{S}(\cdot)$ multi-base-change holds in the sense of Definition 1.10.1.*

Then there exists a (symmetric) oplax left fibered multiderivator

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$$

satisfying the following properties

a) *The corresponding (symmetric) 1-opfibration, and 2-opfibration of 2-multicategories with 1-categorical fibers*

$$\mathbb{E}(\cdot) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(\cdot) = \mathcal{S}^{\mathrm{cor},G}$$

is just (up to equivalence) obtained from $\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$ by the procedure described in Definition 3.3.2.

b) *For every $S \in \mathcal{S}$ there is a canonical equivalence between the fibers (which are usual left and right derivators):*

$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Using standard theorems on Brown representability (cf. Section 8) we can refine this:

Main Theorem 9.2.2. *Let* Dia *be an infinite diagram category (cf. Definition 4.1.1) which contains all finite posets. Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}$ be the corresponding represented (symmetric) pre-multiderivator. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be an* infinite *(symmetric) left and right fibered multiderivator satisfying conditions 1. and 2. of Theorem 9.2.1,* with stable, perfectly generated fibers *(cf. Definition 4.3.19 and Definition 8.1.7).*
Then the restriction of the left fibered multiderivator $\mathbb{E}$ from Theorem 9.2.1 is a (symmetric) left and right *fibered multiderivator*

$$\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}} \to \mathbb{S}^{\mathrm{cor}}$$

and has an extension as a (symmetric) lax *right fibered multiderivator*

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}.$$

In other words, we get a (symmetric) derivator Grothendieck context in the sense of Section 9.1.

We begin by explaining the construction of $\mathbb{E}$. We need some preparation:

9.2.3. Let I be a diagram, n a natural number and $\Xi = (\Xi_1, \dots, \Xi_n) \in \{\uparrow, \downarrow\}^n$ be a sequence of arrow directions. We define a diagram

$${}^{\Xi}I$$

whose objects are sequences of $n-1$ morphisms in I

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_n$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd} i_1 \arrow[r] \arrow[d, leftrightarrow] & i_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & i_n \arrow[d, leftrightarrow] \\ i_1' \arrow[r] & i_2' \arrow[r] & \cdots \arrow[r] & i_n' \end{tikzcd}$$

in which the j-th vertical arrow goes in the direction indicated by Ξ_j. We call such morphisms **of type** j if the morphism $i_k \to i'_k$ is an identity *unless* $k = j$. *From now on we assume that* Dia *permits this construction for any* $I \in$ Dia*, i.e. if* $I \in$ Dia *then also* ${}^{\Xi}I \in$ Dia *for every finite* Ξ.

Example 9.2.4.

$$\begin{aligned} {}^{\downarrow}I &= I \\ {}^{\uparrow}I &= I^{\mathrm{op}} \\ {}^{\downarrow\downarrow}I &= I \times_{/I} I \\ {}^{\downarrow\uparrow}I &= \mathrm{tw}(I) \end{aligned}$$

where $I \times_{/I} I$ *is the comma category (or the arrow category of* I*) and* $\mathrm{tw}(I)$ *is called the* **twisted arrow category**.

9.2.5. For any ordered subset $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, denoting Ξ' the restriction of Ξ to the subset, we get an obvious restriction functor

$$\pi_{i_1,\dots,i_m} : {}^{\Xi}I \to {}^{\Xi'}I.$$

If $\Xi = \Xi' \circ \Xi'' \circ \Xi'''$, where $\circ$ means concatenation, then the projection

$$\pi_{1,\dots,n'} : {}^{\Xi}I \to {}^{\Xi'}I$$

is a *fibration* if the last arrow of Ξ' is $\downarrow$ and an *opfibration* if the last arrow of Ξ' is $\uparrow$ while the projection

$$\pi_{n-n'''+1,\dots,n} : {}^{\Xi}I \to {}^{\Xi'''}I$$

is an *opfibration* if the first arrow of Ξ''' is $\downarrow$ and a *fibration* if the first arrow of Ξ''' is $\uparrow$.

9.2.6. A functor $\alpha : I \to J$ induces an obvious functor

$${}^{\Xi}\alpha : {}^{\Xi}I \to {}^{\Xi}J.$$

A natural transfomation $\mu : \alpha \Rightarrow \beta$ induces functors

$$({}^{\Xi}\mu)_0, \dots, ({}^{\Xi}\mu)_n : {}^{\Xi}I \to {}^{\Xi}J$$

with $({}^{\Xi}\mu)_0 = {}^{\Xi}\alpha$, and $({}^{\Xi}\mu)_n = {}^{\Xi}\beta$, defined by mapping an object $i_1 \xrightarrow{\nu_1} i_2 \longrightarrow \dots \xrightarrow{\nu_{n-1}} i_n$ of ${}^{\Xi}I$ to the sequence:

$$\begin{tikzcd} \alpha(i_1) \arrow[r] & \cdots \arrow[r] & \alpha(i_{n-j}) \arrow[dr, "\beta(\nu_{n-j})\circ\mu(i_{n-j})"] & & & \\ & & & \beta(i_{n-j+1}) \arrow[r] & \cdots \arrow[r] & \beta(i_n) \end{tikzcd}$$

There is a sequence of natural transformations

$$^{\Xi}\alpha = (^{\Xi}\mu)_0 \Leftrightarrow \cdots \Leftrightarrow (^{\Xi}\mu)_n = {}^{\Xi}\beta$$

where the natural transformations at position i (the count starting with 0) goes to the right if $\Xi_{n-i} =\downarrow$ and to the left if $\Xi_{n-i} =\uparrow$. Furthermore, the natural transformation at position i consists element-wise of morphisms of type $n-i$.

9.2.7. If $\alpha : I \to J$ is an opfibration and we form the pull-back

$$\begin{tikzcd}
{}^{\downarrow\uparrow}J \times_J I \arrow[r] \arrow[d] & I \arrow[d, "\alpha"] \\
{}^{\downarrow\uparrow}J \arrow[r, "\pi_1"'] & J
\end{tikzcd}$$

then obviously the left vertical functor is an opfibration as well.

9.2.8. Let $S : I \to \mathcal{S}^{\mathrm{cor}}$ be a pseudo-functor. We can associate to it a natural functor $S' : {}^{\downarrow\uparrow}I \to \mathcal{S}$ such that for each composition of three morphisms $\gamma\beta\alpha$ the commutative diagram

$$\begin{tikzcd}
\gamma\beta\alpha \arrow[r] \arrow[d] & \beta\alpha \arrow[d] \\
\gamma\beta \arrow[r] & \beta
\end{tikzcd} \tag{52}$$

in ${}^{\downarrow\uparrow}I$ is mapped to a Cartesian square in $\mathcal{S}$. We call such diagrams **admissible**.
Note that the horizontal morphisms are of type 2 and the vertical ones of type 1. Conversely every square in ${}^{\downarrow\uparrow}I$ with these properties has the above form.
The construction of S' is as follows. S maps a morphism ν in I to a correspondence in $\mathcal{S}$

$$X_\nu \longleftarrow A_\nu \longrightarrow Y_\nu,$$

and we define $S'(\nu) := A_\nu$. A morphism $\xi : \nu \to \mu$ defined by

$$\begin{tikzcd}
i \arrow[r, "\nu"] \arrow[d, "\alpha"'] & j \\
k \arrow[r, "\mu"'] & l \arrow[u, "\beta"']
\end{tikzcd}$$

induces, by definition of the composition in $\mathcal{S}^{\mathrm{cor}}$, a commutative diagram in which all squares are Cartesian:

$$\begin{tikzcd}[column sep=small]
 & & & A_\nu \arrow[dl] \arrow[dr] & & & \\
 & & A_{\mu\alpha} \arrow[dl] \arrow[dr] & & A_{\beta\mu} \arrow[dl] \arrow[dr] & & \\
 & A_\alpha \arrow[dl] \arrow[dr] & & A_\mu \arrow[dl] \arrow[dr] & & A_\beta \arrow[dl] \arrow[dr] & \\
X & & Y & & Z & & W
\end{tikzcd}$$

We define $S'(\xi)$ to be the induced morphism $A_\nu \to A_\mu$. Note that the square of the form (52) is just mapped to the upper square in the above diagram, thus to a Cartesian square. Hence the so defined functor S' is admissible.

9.2.9. A multimorphism

$$T \longrightarrow S_1, \dots, S_n$$

of admissible diagrams in $\mathbb{S}({}^{\downarrow\uparrow}I)$ is called **type** i **admissible** ($i = 1, 2$), if for any morphism $\xi : \nu \to \mu$ in ${}^{\downarrow\uparrow}I$ of type i the diagram

$$\begin{tikzcd}
T(\nu) \arrow[r] \arrow[d] & (S_1(\nu), \dots, S_n(\nu)) \arrow[d] \\
T(\mu) \arrow[r] & (S_1(\mu), \dots, S_n(\mu))
\end{tikzcd}$$

is a multipullback.
A multimorphism $(X_1, \dots, X_n) \to Y$ in $\mathcal{S}^{\mathrm{cor}}(I)$ can be seen equivalently as a multicorrespondence of admissible diagrams in $\mathcal{S}({}^{\downarrow\uparrow}I)$

$$\begin{tikzcd}
 & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\
(X_1, \dots, X_n) & & Y
\end{tikzcd}$$

where f is type 2 admissible and g is type 1 admissible. In this description, the 2-morphisms are the commutative diagrams

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\
(X_1, \dots, X_n) & & Y \\
 & A' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the morphism h is an isomorphism.
In this way, we see that the 2-multicategory $\mathcal{S}^{\mathrm{cor}}(I)$ is equivalent to the 2-multicategory having as objects admissible diagrams ${}^{\downarrow\uparrow}I \to \mathcal{S}$ with the 1-multimorphisms and 2-morphisms described above.

Lemma 9.2.10. *Type i admissible morphisms $S \to T$ between admissible diagrams $S, T \in \mathbb{S}({}^{\downarrow\uparrow}I)$ satisfy the following property:*
If $h_3 = h_2 \circ h_1$ and h_2 is type i-admissible then h_1 is type i admissible if and only if h_3 is type i admissible.

Proof. This follows immediately from the corresponding property of Cartesian squares. □

9.2.11. The discussion in 9.2.9 has an (op)lax variant. Recall the definition of the category (value of the represented (op)lax 2-pre-multiderivator) $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)$), of pseudo-functors, *(op)lax* natural transformations, and modifications. A *lax* multimorphism of pseudo-functors

$$(X_1, \dots, X_n) \longrightarrow Y$$

can be equivalently seen as a multicorrespondence of admissible diagrams in $\mathbb{S}({}^{\downarrow\uparrow}I)$

$$\begin{tikzcd} & A \arrow[ld, "g"'] \arrow[rd, "f"] & \\ (X_1, \dots, X_n) & & Y \end{tikzcd}$$

where g is type 1 admissible and f is *arbitrary*. Similarly an *oplax* multimorphism can be seen as such a multicorrespondence in which g is *arbitrary* and f is type 2 admissible. In the 2-morphisms the morphism h can be an arbitrary morphism, which is automatically type 1 admissible in the lax case and type 2 admissible in the oplax case (cf. Lemma 9.2.10).

9.2.12. We can therefore describe the represented 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$, $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$, and $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$, respectively, in a different way: A diagram I is mapped to the 2-multicategory of admissible diagrams ${}^{\downarrow\uparrow}I \to \mathcal{S}$ where multimorphisms are multicorrespondences

$$\begin{tikzcd} & A \arrow[ld, "g"'] \arrow[rd, "f"] & \\ (X_1, \dots, X_n) & & Y \end{tikzcd}$$

of admissible diagrams with the corresponding conditions discussed above and where 2-morphisms are the isomorphisms (resp. arbitrary morphisms) between these multicorrespondences.
A functor $\alpha : I \to J$ is mapped to the composition $- \circ ({}^{\downarrow\uparrow}\alpha)$. This is a strict functor and the association is strictly functorial. A natural transformation $\mu : \alpha \Rightarrow \beta$ is mapped to

the following natural transformation. First of all it gives rise (cf. 9.2.6) to a sequence of natural transformations

$$({}^{\downarrow\uparrow}\alpha) \Leftarrow ({}^{\downarrow\uparrow}\mu)_1 \Rightarrow ({}^{\downarrow\uparrow}\beta).$$

For any admissible diagram $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ this defines a diagram

$$\begin{tikzcd} & ({}^{\downarrow\uparrow}\mu)_1^* S \arrow[dl, "g_S"'] \arrow[dr, "f_S"] & \\ ({}^{\downarrow\uparrow}\alpha)^* S & & ({}^{\downarrow\uparrow}\beta)^* S \end{tikzcd}$$

in which the morphism f_S is type 2 admissible and the morphism g_S is type 1 admissible. This defines a 1-morphism

$$({}^{\downarrow\uparrow}\alpha)^* S \to ({}^{\downarrow\uparrow}\beta)^* S$$

in the alternative description (cf. 9.2.9) of $\mathbb{S}^{\mathrm{cor}}(I)$. For any admissible diagram S this defines a pseudo-functor $\alpha \mapsto \alpha^* S$ from the category of functors $\mathrm{Fun}(I, J)$ to the 2-category $\mathbb{S}^{\mathrm{cor}}(I)$.

9.2.13. Let I be a diagram. Consider the category ${}^{\downarrow\uparrow\downarrow}I$ defined in 9.2.3. Recall that its objects are compositions of two morphisms in I and its morphisms $\nu \to \mu$ are commutative diagrams

$$\begin{tikzcd} i \arrow[r, "\nu_1"] \arrow[d] & j \arrow[r, "\nu_2"] & k \arrow[d] \\ i' \arrow[r, "\mu_1"'] & j' \arrow[u] \arrow[r, "\mu_2"'] & k' \end{tikzcd}$$

9.2.14. If $\alpha : I \to J$ is an opfibration and we form the pull-back

$$\begin{tikzcd} {}^{\downarrow\uparrow\downarrow}J \times_J I \arrow[r] \arrow[d] & I \arrow[d, "\alpha"] \\ {}^{\downarrow\uparrow\downarrow}J \arrow[r, "\pi_1"'] & J \end{tikzcd}$$

and

$$\begin{tikzcd} {}^{\downarrow\uparrow\downarrow}J \times_{{}^{\downarrow\uparrow}J} {}^{\downarrow\uparrow}I \arrow[r] \arrow[d] & {}^{\downarrow\uparrow}I \arrow[d, "{}^{\downarrow\uparrow}\alpha"] \\ {}^{\downarrow\uparrow\downarrow}J \arrow[r, "\pi_{12}"'] & {}^{\downarrow\uparrow}J \end{tikzcd}$$

then obviously the left vertical functors are opfibrations as well.

Lemma 9.2.15. *Let $\alpha : I \to J$ be an opfibration, and consider the sequence defined by the universal property of pull-backs*

$${}^{\downarrow\uparrow\downarrow}I \xrightarrow{q_1} {}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I \xrightarrow{q_2} {}^{\downarrow\uparrow\downarrow}J \times_J I.$$

1. *The functor* q_1 *is an opfibration. The fiber of* q_1 *over a pair* $j_1 \to j_2 \to j_3$ *and* $i_1 \to i_2$ *is*

$$i_3 \times_{/I_{j_3}} I_{j_3}$$

where i_3 *is the target of a coCartesian arrow over* $j_2 \to j_3$ *with source* i_2*.*

2. *The functor* q_2 *is a fibration. The fiber of* q_2 *over a pair* $j_1 \to j_2 \to j_3$ *and* i_1 *is*

$$(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$$

where i_2 *is the target of a coCartesian arrow over* $j_1 \to j_2$ *with source* i_1*.*

Proof. Straightforward. □

Recall the following definition (Definition 7.4.1), in which $\mathbb{S}$ can actually be any 2-pre-multiderivator.

Definition 9.2.16. *Let* $\mathbb{D} \to \mathbb{S}$ *be a right (resp. left) fibered (multi)derivator of domain* Dia*. Let* $I, E \in \mathrm{Dia}$ *be diagrams and let* $\alpha : I \to E$ *be a functor in* Dia*. We say that an object*

$$\mathcal{E} \in \mathbb{D}(I)$$

is E-**(co-)Cartesian***, if for any morphism* $\mu : i \to j$ *in* I *mapping to an identity in* E*, the corresponding morphism* $\mathbb{D}(\mu) : i^*\mathcal{E} \to j^*\mathcal{E}$ *is (co-)Cartesian.*
If E *is the trivial category, we omit it from the notation, and talk about (co-)Cartesian objects.*

These notions define full subcategories $\mathbb{D}(I)^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)$, and $\mathbb{D}(I)_S^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)_S^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)_S$ for any $S \in \mathbb{S}(I)$. If we want to specify the functor α, we speak about α-(co)Cartesian objects and denote these e.g. by $\mathbb{D}(I)_S^{\alpha-\mathrm{cart}}$.

Definition 9.2.17. *Let* $\mathcal{S}$ *be an opmulticategory with multipullbacks and let* $\mathbb{S}^{\mathrm{op}}$ *be the pre-multiderivator represented by* $\mathcal{S}^{\mathrm{op}}$*. Let* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ *be a (left and right) fibered multiderivator such that conditions 1. and 2. of Theorem 9.2.1 hold true.*
We define the morphism of 2-pre-multiderivators $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ *of Theorem 9.2.1. The 2-pre-multiderivator* $\mathbb{E}$ *is defined as follows: A diagram* I *is mapped to a 1-opfibered, and 2-opfibered multicategory with 1-categorical fibers* $\mathbb{E}(I) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)$*. We will specify this by giving the pseudo-functor of 2-multicategories*

$$\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)^{2-\mathrm{op}} \to \mathcal{CAT}$$

where we understand $\mathbb{S}^{\mathrm{cor}}(I)$ *(resp.* $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$*) in the form described in 9.2.12. An admissible diagram* $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ *is mapped to the category*

$$\boxed{\mathbb{E}(I)_S := \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}}$$

(cf. Definition 9.2.16). Note that $({}^{\downarrow\uparrow}I)^{\mathrm{op}} = {}^{\uparrow\downarrow}I$*.*

A multicorrespondence

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (S_1, \dots, S_n) & & T \end{tikzcd}$$

where f is type 2 admissible and g is type 1 admissible is mapped to the functor

$$\boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet : \mathbb{E}(I)_{S_1} \times \cdots \times \mathbb{E}(I)_{S_n} \to \mathbb{E}(I)_T}$$

Note that, by Lemma 9.2.18, $(\pi_{23}^* g)_\bullet$ preserves the subcategory of π_{12}-Cartesian objects and, by Lemma 9.2.19, $(\pi_{23}^* f)^\bullet$ preserves the subcategory of π_{13}-coCartesian objects. In the oplax case, the condition on f is repealed and the multicorrespondence is mapped to

$$\Box_* (\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet$$

where $\Box_*$ is the right coCartesian projection *defined and discussed in Section 9.3.*
A 2-morphism, given by a morphism of multicorrespondences

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\ X_1, \dots, X_n & & Y \\ & A' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

where h is an isomorphism, is mapped to the natural transformation given by the unit

$$\boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet \cong (\pi_{23}^* f')^\bullet (\pi_{23}^* h)^\bullet (\pi_{23}^* h)_\bullet (\pi_{23}^* g')_\bullet \leftarrow (\pi_{23}^* f')^\bullet (\pi_{23}^* g')_\bullet}$$

In the oplax case, h can be an arbitrary morphism (which will be automatically type 1 admissible). The 2-morphism is then mapped to the natural transformation given by the unit

$$\Box_* (\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet \cong \Box_* (\pi_{23}^* f')^\bullet \Box_* (\pi_{23}^* h)^\bullet (\pi_{23}^* h)_\bullet (\pi_{23}^* g')_\bullet \leftarrow \Box_* (\pi_{23}^* f')^\bullet (\pi_{23}^* g')_\bullet .$$

A functor $\alpha : I \to J$ is mapped to the functor

$$\boxed{({}^{\downarrow\uparrow\downarrow}\alpha)^*}$$

which obviously preserves the (co)Cartesianity conditions. This is strictly compatible with composition of functors between diagrams. A natural transformation $\mu : \alpha \to \beta$ is mapped to the following natural transformation $({}^{\downarrow\uparrow\downarrow}\alpha)^* \to ({}^{\downarrow\uparrow\downarrow}\beta)^*$: We have the correspondence (cf. 9.2.12)

$$\begin{tikzcd} & ({}^{\downarrow\uparrow}\mu)_1^* S \arrow[dl, "g_\mu"'] \arrow[dr, "f_\mu"] & \\ ({}^{\downarrow\uparrow}\alpha)^* S & & ({}^{\downarrow\uparrow}\beta)^* S \end{tikzcd}$$

where f_μ is type 2 admissible and g_μ is type 1 admissible by the definition of admissible diagram. On the other hand, there are natural transformations (cf. 9.2.6)

$$ {}^{\downarrow\uparrow\downarrow}\alpha \Rightarrow ({}^{\downarrow\uparrow\downarrow}\mu)_1 \Leftarrow ({}^{\downarrow\uparrow\downarrow}\mu)_2 \Rightarrow {}^{\downarrow\uparrow\downarrow}\beta. $$

Inserting $\pi_{23}^*(S^{\mathrm{op}})$ into this, we get

$$ \pi_{23}^*({}^{\downarrow\uparrow}\alpha)^*(S^{\mathrm{op}}) \xrightarrow{\pi_{23}^* g_\mu} \pi_{23}^*({}^{\downarrow\uparrow}\mu)_1^*(S^{\mathrm{op}}) \xleftarrow{\pi_{23}^* f_\mu} \pi_{23}^*({}^{\downarrow\uparrow}\beta)^*(S^{\mathrm{op}}) = \pi_{23}^*({}^{\downarrow\uparrow}\beta)^*(S^{\mathrm{op}}). \quad (53) $$

The natural transformation $\mu : \alpha \to \beta$ may be seen as a functor $\Delta_1 \times I \to J$ and therefore we get a functor

$$ {}^{\downarrow\uparrow\downarrow}\mu : {}^{\downarrow\uparrow\downarrow}\Delta_1 \times {}^{\downarrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\downarrow}J. $$

Applying the (pre-)derivator $\mathbb{D}$ and partially evaluating at the objects and morphisms of ${}^{\downarrow\uparrow\downarrow}\Delta_1$ we get natural transformations

$$ \begin{array}{rcl} (\pi_{23}^* g_\mu)_\bullet ({}^{\downarrow\uparrow\downarrow}\alpha)^* & \to & ({}^{\downarrow\uparrow\downarrow}\mu)_1^* \\ ({}^{\downarrow\uparrow\downarrow}\mu)_2^* & \to & (\pi_{23}^* f_\mu)^\bullet ({}^{\downarrow\uparrow\downarrow}\mu)_1^* \\ ({}^{\downarrow\uparrow\downarrow}\mu)_2^* & \to & ({}^{\downarrow\uparrow\downarrow}\beta)^* \end{array} $$

where the $(-)^*$-functors are now considered to be functors between the respective fibers over the objects of (53). Clearly the first two morphisms (in particular the second) are isomorphisms when restricted to the respective categories of (co)Cartesian objects. Therefore we can form their composition:

$$ \boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet ({}^{\downarrow\uparrow\downarrow}\alpha)^* \to ({}^{\downarrow\uparrow\downarrow}\beta)^*} $$

which will be the image of μ under the 2-pre-multiderivator $\mathbb{E}$. One checks that for any admissible diagram $S \in \mathbb{S}({}^{\downarrow\uparrow}I)$, this defines a pseudo-functor from the category of functors $\mathrm{Fun}(I, J)$ *to the 2-category of functors of the 2-category $\mathbb{E}(I)$ to the 2-category $\mathbb{E}(J)$, pseudo-natural transformations and modifications.*

Lemma 9.2.18. *Under the conditions of Theorem 9.2.1, let $S, T : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be admissible diagrams and let $f : S \to T$ be any morphism in $\mathbb{S}({}^{\downarrow\uparrow}I)$. Then the functor*

$$ (\pi_{23}^* f)^\bullet : \mathbb{D}(I)_{\pi_{23}^* T^{\mathrm{op}}} \to \mathbb{D}(I)_{\pi_{23}^* S^{\mathrm{op}}} $$

maps always π_{13}-Cartesian objects to π_{13}-Cartesian objects, and maps π_{12}-coCartesian objects to π_{12}-coCartesian if f is type 2 admissible.

Proof. This follows immediately from base-change and from the definition of type 2 admissible. ☐

Lemma 9.2.19. *Under the conditions of Theorem 9.2.1, let $S_1, \dots, S_n, T : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be admissible diagrams and let $g : S_1, \dots, S_n \to T$ be any multimorphism in $\mathbb{S}({}^{\downarrow\uparrow}I)$. Then the functor*

$$ (\pi_{23}^* g)_\bullet : \mathbb{D}(I)_{\pi_{23}^* S_1^{\mathrm{op}}} \times \cdots \times \mathbb{D}(I)_{\pi_{23}^* S_n^{\mathrm{op}}} \to \mathbb{D}(I)_{\pi_{23}^* T^{\mathrm{op}}} $$

maps always π_{12}-coCartesian objects to π_{12}-coCartesian objects, and maps π_{13}-Cartesian objects to π_{13}-Cartesian objects if g is type 1 admissible.

Proof. This follows immediately from multi-base-change and from the definition of type 1 admissible. □

9.2.20. Recall that a diagram I is called contractible, if

$$\mathrm{id} \Rightarrow p_{I,*}(p_I)^*,$$

or equivalently

$$p_{I,!}(p_I)^* \Rightarrow \mathrm{id},$$

is an isomorphism for all derivators. Cisinski showed that this is the case if and only if $N(I)$ is weakly contractible in the sense of simplicial sets. For instance, any diagram possessing a final or initial object is contractible. The following lemma was shown in Chapter 7 for the case of all contractible diagrams for a restricted class of stable derivators. We will only need the mentioned special case which is easy to prove in full generality:

Lemma 9.2.21. *If $\mathbb{D}$ is a left derivator and I has a final object, or $\mathbb{D}$ is a right derivator and I has an initial object, then the functor*

$$p_I^* : \mathbb{D}(\cdot) \to \mathbb{D}(I)^{\mathrm{cart}} = \mathbb{D}(I)^{\mathrm{cocart}}$$

is an equivalence.

Note that Cartesian=coCartesian here only means that all morphisms in the underlying diagram in $\mathrm{Fun}(I, \mathbb{D}(\cdot))$ are isomorphisms.

Proof. Assume we have a left derivator and I has a final object (the other statement is dual). It suffices to show that the counit

$$p_{I,!}p_I^* \Rightarrow \mathrm{id}$$

is an isomorphism and that the unit

$$\mathrm{id} \Rightarrow p_I^* p_{I,!}$$

is an isomorphism when restricted to the subcategory of Cartesian objects. Since I has a final object i we have an isomorphism

$$p_{I,!} \cong i^*$$

and the unit and counit become the morphisms induced by the natural transformations $p_I \circ i = \mathrm{id}$ and $\mathrm{id} \Rightarrow i \circ p_I$. Hence we have

$$i^* p_I^* = \mathrm{id}$$

and the morphism

$$\mathrm{id} \Rightarrow p_I^* i^*$$

is an isomorphism on (co)Cartesian objects by definition of (co)Cartesian. □

Corollary 9.2.22. *If $\mathbb{D}$ is a left and right derivator and I has a final* or *initial object then*

$$p_I^* : \mathbb{D}(\cdot) \to \mathbb{D}(I)^{\mathrm{cart}} = \mathbb{D}(I)^{\mathrm{cocart}}$$

is an equivalence, whose inverse is given by $p_{I,!}$ or equivalently by $p_{I,*}$.

Proof. The first part is just restating the above lemma. The fact that both the restriction of $p_{I,!}$, and the restriction of $p_{I,*}$, to the subcategory $\mathbb{D}(I)^{\mathrm{cart}}$ are an inverse to p_I^* follows because these restrictions are obviously still left, resp. right, adjoints to the equivalence p_I^*, hence both inverses, because of the uniqueness of adjoints (up to unique isomorphism). □

Lemma 9.2.23. *Under the assumptions of Theorem 9.2.1, if $\alpha : I \to J$ is an opfibration then the functors*

$$\mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_J I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \xrightarrow{q_2^*} \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \xrightarrow{q_1^*} \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

are equivalences. In particular (applying this to $J = \cdot$ and variable I) we have an equivalence of fibers:

$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Proof. We first treat the case of q_1^*. We know by Lemma 9.2.15 that q_1 is an opfibration with fibers of the form $i_3 \times_{/I_{j_3}} I_{j_3}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a left adjoint:

$$q_{1,!} : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

We will show that the unit and counit

$$\mathrm{id} \Rightarrow q_1^* q_{1,!} \qquad q_{1,!} q_1^* \Rightarrow \mathrm{id}$$

are isomorphisms *when restricted to the subcategory of π_{12}-coCartesian objects.* Since the conditions of being π_{13}-Cartesian match under q_1^* this shows the first assertion. Since q_1 is an opfibration this is the same as to show that for any $\gamma \in {}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I$ with fiber $F = i_3 \times_{/I_{j_3}} I_{j_3}$ the unit and counit

$$\mathrm{id} \Rightarrow p_F^* p_{F,!} \qquad p_{F,!} p_F^* \Rightarrow \mathrm{id} \tag{54}$$

are isomorphisms when restricted to the subcategory of π_{12}-coCartesian objects. Since π_{12} maps all morphisms in the fiber F to an identity, we have to show that the morphisms in (54) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that F has an initial object (Lemma 9.2.21 and Corollary 9.2.22). We now treat the case of q_2^*. We know by Lemma 9.2.15 that q_2 is a fibration with fibers of the form $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a right adjoint:

$$q_{2,*} : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_J I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

We will show that the unit and counit

$$\mathrm{id} \Rightarrow q_{2,*}q_2^* \qquad q_2^* q_{2,*} \Rightarrow \mathrm{id}$$

are isomorphisms *when restricted to the subcategory of π_{13}-Cartesian objects.* Since the conditions of being π_{12}-coCartesian match under q_2^* this shows the second assertion. Since q_2 is a fibration this is the same as to show that for any $\gamma \in {}^{\downarrow\uparrow\downarrow}J \times_J I$ with fiber $F = (i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ the the unit and counit

$$\mathrm{id} \Rightarrow p_{F,*}p_F^* \qquad p_F^* p_{F,*} \Rightarrow \mathrm{id} \tag{55}$$

are isomorphisms when restricted to the subcategory of π_{13}-Cartesian objects. Since π_{13} maps all morphisms in the fiber $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ to an identity, this means that we have to show that (55) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ has a final object (Lemma 9.2.21 and Corollary 9.2.22). □

Lemma 9.2.24. *Let the situation be as in Theorem 9.2.1 and let $p' : \mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ be the morphism of 2-pre-multiderivators defined in 9.2.17. Let $\alpha : I \to J$ be an opfibration. Then $\alpha^* : \mathbb{E}(J)_{\alpha^* S} \to \mathbb{E}(I)_S$ has a left adjoint $\alpha_!^{(S)}$.*

Proof. We have to show that

$$({}^{\downarrow\uparrow\downarrow}\alpha)^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

has a left adjoint. The right hand side category is by Lemma 9.2.23 equivalent to

$$\mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^* S},$$

hence we have to show that

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

has a left adjoint. By assumption the functor

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

has a left adjoint $\mathrm{pr}_{2,!}$. We claim that it maps π_{12}-coCartesian objects to π_{12}-coCartesian objects and π_{13}-Cartesian objects to π_{13}-Cartesian objects. The statement then follows. Let $\kappa : \nu \to \nu'$

$$\begin{tikzcd} j_1 \arrow[r, "\nu_1"] & j_2 \arrow[r, "\nu_2"] & j_3 \\ j_1 \arrow[u, equal] \arrow[r, "\nu_1'"'] & j_2' \arrow[u, "\kappa_2"'] \arrow[r, "\nu_2'"'] & j_3 \arrow[u, equal] \end{tikzcd}$$

be a morphism in ${}^{\downarrow\uparrow\downarrow}J$ such that π_{13} maps it to an identity. Denote

$$f := S(\pi_{23}(\kappa)) : S(\pi_{23}(\nu)) \to S(\pi_{23}(\nu'))$$

the corresponding morphism in $\mathbb{S}(\cdot)^{\mathrm{op}}$. Denote by (ν), resp. (ν') the inclusion of the one element category mapping to ν, resp. ν' in ${}^{\downarrow\uparrow\downarrow}I$. We have to show that the induced map

$$(\nu)^* \operatorname{pr}_{1,!} \to f^\bullet (\nu')^* \operatorname{pr}_{1,!}$$

is an isomorphism on π_{13}-Cartesian objects. Since pr_1 is an opfibration, this is the same as to show that the natural morphism

$$p_! \iota_\nu^* \to f^\bullet p_! \iota_{\nu'}^*$$

is an isomorphism on π_{13}-Cartesian objects where $p : I_{j_1} \to \cdot$ is the projection. Since $f^\bullet$ commutes with homotopy colimits by assumption 1. of Theorem 9.2.1, this is to say that

$$p_! \iota_\nu^* \to p_! (p^* f)^\bullet \iota_{\nu'}^*$$

is an isomorphism. However the fibers over ν and ν' in $({}^{\downarrow\uparrow\downarrow}J) \times_J I$ are both equal to I_{j_1} and the natural morphism

$$\iota_\nu^* \to (p^* f)^\bullet \iota_{\nu'}^*$$

is already an isomorphism on Cartesian objects by definition.
Let $\kappa : \nu_1 \to \nu_2$

$$\begin{tikzcd}
j_1 \arrow[r, "\nu_1"] \arrow[d, equal] & j_2 \arrow[r, "\nu_2"] \arrow[d, equal] & j_3 \arrow[d] \\
j_1 \arrow[r, "\nu_1'"'] & j_2' \arrow[r, "\nu_2'"'] & j_3'
\end{tikzcd}$$

be a morphism in ${}^{\downarrow\uparrow\downarrow}J$ such that π_{12} maps it to an identity. And denote

$$g := S(\pi_{23}(\kappa)) : S(\pi_{23}(\nu)) \to S(\pi_{23}(\nu'))$$

the corresponding morphism in $\mathbb{S}(\cdot)$. Denote by (ν), resp. (ν') the inclusion of the one element category mapping to ν, resp. ν'. We have to show that the induced map

$$g_\bullet (\nu)^* \operatorname{pr}_{1,!} \to (\nu')^* \operatorname{pr}_{1,!}$$

is an isomorphism on π_{12}-coCartesian objects. This is the same as to show that the natural morphism

$$g_\bullet p_! \iota_\nu^* \to p_! \iota_{\nu'}^*$$

is an isomorphism on π_{12}-coCartesian objects where $p : I_{j_1} \to \cdot$ is the projection. Since $g_\bullet$ commutes with homotopy colimits, this is to say that

$$p_! (p^* g)_\bullet \iota_\nu^* \to p_! \iota_{\nu'}^*$$

is an isomorphism. However the fibers over ν and ν' in $({}^{\downarrow\uparrow\downarrow}J) \times_J I$ are both equal to I_j and the natural morphism

$$(p^* g)_\bullet \iota_\nu^* \to \iota_{\nu'}^*$$

is already an isomorphism on coCartesian objects by definition of coCartesian. □

Proof of Theorem 9.2.1. It is clear that the 2-pre-multiderivator $\mathbb{E}$ as defined in 9.2.17 satisfies axioms (Der1) and (Der2) because $\mathbb{D}$ satisfies them. Axiom (FDer0 left) holds by construction of $\mathbb{E}$. Instead of Axiom (FDer3 left) it is sufficient to show Axiom (FDer3 left') which follows from Lemma 9.2.24. Axiom (FDer4 left') follows from the proof of Lemma 9.2.24. (FDer5 left) follows from the corresponding axiom for $\mathbb{D}$ and the fact that pull-back along 1-ary morphisms in $\mathbb{S}^{\mathrm{op}}$ commutes with homotopy colimits as well, by assumption. □

9.2.25. Let $\alpha : K \to L$ be a functor in Dia and let $\xi : (I_1, S_1), \dots, (I_n, S_n) \to (J, T)$ be a 1-morphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$. If we have a 1-opfibration and 2-opfibration

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

then the isomorphism of Lemma 5.5.6, 3. is transformed into an isomorphism

$$(\alpha \times \mathrm{id})^* \circ (\xi \times L)_\bullet \to (\xi \times K)_\bullet \circ ((\alpha \times \mathrm{id})^*, \dots, (\alpha \times \mathrm{id})^*)$$

which turns $K \mapsto (\xi \times K)_\bullet$ into a morphism of usual derivators

$$\xi_\bullet : \mathbb{D}_{I_1,S_1} \times \cdots \times \mathbb{D}_{I_n,S_n} \to \mathbb{D}_{J,T}. \tag{56}$$

Lemma 9.2.26. *The morphism of derivators (56) is left exact in each variable, i.e. the exchange*

$$(\xi \times_j L)_\bullet \circ_j (\alpha \times \mathrm{id})_! \to (\alpha \times \mathrm{id})_! \circ (\xi \times_j K)_\bullet$$

is an isomorphism for any $\alpha : K \to L$.

Proof. This follows from Lemma 5.5.6, 4. □

Proof of Theorem 9.2.2. The first assertion is a slight generalization of Theorem 8.2.3. Using Definition 5.4.1 of a left, resp. right fibered multiderivator over 2-pre-multiderivators we give a different slicker proof. We have to show that, under the conditions of Theorem 9.2.2, the constructed 1-opfibration

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

is a 1-fibration as well. The conditions imply:

1. Dia, $\mathbb{E}$ and $\mathbb{S}^{\mathrm{cor}}$ are infinite,
2. the fibers of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ (which are the same as those of $\mathbb{D} \to \mathbb{S}$) are stable and perfectly generated infinite left derivators with domain Dia, and also right derivators with domain (at least) Posf.

Any multimorphism in $(I_1, S_1), \dots, (I_n, S_n) \to (J, T)$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ gives actually a morphism between fibers which are usual left and right stable derivators which are perfectly generated:

$$\mathbb{D}_{I_1,S_1} \times \cdots \times \mathbb{D}_{I_n,S_n} \to \mathbb{D}_{J,T}.$$

Lemma 9.2.26 shows that this morphism commutes with homotopy colimits in each variable. Thus by Theorem 8.2.1 it has a right adjoint in each slot j, which, in particular, evaluated at $\cdot$ yields a right adjoint functor in the slot j:

$$\mathbb{D}(I_1)^{\mathrm{op}}_{S_1} \times \cdots \times \mathbb{D}(J)_T \times \cdots \times \mathbb{D}(I_n)^{\mathrm{op}}_{S_n} \to \mathbb{D}(I_j)_{S_j}$$

for each j. This establishes that the morphism

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

is 1-fibered as well.

The lax extension of this 1-fibration is given as follows. For each diagram I we again specify a 1-fibered, and 2-opfibered multicategory with 1-categorical fibers $\mathbb{E}'(I) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$. The category

$$\mathbb{E}'(I)$$

has the same objects as $\mathbb{E}(I)$, i.e. pairs $(S, \mathcal{E})$ consisting of an admissible diagram $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ and an object

$$\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}.$$

The 1-morphisms are the morphisms in $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$, i.e. lax morphisms, which can be given by a multicorrespondence

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (S_1, \dots, S_n) & & T \end{tikzcd}$$

in which f is type 2 admissible, and g is arbitrary, together with a morphism

$$\rho \in \mathrm{Hom}_{\mathbb{E}(I)}\left((\mathcal{E}_1, S_1), \dots, (\mathcal{E}_n, S_n), (\mathcal{F}, T)\right) = \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(T^{\mathrm{op}})}}\left((\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet (\mathcal{E}_1, \dots, \mathcal{E}_n), \mathcal{F}\right).$$

Note that the multivalued functor $(\pi_{23}^* g)_\bullet$ does not necessarily have values in the subcategory of π_{13}-Cartesian objects.

A 2-morphism $(f, g, \rho) \Rightarrow (f', g', \rho')$ is given by a morphism of multicorrespondences

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\ X_1, \dots, X_n & & Y \\ & A' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

where h is an arbitrary morphism (which is automatically type 2 admissible, cf. Lemma 9.2.10) such that the diagram

$$\begin{tikzcd}
(\pi_{23}^*f)^\bullet(\pi_{23}^*g)_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \arrow[drr, "\rho"] & & \\
(\pi_{23}^*f')^\bullet(\pi_{23}^*h)^\bullet(\pi_{23}^*h)_\bullet(\pi_{23}^*g')_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \arrow[u, "\sim"'] & & \mathcal{F} \\
(\pi_{23}^*f')^\bullet(\pi_{23}^*g')_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \arrow[u] \arrow[urr, "\rho'"'] & &
\end{tikzcd}$$

commutes, where the lower left vertical morphism is the unit.
A functor $\alpha : I \to J$ is mapped to the functor $({}^{\downarrow\uparrow\downarrow}\alpha)^*$ which obviously preserves the (co)Cartesianity conditions. Natural morphisms are treated in the same way as in the plain case because no lax morphisms are involved.
We will now discuss the axioms:
(FDer0 right): It is clear from the definition that

$$\mathbb{E}'(I) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$$

is 2-opfibered and has 1-categorical fibers. It is also 1-fibered because we have

$$\begin{aligned}
& \mathrm{Hom}_{\mathbb{E}(I)}((\mathcal{E}_1,S_1),\dots,(\mathcal{E}_n,S_n),(\mathcal{F},T)) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*T^{\mathrm{op}}}}((\pi_{23}^*f)^\bullet(\pi_{23}^*g)_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n),\mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*S_j^{\mathrm{op}}}}(\mathcal{E}_j,\Box_*(\pi_{23}^*g)^{\bullet,j}(\mathcal{E}_1,\overset{\widehat{j}}{\dots},\mathcal{E}_n;(\pi_{23}^*f)_?\mathcal{F})).
\end{aligned}$$

Here $\Box_*$ is the right coCartesian projection defined and discussed in Section 9.3 and $(\pi_{23}^*f)_?$ is a right adjoint of $(\pi_{23}^*f)^\bullet$, which exists by the reasoning in the first part of the proof. (Note that $(\pi_{23}^*f)_?$ would be denoted $f^!$, i.e. exceptional pull-back, in the usual language of six-functor-formalisms. Our notation, unfortunately, has reached its limit here.) Therefore Cartesian morphisms exist w.r.t. to any slot j with pull-back functor explicitly given by

$$\Box_*(\pi_{23}^*g)^{\bullet,j}(-,\overset{\widehat{j}}{\dots},-;(\pi_{23}^*f)_?-).$$

The second part of (FDer0 right) follows from the corresponding statement for $\mathbb{D}$ and the fact that $\Box_*$ is "point-wise the identity" (cf. Proposition 9.3.5). The axioms (FDer3–4 right) do not involve lax morphisms. (FDer5 right) follows because the corresponding axiom holds for $\mathbb{D}$, because $(\pi_{23}^*f)_?$, as right adjoint, commutes with homotopy limits, and because $\Box_*$ is "point-wise the identity" (cf. Proposition 9.3.5). $\square$

9.3 Cocartesian projectors

9.3.1. We will show in this section that the fully-faithful inclusion

$$\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

(cf. Definitions 9.2.16, 9.2.17) has a right adjoint $\Box_*$ which we will call a **right coCartesian projection** (cf. also Definition 7.4.4).
A right coCartesian projection (or rather its composition with the fully-faithful inclusion) can be specified by an endofunctor $\Box_*$ of $\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$ together with a natural transformation

$$\nu : \Box_* \Rightarrow \mathrm{id}$$

such that

1. $\Box_*$ has values in the subcategory of π_{12}-coCartesian objects and
2. $\nu_{\Box_* \mathcal{E}} = \Box_* \nu_{\mathcal{E}}$ holds true.

This, in particular, gives a pullback functor

$$\Box_* f^\bullet : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(T^{\mathrm{op}})}$$

for *any* morphism (not necessarily type 2 admissible)

$$f : S \to T$$

of admissible diagrams in $\mathcal{S}({}^{\downarrow\uparrow}I)$.
Note that, of course, $f^\bullet$ preserves automatically the condition of being π_{13}-Cartesian. Proposition 9.3.5 below shows that this is still computed point-wise, i.e. that we have for any $\alpha : I \to J$

$$\alpha^* \Box_* f^\bullet \cong \Box_* (\alpha^* f)^\bullet.$$

9.3.2. We need some technical preparation. Consider the projections:

$$\pi_{123}, \pi_{125}, \pi_{145}, \pi_{345} : {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\downarrow}I.$$

We have obvious natural transformations

$$\pi_{123} \Rightarrow \pi_{125} \Leftarrow \pi_{145} \Rightarrow \pi_{345}$$

and therefore

$$\pi_{123}^* \Rightarrow \pi_{125}^* \Leftarrow \pi_{145}^* \Rightarrow \pi_{345}^*$$

If we plug in $\pi_{23}^*(S^{\mathrm{op}})$ for an admissible diagram $S \in \mathbb{S}({}^{\downarrow\uparrow}I)$, we get morphisms of diagrams in $\mathbb{S}^{\mathrm{op}}$:

$$\pi_{23}^*(S^{\mathrm{op}}) \xrightarrow{g} \pi_{25}^*(S^{\mathrm{op}}) \xleftarrow{f} \pi_{45}^*(S^{\mathrm{op}}) = \pi_{45}^*(S^{\mathrm{op}})$$

and therefore natural transformations

$$\begin{array}{ccc} g_\bullet \pi_{123}^* & \Rightarrow & \pi_{125}^* \\ f^\bullet \pi_{125}^* & \Leftarrow & \pi_{145}^* \end{array}$$

of functors between fibers.

Lemma 9.3.3. *π_{123} and π_{345} are opfibrations.*

Proof. This was explained in 9.2.5. □

Lemma 9.3.4. *The natural transformation*

$$\pi_{345,!}^{(\pi_{45}^* S)} \pi_{145}^* \Rightarrow \mathrm{id}$$

induced by the natural transformation

$$\pi_{145}^* \Rightarrow \pi_{345}^*$$

of functors

$$\pi_{145}^*, \pi_{345}^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I)_{\pi_{45}^*(S^{\mathrm{op}})}$$

is an isomorphism.

Proof. Since π_{345} is an opfibration, we have for any object $\alpha = \{i \to j \to k\}$ in ${}^{\downarrow\uparrow\downarrow}I$:

$$\alpha^* \pi_{345,!} \pi_{145}^* = p_! \pi_{145}^*$$

where $p : {}^{\downarrow\uparrow}(I \times_{/I} i) \to \{\cdot\}$. We can factor p in the following way:

$${}^{\downarrow\uparrow}(I \times_{/I} i) \xrightarrow{\pi_1} I \times_{/I} i \xrightarrow{P} \{\cdot\}$$

The functor π_1 is an opfibration with fibers of the form $\beta \times_{/(I \times_{/I} i)} (I \times_{/I} i)$. Since these fibers have an initial object, and the objects in the image of π_{145}^* are constant along it, the homotopy colimit over objects in the image of π_{145}^* along it are equal to this constant value by Corollary 9.2.22. Furthermore, the homotopy colimit over $I \times_{/I} i$ is the same as evaluation at id_i because id_i is the final object. □

If $\mathcal{E}$ is an object in $\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})}^{\pi_{13}-\mathrm{cocart}}$ we have that the morphism

$$f^\bullet \pi_{125}^* \mathcal{E} \leftarrow \pi_{145}^* \mathcal{E}$$

is an isomorphism.

Proposition 9.3.5. *Using the notation of 9.3.2, denote $\square_* := \pi_{345;!} f^\bullet g_\bullet \pi_{123}^*$. This functor, together with the composition*

$$\begin{tikzcd} \mathcal{E} & \pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_{345;!} f^\bullet \pi_{125}^* \mathcal{E} & \pi_{345;!} f^\bullet g_\bullet \pi_{123}^* \mathcal{E} = \square_* \mathcal{E} \arrow[l] \arrow[lll, bend left, "\nu_{\mathcal{E}}"] \end{tikzcd},$$

defines a right coCartesian projection:

$$\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* S}^{\pi_{13}-\mathrm{cart}} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* S}^{\pi_{13}-\mathrm{cart}, \pi_{12}-\mathrm{cocart}}.$$

This projection has the following property:

- *For each $i \in I$ the natural transformation*

$$({}^{\downarrow\uparrow\downarrow}i)^* \Box_* \to ({}^{\downarrow\uparrow\downarrow}i)^*$$

is an isomorphism. (Here i denotes, by abuse of notation, the subcategory of I consisting of i and id_i*. Hence ${}^{\downarrow\uparrow\downarrow}i$ is the subcategory of ${}^{\downarrow\uparrow\downarrow}I$ consisting of $i = i = i$ and its identity.)*

Proof. We have to show that $\Box_*\mathcal{E}$ is coCartesian and that

$$\Box_* \nu_{\mathcal{E}} = \nu_{\Box_*\mathcal{E}}.$$

The first assertion follows immediately from the fact that the values of $\pi_{345,!}$ at an object $i \to j \to k$ of ${}^{\downarrow\uparrow\downarrow}I$ are the homotopy colimits over the diagram $\iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$ for $\iota_{i,j,k} : {}^{\downarrow\uparrow}(I \times_{/I} i) \hookrightarrow {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I$ as follows: For any morphism *to* $i \to j \to k'$, or any morphism *from* $i \to j' \to k$, i.e. any morphism such that π_{12} (resp. π_{13}) maps it to an identity, the induced morphism

$$g_\bullet \iota_{i,j,k'}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E} \to \iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

resp.

$$f^\bullet \iota_{i,j',k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E} \leftarrow \iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

is obviously an isomorphism. Therefore the statement follows because base change holds and homotopy colimits commute with pull-back and push-forward by assumption.
To see the second equation, consider the diagram:

$$
\begin{tikzcd}
\mathcal{E} & \pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l] \arrow[r] & \pi_{345;!}f^\bullet\pi_{125}^*\mathcal{E} & \pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\mathcal{E} \arrow[l] \\
\pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[u] \arrow[d] & \pi_{345;!}\pi_{145}^*\pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \pi_{345;!}\pi_{145}^*\pi_{345;!}f^\bullet\pi_{125}^*\mathcal{E} \arrow[u] \arrow[d] & \pi_{345;!}\pi_{145}^*\pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\mathcal{E} \arrow[l] \arrow[u] \arrow[d] \\
\pi_{345;!}f^\bullet\pi_{125}^*\mathcal{E} & \pi_{345;!}f^\bullet\pi_{125}^*\pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l] \arrow[r] & \pi_{345;!}f^\bullet\pi_{125}^*\pi_{345;!}f^\bullet\pi_{125}^*\mathcal{E} & \pi_{345;!}f^\bullet\pi_{125}^*\pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\mathcal{E} \arrow[l] \\
\pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\mathcal{E} \arrow[u] & \pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l] \arrow[r] \arrow[u] & \pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\pi_{345;!}f^\bullet\pi_{125}^*\mathcal{E} \arrow[u] & \pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\pi_{345;!}f^\bullet g_\bullet\pi_{123}^*\mathcal{E} \arrow[l] \arrow[u]
\end{tikzcd}
$$

We have to show that all squares commute. This is immediate for all squares except the (2,2) and (3,3)-squares on the diagonal.

Note that the following diagrams are Cartesian

$$
\begin{tikzcd}
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12345}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{123}"] & {}^{\downarrow\uparrow\downarrow}I
\end{tikzcd}
\quad
\begin{tikzcd}
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12347}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{125}"] & {}^{\downarrow\uparrow\downarrow}I
\end{tikzcd}
\quad
\begin{tikzcd}
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12367}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\
{}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{145}"] & {}^{\downarrow\uparrow\downarrow}I
\end{tikzcd}
$$

Consider also the following commutative diagram in which the square is Cartesian:

$$
\begin{tikzcd}
 & & \pi_{67}^*(S^{\mathrm{op}}) \arrow[d, "f_2"] \arrow[dd, bend left, "f_3"] \\
 & \pi_{45}^*(S^{\mathrm{op}}) \arrow[r, "g_4"] \arrow[d, "f_5"] & \pi_{47}^*(S^{\mathrm{op}}) \arrow[d, "f_1"] \\
\pi_{23}^*(S^{\mathrm{op}}) \arrow[r, "g_3"'] \arrow[rr, bend right, "g_2"'] & \pi_{25}^*(S^{\mathrm{op}}) \arrow[r, "g_5"'] & \pi_{27}^*(S^{\mathrm{op}})
\end{tikzcd}
$$

The (2,2)-square is identified with

$$
\begin{tikzcd}
\pi_{567,!}\pi_{167}^*\mathcal{E} \arrow[r] \arrow[d] & \pi_{567,!}f_3^\bullet\pi_{127}^*\mathcal{E} \arrow[d] \\
\pi_{345,!}f^\bullet\pi_{34567,!}\pi_{147}^*\mathcal{E} \arrow[r] & \pi_{345,!}f^\bullet\pi_{34567,!}f_2^\bullet\pi_{127}^*\mathcal{E}
\end{tikzcd}
$$

and, using the commutation of pull-back with left Kan extensions, also with

$$
\begin{tikzcd}
\pi_{567;!}\pi_{167}^*\mathcal{E} \arrow[r] \arrow[d] & \pi_{567;!}f_3^\bullet\pi_{127}^*\mathcal{E} \arrow[d] \\
\pi_{567;!}f_2^\bullet\pi_{147}^*\mathcal{E} \arrow[r] & \pi_{567;!}f_2^\bullet f_1^\bullet\pi_{127}^*\mathcal{E}
\end{tikzcd}
$$

This last diagram is clearly commutative.
The (3,3)-square is identified with

$$
\begin{tikzcd}
\pi_{345,!}f^\bullet\pi_{34567,!}f_1^\bullet\pi_{127}^*\mathcal{E} \arrow[r] \arrow[d] & \pi_{345,!}f^\bullet\pi_{34567,!}f_1^\bullet g_{2,\bullet}\pi_{123}^*\mathcal{E} \arrow[d] \\
\pi_{345,!}f^\bullet g_\bullet\pi_{34567,!}f_5^\bullet\pi_{125}^*\mathcal{E} \arrow[r] & \pi_{345,!}f^\bullet g_\bullet\pi_{34567,!}f_5^\bullet g_{3,\bullet}\pi_{123}^*\mathcal{E}
\end{tikzcd}
$$

and, using the commutation of pull-back with left Kan extensions, also with

$$
\begin{tikzcd}
\pi_{345,!}f_2^\bullet f_1^\bullet\pi_{127}^*\mathcal{E} \arrow[r] \arrow[d] & \pi_{345,!}f_2^\bullet f_1^\bullet g_{2,\bullet}\pi_{123}^*\mathcal{E} \arrow[d] \\
\pi_{345,!}f_2^\bullet g_{4,\bullet}f_5^\bullet\pi_{125}^*\mathcal{E} \arrow[r] & \pi_{345,!}f_2^\bullet g_{4,\bullet}f_5^\bullet g_{3,\bullet}\pi_{123}^*\mathcal{E}
\end{tikzcd}
$$

and using base change

$$\begin{tikzcd}
\pi_{345,!}f_2^\bullet f_1^\bullet \pi_{127}^*\mathcal{E} \arrow[r] \arrow[d] & \pi_{345,!}f_2^\bullet f_1^\bullet g_{2,\bullet}\pi_{123}^*\mathcal{E} \arrow[d] \\
\pi_{345,!}f_2^\bullet f_1^\bullet g_{5,\bullet}\pi_{125}^*\mathcal{E} \arrow[r] & \pi_{345,!}f_2^\bullet f_1^\bullet g_{5,\bullet}g_{3,\bullet}\pi_{123}^*\mathcal{E}
\end{tikzcd}$$

This diagram is clearly commutative.
For the additional statement observe that $\pi_{345,!}\mathcal{E}$ at an arrow $i \to i \to i$ is the homotopy colimit over the diagram $\iota^*\mathcal{E}$ for $\iota : {}^{\downarrow\uparrow}(I \times_{/I} i) \hookrightarrow {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I$ pulled back to $S(i)$. The projection $\mathrm{pr}_1 : {}^{\downarrow\uparrow}(I \times_{/I} i) \to (I \times_{/I} i)$ is an opfibration with fibers of the form $\beta \times_{/(I \times_{/I} i)} (I \times_{/I} i)$. These categories have an initial object and the restriction of the diagram $\pi_{123}^*\mathcal{E}$ is constant on it, because of the assumption that $\mathcal{E}$ is π_{13}-Cartesian already. Hence the homotopy colimit over the restriction of $\pi_{123}^*\mathcal{E}$ to these fibers is the corresponding constant value by Lemma 9.2.21. The colimit over $(I \times_{/I} i)$, furthermore, is evaluation at id_i because it is a final object. In total, the natural morphism

$$({}^{\downarrow\uparrow\downarrow}i)^* \,\Box_*\, \mathcal{E} \to ({}^{\downarrow\uparrow\downarrow}i)^*\mathcal{E}$$

is an isomophism. □

9.4 The (co)localization property of a derivator six-functor-formalism and n-angels

9.4.1. Let $\mathcal{S}$ be a category and $\mathcal{S}_0$ a class of "proper" morphisms. Let

$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}} \quad \text{resp.} \quad \mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

be a proper derivator six-functor-formalism (cf. Definition 9.1.1) with stable fibers (cf. Definition **??**). The multi-aspect will not play any role in this section. The reasoning in this section has an "etale" analogue that we leave to the reader to state.

9.4.2. If $\mathcal{S}$ is a category of some kind of spaces, we are often given a class of elementary squares as follows. Assume that in $\mathcal{S}$ there are certain distinguished morphisms called "closed immersions" or "open immersions" respectively, with an operation of taking complements. For a morphism $f : X \to Y$ in $\mathcal{S}$ we denote by f, resp. f^{op} the correspondences

$$f: \begin{tikzcd} & S \arrow[dl, equal] \arrow[dr, "f"] & \\ S & & T \end{tikzcd} \qquad f^{\mathrm{op}}: \begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, equal] & \\ T & & S \end{tikzcd}$$

in $\mathcal{S}^{\mathrm{cor}}$. Let

$$U \overset{i}{\hookrightarrow} V \overset{j}{\hookrightarrow} X$$

be a sequence of “open embeddings”. And let $\bar{i} : V \smallsetminus U \hookrightarrow V$, resp. $\overline{j \circ i} : X \smallsetminus U \hookrightarrow X$ be “closed embeddings of the complements”. For now these morphisms can be arbitrary, but to make sense of these definitions in applications they should satisfy the properties of 9.4.3 below.

We then have the following square in $\Xi_{U,V,X} \in \mathbb{S}^{\mathrm{cor}}(\square)$:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "\bar{i}^{\mathrm{op}}"] & X \arrow[d, "\overline{j \circ i}^{\mathrm{op}}"] \\ V \smallsetminus U \arrow[r, "j"] & X \smallsetminus U \end{tikzcd}$$

Assume that the “closed embeddings” lie in the class $\mathcal{S}_0$ which was fixed to define the notion of proper derivator six-functor-formalism. Then the above square comes equipped with a morphism $\xi : \Xi_{U,V,X} \to p^* X$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}(\square)$ represented by the cube (as a morphism from the front face to the back face):

$$\begin{tikzcd} & X \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[dd, equal] \\ V \arrow[ur, "j"] \arrow[rr, "j"] \arrow[dd, "\bar{i}^{\mathrm{op}}"] & & X \arrow[ur, equal] \arrow[dd, "\overline{j \circ i}^{\mathrm{op}}"] & \\ & X \arrow[rr, equal] & & X \\ V \smallsetminus U \arrow[ur, "j \circ \bar{i}"] \arrow[rr, "j"] & & X \smallsetminus U \arrow[ur, "\overline{j \circ i}"] & \end{tikzcd}$$

The top and bottom squares are 2-commutative, whereas the left and right squares are only oplax 2-commutative, e.g. there is a 2-morphism making the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{j \circ i}^{\mathrm{op}}"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & X \smallsetminus U \arrow[d, "\overline{j \circ i}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutative, which is given by the morphism of correspondences

$$\begin{tikzcd} & X \smallsetminus U \arrow[dl, "j \circ i"'] \arrow[dr, "j \circ i"] \arrow[dd, "j \circ i"] & \\ X \arrow[dr, equal] & & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

From now on, we forget about the provenance of these squares and just consider a proper derivator six-functor-formalism (more precisely, its oplax left fibered derivator)

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

with a class of distinguised squares $\Xi \in \mathbb{S}^{\mathrm{cor}}(\square)$ with given morphisms $\xi : \Xi \to p^*X$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}(\square)$.

Definition 9.4.3. *Let $\mathbb{S}$ be a 2-pre-derivator with all 2-morphisms invertible. We call a square $\Xi \in \mathbb{S}(\square)$* **Cartesian***, if the natural functor*

$$\mathrm{Hom}(X, (0,0)^*\Xi) \to \mathrm{Hom}(p^*X, i_{\lrcorner}^*\Xi)$$

is an equivalence of groupoids for all $X \in \mathbb{S}(\cdot)$, and **coCartesian** *if the natural functor*

$$\mathrm{Hom}((1,1)^*\Xi, X) \to \mathrm{Hom}(i_{\ulcorner}^*\Xi, p^*X)$$

is an equivalence of groupoids for all $X \in \mathbb{S}(\cdot)$. We call a square $\Xi \in \mathbb{S}(\square)$ **biCartesian** *if it is Cartesian and coCartesian.*

Remark 9.4.4. *If $\mathbb{S}$ is a usual derivator then this notion coincides with the usual notion [Gro13].*

9.4.5. One can show that the squares $\Xi_{U,V,X} \in \mathbb{S}^{\mathrm{cor}}(\square)$ constructed in the last paragraph are actually Cartesian in $\mathbb{S}^{\mathrm{cor}}$ provided that for all pairs $U, X \smallsetminus U$ of "open and closed embeddings" used above we have

$$\mathrm{Hom}_{\mathcal{S}}(A, U) = \{\alpha \in \mathrm{Hom}_{\mathcal{S}}(A, X) \mid A \times_{\alpha,X} (X \smallsetminus U) = \varnothing\}$$

and coCartesian provided that we have

$$\mathrm{Hom}_{\mathcal{S}}(A, X \smallsetminus U) = \{\alpha \in \mathrm{Hom}_{\mathcal{S}}(A, X) \mid A \times_{\alpha,X} U = \varnothing\}$$

where $\varnothing$ is the initial object.

9.4.6. There is a dual variant of the previous construction (not to be confused with the transition to an etale six-functor-formalism). We consider instead the square $\Xi'_{U,V,X}$ with morphism

$$\begin{tikzcd}
& X \arrow[rr, equal] \arrow[dl, "\overline{j\circ i}^{\mathrm{op}}"'] \arrow[dd, equal] & & X \arrow[dl, equal] \arrow[dd, equal] \\
X \smallsetminus U \arrow[rr, "\overline{j\circ i}"] \arrow[dd, "i^{\mathrm{op}}"'] & & X \arrow[dd, "j^{\mathrm{op}}"] & \\
& X \arrow[rr, equal] \arrow[dl, "(j\circ\bar{i})^{\mathrm{op}}"] & & X \arrow[dl, "j^{\mathrm{op}}"] \\
V \smallsetminus U \arrow[rr, "\bar{i}"'] & & V &
\end{tikzcd}$$

In this case the top and bottom squares are only *lax* 2-commutative, e.g. there is a 2-morphism making the diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\overline{j\circ i}^{\mathrm{op}}"'] \arrow[dr, phantom, "\Nearrow"] & X \arrow[d, equal] \\
X \smallsetminus U \arrow[r, "\overline{j\circ i}"] & X
\end{tikzcd}$$

2-commutative. This means that for a stable proper derivator six-functor-formalism it is also reasonable to consider a class of distinguished squares with given morphisms $\xi' : p^*X \to \Xi$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}(\square)$. The morphism $p^*X \to \Xi'_{U,V,X}$ is just the *dual* of the morphism $\Xi_{U,V,X} \to p^*X$ for the absolute duality on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{cor})$ (cf. 9.1.2).

Let $i_\ulcorner : \ulcorner \hookrightarrow \square$ and $i_\lrcorner : \lrcorner \hookrightarrow \square$ be the inclusions. Analogously to the situation for stable derivators [Gro13, 4.1] we define:

Definition 9.4.7. *A square $\mathcal{E} \in \mathbb{D}(\square)$ over $\Xi \in \mathbb{S}(\square)$ is called* **relatively coCartesian**, *if for the inclusion $i_\ulcorner : (\ulcorner, i_\ulcorner^*\Xi) \to (\square, \Xi)$ the unit $\mathcal{E} \to i_{\ulcorner,*} i_\ulcorner^* \mathcal{E}$ is an isomorphism, and it is called* **relatively Cartesian** *if for the inclusion $i_\lrcorner : (\lrcorner, i_\lrcorner^*\Xi) \to (\square, \Xi)$ the counit $i_{\lrcorner,!} i_\lrcorner^* \mathcal{E} \to \mathcal{E}$ is an isomorphism*[30]*. $\mathcal{E}$ is called* **relatively biCartesian** *if it is relatively Cartesian and relatively coCartesian.*

If Ξ is itself (co)Cartesian in the sense of Definition 9.4.3 then $\mathcal{E}$ relatively (co)Cartesian implies (co)Cartesian in the sense of Definition 9.4.3.

Lemma 9.4.8. *Assume $V = U$ and let $\mathbb{D}(\square)^{\mathrm{bicart}}_{\Xi_{U,U,X}}$ be the full subcategory of relatively biCartesian squares. Let $(1,0) : (\cdot, X) \to (\square, \Xi_{U,U,X})$ be the inclusion. Then the functor*

$$(1,0)^* : \mathbb{D}(\square)^{\mathrm{bicart}}_{\Xi_{U,U,X}} \to \mathbb{D}(\cdot)_X$$

and the composition

$$\mathbb{D}(\cdot)_X \xrightarrow{1_*} \mathbb{D}(\to)_{U \to X} \xrightarrow{0_!} \mathbb{D}(\square)^{\mathrm{bicart}}_{\Xi_{U,U,X}}$$

define an equivalence of categories.
Also if $V \neq U$ the functor $1_* 0_!$ takes values in relatively biCartesian squares.

Recall that the functor

$$0^* : \mathbb{D}(\square)^{\mathrm{bicart},0}_{p^*X} \to \mathbb{D}(\to)_{p^*X}$$

is an equivalence, where $\mathbb{D}(\square)^{\mathrm{bicart},0}_{p^*X}$ is the full subcategory of (relatively) biCartesian objects whose $(1,0)$-entry is zero. (This is a statement about usual derivators.)

Definition 9.4.9. *We say that a distinguished square Ξ together with $\xi : \Xi \to p^*X$ is a* **localizing square** *if the push-forward $\xi_\bullet$ maps relatively biCartesian squares to relatively biCartesian squares. We say that a distinguished square Ξ together with $\xi : p^*X \to \Xi$ is a* **colocalizing square** *if the pull-back $\xi^\bullet$ maps relatively biCartesian squares to relatively biCartesian squares.*

If every object in $\mathbb{D}(\cdot)$ is dualizable w.r.t. the absolute monoidal product in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ then $\xi : \Xi \to p^*X$ is localizing if and only if $\xi^\vee : p^*X \to \Xi^\vee$ is colocalizing.

[30] The functors $i_{\lrcorner,!}$ and $i_{\ulcorner,*}$ are in both cases considered w.r.t. the base Ξ.

Remark 9.4.10. *If the proper derivator six-functor-formalism with its oplax extension*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

has stable *fibers, and the square $\Xi_{U,V,X}$ constructed above is distinguished, then the property of being a localizing square implies that for $\mathcal{E} \in \mathbb{D}(\cdot)_X$ the triangle*

$$j_! j^! \mathcal{E} \longrightarrow (j \circ \bar{i})_! \bar{i}^* j^! \mathcal{E} \oplus \mathcal{E} \longrightarrow \overline{j \circ i}_* \overline{j \circ i}^* \mathcal{E} \xrightarrow{[1]}$$

is distinguished. If $U = V$ this is just the sequence

$$j_! j^! \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \bar{j}_* \bar{j}^* \mathcal{E} \xrightarrow{[1]}$$

Remark 9.4.11. *If the proper derivator six-functor-formalism with its lax extension*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

has stable *fibers, and the square $\Xi_{U,V,X}$ constructed above is distinguished, then the property of being a colocalizing square implies that for $\mathcal{E} \in \mathbb{D}(\cdot)_X$ the triangle*

$$\overline{j \circ i}_! \overline{j \circ i}^! \mathcal{E} \longrightarrow (j \circ \bar{i})_* i^* (\overline{j \circ i})^! \mathcal{E} \oplus \mathcal{E} \longrightarrow j_* j^* \mathcal{E} \xrightarrow{[1]}$$

is distinguished. If $U = V$ this is just the sequence:

$$\bar{j}_! \bar{j}^! \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow j_* j^* \mathcal{E} \xrightarrow{[1]}$$

Definition 9.4.12. *We say that the proper derivator six-functor-formalism with its extension as oplax left fibered derivator*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

satisfies **the localization property** *w.r.t. a class of distinguished squares $\xi : \Xi \to p^* X$ if these are localizing squares.*
We say that the proper derivator six-functor-formalism with its extension as lax right fibered derivator

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

satisfies **the colocalization property** *w.r.t. a class of distinguished squares $\xi : p^* X \to \Xi$ if these are colocalizing squares.*

There is an analogous notion in which an *etale* derivator-six-functor-formalism w.r.t. a class of "etale morphisms" $\mathcal{S}_0$ in $\mathcal{S}$ satisfies the (co)localization property

9.4.13. Consider again the situation in 9.4.2. More generally we may consider a sequence

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$$

of open embeddings. They lead to a diagram Ξ

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[d] \\
\varnothing \arrow[r] & X_2 \smallsetminus X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \smallsetminus X_1 \arrow[d] \\
& \varnothing \arrow[r] & \ddots \arrow[d] & \vdots \arrow[d] \\
& & \varnothing \arrow[r] & X_n \smallsetminus X_{n-1}
\end{tikzcd}$$

in which all squares are biCartesian in $\mathbb{S}^{\mathrm{cor}}$. Starting from an object $\mathcal{E} \in \mathbb{D}(\cdot)_{X_n}$ we may form again

$$0_*(n)_!\mathcal{E}$$

where $(n) : \cdot \to [n]$ is the inclusion of the last object and $0 : [n] \to \Xi$ is the inclusion of the first line. It is easy to see that in the object $0_*(n)_!\mathcal{E}$ all squares are biCartesian. There is furthermore again a morphism $\xi : \Xi \to p^* X_n$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$ such that all squares in $\xi_\bullet 0_*(n)_!\mathcal{E}$ are biCartesian with zero's along the diagonal. This category is equivalent to $\mathbb{D}([n])_{p^*X_n}$ by the embedding of the first line. It can be seen as a category of n-angels in the stable derivator $\mathbb{D}_{X_n}$ (the fiber of $\mathbb{D}$ over X).

Hence for an oplax derivator six-functor-formalism with localization property, and for any filtration of a space X by n open subspaces, and for any object $\mathcal{E} \in \mathbb{D}(\cdot)_X$ we get a corresponding $(n+1)$-angle in the derivator $\mathbb{D}_X$ in the sense of [GS14, §13].

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