DISJOINT STATIONARY SEQUENCES ON AN INTERVAL OF CARDINALS

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Abstract. We introduce strong distributivity, a strengthening of distributivity, which implies preservation of ccness and stationarity, afterwards showing a stronger version of the Easton Lemma. We also introduce a new framework for working with arbitrary orders on products of sets. Both concepts are applied together to answer two questions of Krueger using a new version of Mitchell's Forcing.

1. Introduction

Disjoint stationary sequences were first introduced by Krueger to answer a question posed by Abraham and Shelah about forcing clubs through stationary sets. Later on, this concept was connected to the seemingly unrelated area of internal unboundedness, stationarity, clubness and approachability. Most of this was accomplished by Krueger using the notion of a mixed support iteration, which is very similar to the approach Mitchell first used to show the relative consistency of the tree property at $\aleph_2$.

In his paper introducing disjoint stationary sequences, see [5], Krueger asked if it is consistent that DSS holds simultaneously at two successive cardinals or even an interval of cardinals. Levine partially answered this in [7] by using a modified version of Mitchell forcing to construct a model where $\mathrm{DSS}(\aleph_2) \wedge \mathrm{DSS}(\aleph_3)$ holds. Using another different version of Mitchell forcing, we will answer the rest of the question by producing a model where $\mathrm{DSS}(\aleph_n)$ holds for any $n \in \omega$, $n \geq 2$:

Theorem 1.1. *Assume $(\kappa_n)_{n\in\omega}$ is an increasing sequence of Mahlo cardinals. There exists a forcing extension in which, for every $n \in \omega$, $\kappa_n = \aleph_{n+2}$ and there exists a disjoint stationary sequence on κ_n:*

In the same paper, Levine noticed that in the model constructed to have disjoint stationary sequences on both $\aleph_2$ and $\aleph_3$, the notions of internal stationarity and clubness are distinct for $[\aleph_2]^{<\aleph_2}$ and $[\aleph_3]^{<\aleph_3}$, partially answering another question of Krueger from [5]. The same is true in our case, i.e. for any $n \in \omega$, $n \geq 2$, the notions of internal stationarity and clubness are distinct for $[\aleph_n]^{<\aleph_n}$. This answers the rest of the question posed by Krueger.

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Theorem 1.2. *Assume $(\kappa_n)_{n\in\omega}$ is an increasing sequence of Mahlo cardinals. There exists a forcing extension in which, for every $n\in\omega$, $\kappa_n=\aleph_{n+2}$ and there are stationarily many $Z\in[\kappa_n]^{<\kappa_n}$ which are internally stationary but not internally club.*

The paper is organised as follows: After reviewing a bit of background information, we introduce a new kind of closure for forcing notions, called *strong $<\kappa$-distributivity*, which lies in strength between $<\kappa$-strategic closure and $<\kappa$-distributivity (and has much better preservation properties than $<\kappa$-distributivity) We will also prove a stronger version of Easton's Lemma. We then introduce a general way of working with arbitrary orders on products of sets. In section 5, we apply the previous concepts to prove our main theorems.

2. Preliminaries

We assume the reader is familiar with the basics of forcing. Good introductory material can be found in the books by Jech (cf. [4]) and Kunen (cf. [6]).

Our notation is fairly standard. To reduce confusion, we write $<\tau$-closed (strategically closed; (strongly) distributive). $p\leq q$ means p is stronger than q (forces more). $\mathbb{P}\restriction p$ is defined as $\{q\in\mathbb{P}\mid q\leq p\}$. $V[\mathbb{P}]$ denotes an arbitrary extension by $\mathbb{P}$, i.e. "$V[\mathbb{P}]$ has property P" means that for every $\mathbb{P}$-generic G, $V[G]$ has property P.

Definition 2.1. Let κ be a cardinal. A pair (V,W) of models of set theory with the same ordinals has the *$<\kappa$-covering property* if for any $x\in W$ of size $<\kappa$ there is $y\in V$ of size $<\kappa$ such that $x\subseteq y$. A forcing order $\mathbb{P}$ has the κ-covering property if $1_{\mathbb{P}}$ forces that $(V,V[\mathbb{P}])$ has it.

If $\mathbb{P}$ is κ-cc. and $\dot{f}$ any $\mathbb{P}$-name for a function from an ordinal into V, then there are $<\kappa$ possibilities for any value $\dot{f}(\check{\alpha})$, so we obtain:

Fact 2.2. *If $\mathbb{P}$ is κ-cc., $\mathbb{P}$ has the κ-covering property.*

When arguing the preservation of properties which are downwards absolute, we will frequently make use of projections, which are a way of showing that an extension by some order $\mathbb{Q}$ is contained in an extension by another order $\mathbb{P}$.

Definition 2.3. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders. A function $\pi:\mathbb{P}\longrightarrow\mathbb{Q}$ is a *projection* if the following hold:

(1) $\pi(1_{\mathbb{P}})=1_{\mathbb{Q}}$.
(2) For all $p\leq q$, $\pi(p)\leq\pi(q)$
(3) For all $p\in\mathbb{P}$, if $q\leq\pi(p)$, there is some $p'\leq p$ such that $\pi(p')\leq q$.

A projection π is trivial if for all $p,p'\in\mathbb{P}$, if $\pi(p)=\pi(p')$, p and p' are compatible.

If there exists a projection from $\mathbb{P}$ to $\mathbb{Q}$, any extension by $\mathbb{Q}$ can be forcing extended to an extension by $\mathbb{P}$.

Definition 2.4. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders, $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ a projection. Let H be $\mathbb{Q}$-generic. In $V[H]$, the forcing order $\mathbb{P}/H$ consists of all $p \in \mathbb{P}$ such that $\pi(p) \in H$. We let $\mathbb{P}/\mathbb{Q}$ be a $\mathbb{Q}$-name for $\mathbb{P}/\dot{H}$ and call $\mathbb{P}/\mathbb{Q}$ the *quotient forcing* of $\mathbb{P}$ and $\mathbb{Q}$.

Fact 2.5. *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders and $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ a projection. If H is $\mathbb{Q}$-generic over V and G is $\mathbb{P}/H$-generic over $V[H]$, then G is $\mathbb{P}$-generic over V and $H \subseteq \pi[G]$. In particular, $V[H][G] = V[G]$.*

One checks that if $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is trivial, then $\mathbb{P}/\mathbb{Q}$ is forced to be centered and thus:

Fact 2.6. *If there exists a trivial projection $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$, $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent.*

For completeness, we state the Product Lemma, which states that forcing with a product of orders can be viewed as successive forcing.

Lemma 2.7 (Product Lemma)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of forcing. For $G \subseteq \mathbb{P}$, $H \subseteq \mathbb{Q}$, the following are equivalent:*

(1) *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V.*
(2) *G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$.*
(3) *H is $\mathbb{Q}$-generic over V and G is $\mathbb{P}$-generic over $V[H]$.*

3. Strongly Distributive Forcings

$<\kappa$-closed forcings have some nice regularity properties which do not hold for forcings which are merely $<\kappa$-distributive (e.g. preserving stationary subsets of κ). In this section, we will introduce a strengthening of $<\kappa$-distributivity, which $<\kappa$-closure turns into after forcing with κ-cc. forcing and show that it can replace $<\kappa$-closure in some important applications.

Definition 3.1. A notion of forcing $\mathbb{P}$ is *strongly $<\kappa$-distributive* if for any sequence $(D_\alpha)_{\alpha<\kappa}$ of open dense sets and any $p \in \mathbb{P}$, there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_0 \leq p$ and $\forall \alpha < \kappa$, $p_\alpha \in D_\alpha$. Such a sequence will be called a *thread* through $(D_\alpha)_{\alpha<\kappa}$.

Strong $<\kappa$-distributivity can be thought of as having $<\kappa$-distributivity witnessed in a uniform way: If $(D_\alpha)_{\alpha<\kappa}$ is a sequence of open dense subsets of some $<\kappa$-distributive forcing notion, there is a sequence $(p_\alpha)_{\alpha<\kappa}$ such that for all $\alpha < \kappa$, $p_\alpha \leq p_0$ and $p_\alpha \in \bigcap_{\beta<\alpha} D_\beta$ (since the intersection of $<\kappa$ open dense sets is open dense). However, we cannot in general find such a sequence in a uniform way, i.e. such that it is descending.

Obviously strong $<\kappa$-distributivity implies $<\kappa$-distributivity. Note that strong $<\kappa$-distributivity and $<\kappa$-distributivity are not equivalent: If $(T, \leq)$ is a Suslin tree, $(T, \geq)$ is $<\omega_1$-distributive (cf. Lemma 15.28 in [4]) but not strongly $<\omega_1$-distributive (let D_α

consists of all nodes of height α so that a thread through all D_α is a cofinal branch in T). There are also ZFC examples: If $S \subseteq \omega_1$ is stationary and costationary, the forcing shooting a club through S by countable approximations is $<\omega_1$-distributive (cf. Lemma 23.9 in [4]). However, as we will later show, it cannot be strongly $<\omega_1$-distributive as it destroys a stationary subset of ω_1.

Keeping with the theme of strong $<\kappa$-distributivity being a uniform version of $<\kappa$-distributivity, we have the following characterisations of strong $<\kappa$-distributivity: Recall that for antichains A, B we say that A refines B if for every $q \in A$ there is $q' \in B$ with $q \leq q'$.

Lemma 3.2. *For a forcing order $\mathbb{P}$, the following are equivalent:*

(1) *$\mathbb{P}$ is strongly $<\kappa$-distributive.*
(2) *$\mathbb{P}$ is $<\kappa$-distributive and for $p \in \mathbb{P}$ and any descending sequence $(A_\alpha)_{\alpha<\kappa}$ (with regards to refinement) of maximal antichains below p, there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_0 \leq p$ and for any α, $p_\alpha \in A_\alpha$.*

Proof. Assume $\mathbb{P}$ is strongly $<\kappa$-distributive. Of course, this implies that $\mathbb{P}$ is $<\kappa$-distributive. Let $(A_\alpha)_{\alpha<\kappa}$ be a sequence of maximal antichains in $\mathbb{P}$ such that for $\beta < \alpha$, A_α refines A_β. For $\alpha < \kappa$, let D_α be the downward closure of A_α and consider a thread $(q_\alpha)_{\alpha<\kappa}$ through $(D_\alpha)_{\alpha<\kappa}$. For any $\alpha < \kappa$, let p_α be the unique (by pairwise incompatibility) element of A_α that is above q_α. We are done after showing

Claim. *The sequence $(p_\alpha)_{\alpha<\kappa}$ is descending.*

Proof. Let $\beta < \alpha$ be arbitrary. Because A_α refines A_β, there exists p'_β such that $p_\alpha \leq p'_\beta$. Thus, $q_\alpha \leq p_\alpha \leq p'_\beta$ and $q_\alpha \leq q_\beta \leq p_\beta$. In summary, p'_β and p_β are compatible and therefore equal. □

Now assume condition (2) holds. Let $(D_\alpha)_{\alpha<\kappa}$ be a sequence of open dense subsets of $\mathbb{P}$. Inductively, and using $<\kappa$-distributivity, construct a sequence $(A_\alpha)_{\alpha<\kappa}$ such that $A_\alpha \subseteq D_\alpha$ is a maximal antichain and for $\beta < \alpha$, A_α refines A_β. It follows that a thread through $(A_\alpha)_{\alpha<\kappa}$ is also one through $(D_\alpha)_{\alpha<\kappa}$. □

While $<\kappa$-distributivity means that every $<\kappa$-sequence of ground-model elements is in the ground model, strong $<\kappa$-distributivity means that we can uniformly approximate κ-sequences of ground-model elements.

Lemma 3.3. *If $\mathbb{P}$ is strongly $<\kappa$-distributive, $p \in \mathbb{P}$ and $\dot{f}$ is a $\mathbb{P}$-name such that $p \Vdash \dot{f} : \check{\kappa} \longrightarrow V$, there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ with $p_0 \leq p$ such that for every $\alpha < \kappa$, p_α decides $\dot{f}(\check{\alpha})$.*

Proof. Consider $D_\alpha := \{q \in \mathbb{P} \mid q \text{ decides } \dot{f}(\check{\alpha})\}$. □

As is the case for $<\kappa$-distributivity, the converse holds for separative forcing orders (but we will never use this).

We even obtain new version of Foreman's Theorem from [1], relating strong $<\kappa$-distributivity to the non-existence of a winning strategy for INC in the completeness game.

Definition 3.4. Let $\mathbb{P}$ be a forcing order, δ an ordinal. The *completeness game* $G(\mathbb{P},\delta)$ on $\mathbb{P}$ with length δ has players COM (complete) and INC (incomplete) playing elements of $\mathbb{P}$ with COM playing at even ordinals (and limits) and INC playing at odd ordinals. COM starts by playing $1_{\mathbb{P}}$, afterwards p_α has to be a lower bound of $(p_\beta)_{\beta<\alpha}$. INC wins if COM is unable to play at some point $<\delta$. Otherwise, COM wins.

The following theorem was shown by Foreman in [1]. Note that this theorem can also work for regular limit cardinals: It implies that, for κ a regular limit, $\mathbb{P}$ is $<\kappa$-distributive if and only for all ordinals $\mu<\kappa$, INC does not have a winning strategy in $G(\mathbb{P},\mu)$: If $\mathbb{P}$ is not $<\kappa$-distributive, it is not $<\mu$-distributive for some $\mu<\kappa$ and this implies that INC has a winning strategy in $G(\mathbb{P},\mu)$. If INC has a winning strategy in some $G(\mathbb{P},\mu)$, he has one in $G(\mathbb{P},\mu^++1)$, so $\mathbb{P}$ is not $<\mu^{++}$-distributive and not $<\kappa$-distributive.

Theorem 3.5 (Foreman)**.** *If $\kappa=\lambda^+$ is a successor, $\mathbb{P}$ is $<\kappa$-distributive if and only if INC does not have a winning strategy in $G(\mathbb{P},\lambda+1)$.*

if INC does not have a winning strategy in $G(\mathbb{P},\lambda+1)$, he does not have one in any $G(\mathbb{P},\mu)$ for $\mu<\lambda^+$. Having this witnessed uniformly suggests the following statement:

Theorem 3.6. *$\mathbb{P}$ is strongly $<\kappa$-distributive if and only if INC does not have a winning strategy in $G(\mathbb{P},\kappa)$.*

Proof. For one direction, if $(D_\alpha)_{\alpha<\kappa}$ is a sequence of open dense sets without a thread below some $p\in\mathbb{P}$, let INC first play p and then, at each stage $\alpha=\gamma+2n$, an element of $D_{\gamma+n}$.

For the other direction, let σ be a winning strategy for INC in $G(\mathbb{P},\kappa)$. Let $\sigma(1_{\mathbb{P}})=p$. We will construct a sequence $(A_\alpha)_{\alpha\in\kappa}$ such that the following holds:

(1) For each $\alpha\in\kappa$ and $p_\alpha\in A_\alpha$, there exists a unique sequence $(p_\beta)_{\beta<\alpha}$ such that for all $\beta\leq\alpha$, $p_\beta\in A_\beta$ and if β is odd, $p_\beta=\sigma((p_\delta)_{\delta<\beta})$.
(2) If $\alpha\in\kappa$ is odd, A_α is a maximal antichain below p (for even α, we carefully choose A_α to obtain uniqueness in (1)).

To begin, let $A_0:=\{1_{\mathbb{P}}\}$ and $A_1:=\{p\}$. Assume the sequence has been constructed until some even successor ordinal γ. We will construct A_γ and $A_{\gamma+1}$ simultaneously. Let $D_{\gamma+1}$ consist of all $p\in\mathbb{P}$ such that there exists a sequence $(p_\alpha)_{\alpha<\gamma+1}$ such that for all $\alpha<\gamma$, $p_\alpha\in A_\alpha$ and if α is odd, $p_\alpha=\sigma((p_\beta)_{\beta<\alpha})$.

Claim. *$D_{\gamma+1}$ is dense below p.*

Proof. Let $p' \leq p$ be arbitrary. By maximality of $A_{\gamma-1}$, there exists $a_{\gamma-1} \in A_{\gamma-1}$ compatible with p', witnessed by some p''. By the inductive hypothesis, there is a sequence $\overline{p} = (p_\beta)_{\beta<\gamma-1}$ with $p_\beta \in A_\beta$ for $\beta < \gamma - 1$ and $p_\beta = \sigma((p_\delta)_{\delta<\beta})$ for all odd $\beta \leq \gamma - 1$. Hence, letting $s := \overline{p}^\frown a_{\gamma-1}{}^\frown p''$, $\sigma(s)$ witnesses density. □

Let $A_{\gamma+1} \subseteq D_{\gamma+1}$ be a maximal antichain. For each $a_{\gamma+1} \in A_{\gamma+1}$, by the definition of $D_{\gamma+1}$, there exists a sequence $(p_\alpha)_{\alpha<\gamma+1}$ such that for all $\alpha < \gamma$, $p_\alpha \in A_\alpha$ and if α is odd, $p_\alpha = \sigma((p_\beta)_{\beta<\alpha})$. Choose such a sequence for each $a_{\gamma+1} \in A_{\gamma+1}$ and let A_γ consist of the γth entries of these sequences.

Claim. *For each $p_{\gamma+1} \in A_{\gamma+1}$, there exists a unique sequence $(p_\beta)_{\beta<\gamma+1}$ such that for all $\beta < \alpha$, $p_\beta \in A_\beta$ and if $\beta \leq \gamma + 1$ is odd, $p_\beta = \sigma((p_\delta)_{\delta<\beta})$*

Proof. Let $a_{\gamma+1} \in A_{\gamma+1}$. The existence of such a sequence s follows from the definition of $D_{\gamma+1}$. Let $s' = (p'_\beta)_{\beta<\gamma+1}$ be another sequence such that for all $\beta < \gamma$, $p_\beta \in A_\beta$ and for odd $\beta \leq \gamma + 1$, $p_\beta = \sigma((p_\delta)_{\delta<\beta})$. By construction, this sequence is descending, so p_β and p'_β are compatible (witnessed by $a_{\gamma+1}$), which implies for odd β that they are equal. However, this also means that for each odd $\beta < \gamma$, $s' \upharpoonright \beta$ and $s \upharpoonright \beta$ are equal by the inductive hypothesis.

So if $s' \neq s$, then $p'_\beta = p_\beta$ for all $\beta < \gamma$ and $p'_\gamma \neq p_\gamma$. Since $p'_\gamma \in A_\gamma$, there exists $a_{\gamma+1} \neq a'_{\gamma+1} \in A_{\gamma+1}$ and a sequence s'' such that $s''{}^\frown p'_\gamma$ witnesses $a'_{\gamma+1} \in A_{\gamma+1}$. Because σ is a function, $s'' \neq s' \upharpoonright \gamma$ and by the inductive hypothesis, $s''(\gamma-1) \neq s'(\gamma-1)$. However, p'_γ witnesses their compatibility, which is a contradiction. □

Assume γ is a limit. Let A'_γ be a common refinement of A_α for odd $\alpha < \gamma$. Given $p \in A'_\gamma$, let $p_\alpha \in A_\alpha$ witness refinement for odd α and let $p_\alpha \in A_\alpha$ witness $p_{\alpha+1} \in A_{\alpha+1}$ for even α. Then $(p_\alpha)_{\alpha<\gamma}$ is a play according to σ by uniqueness (which implies that the sequences witnessing $p_\alpha \in A_\alpha$ are coherent). Let D_γ be the downward closure of A'_γ and let $D_{\gamma+1}$ consist of $\sigma(s)$ for sequences $s = (p_\alpha)_{\alpha<\gamma+1}$ with $p_\gamma \in D_\gamma$ and $s \upharpoonright \gamma$ witnessing this. Thus, D_γ is dense and we can proceed as in the previous step.

Lastly, there exists a thread through $(A_\alpha)_{\alpha\in\kappa\cap\mathrm{Odd}}$, i.e. a sequence $(p_\alpha)_{\alpha\in\kappa\cap\mathrm{Odd}}$ such that for odd α, $p_\alpha \in A_\alpha$. For even α, let $p_\alpha \in A_\alpha$ witness $p_{\alpha+1} \in A_{\alpha+1}$. By uniqueness, $(p_\alpha)_{\alpha<\kappa}$ is a play in $G(\mathbb{P}, \kappa)$ according to σ. But this contradicts our assumption that σ was a winning strategy. □

If COM even has a winning strategy in $G(\mathbb{P}, \delta)$, we say that $\mathbb{P}$ is *weakly $<\delta$-strategically closed*.

The main point for introducing strong $<\kappa$-distributivity is the following strengthening of the Easton Lemma, showing that in many cases where we previously only had $<\kappa$-distributivity, we actually have strong $<\kappa$-distributivity. The second statement in the Lemma was also noticed (in a different form) by Andreas Lietz on Mathoverflow after a question by the author (see [8]).

Lemma 3.7. *Let κ be a regular cardinal. Assume $\mathbb{P}$ is κ-cc. and $\mathbb{Q}$ is strongly $<\kappa$-distributive.*

(1) $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ *is $\check{\kappa}$-cc.*
(2) $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ *is strongly $<\check{\kappa}$-distributive.*

Proof. (1) Assume $\dot{f}$ is forced by some q to be an enumeration of an antichain in $\mathbb{P}$ of size κ. Thus, $q \Vdash \dot{f} : \check{\kappa} \longrightarrow V$. Hence, there exists a descending sequence $(q_\alpha)_{\alpha<\kappa}$ with $q_0 \leq q$ such that $q_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha$ for some p_α.

Claim. *$\{p_\alpha \mid \alpha < \kappa\}$ is an antichain in $\mathbb{P}$.*

Proof. Let $\beta < \alpha$ be arbitrary. Thus $q_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha \wedge \dot{f}(\check{\beta}) = \check{p}_\beta$ because $q_\alpha \leq q_0$, $q_\alpha \Vdash \check{p}_\alpha \perp \check{p}_\beta$ and thus $p_\alpha \perp p_\beta$. □

This claim directly contradicts our assumption.

(2) We first show a helpful claim

Claim. *If $D \subseteq \mathbb{P}\times\mathbb{Q}$ is open dense and $p \in \mathbb{P}$, the set D_p consisting of all $q \in \mathbb{Q}$ such that for some $A \subseteq \mathbb{P}$ that is a maximal antichain below p, $A \times \{q\} \subseteq D$, is open dense in $\mathbb{Q}$.*

Proof. Openness is clear: If A witnesses $q \in D_p$ and $q' \leq q$, A also witnesses $q' \in D_p$.

Thus, assume the set is not dense and there is $q \in \mathbb{Q}$ such that for every $q' \leq q$, $q' \notin D_p$. We will give a winning strategy for INC in $G(\mathbb{P},\kappa)$. In every run of the game, we construct an antichain $\{p_\alpha \mid \alpha \in \gamma \cap \mathrm{Odd}\}$ below p such that $(p_\alpha, q_\alpha) \in D$. To begin, let INC find a pair $(p_1, q_1) \leq (p, q)$ with $(p_1, q_1) \in D$ and play q_1.

Assume the game has lasted until γ, $\gamma+1$ is Odd and COM has just played q_γ. If $\{p_\alpha \mid \alpha \in \gamma\cap\mathrm{Odd}\}$ is maximal, it witnesses $q_\gamma \in D$ by openness: For every $\alpha \in \gamma \cap \mathrm{Odd}$, $(p_\alpha, q_\alpha) \in D$ and thus $(p_\alpha, q_\gamma) \in D$. This contradicts our assumption, since $q_\gamma \leq q$. It follows that there exists some $p'_{\gamma+1}$ which is incompatible with every p_α. By open density, there exists $(p_{\gamma+1}, q_{\gamma+1}) \leq (p'_{\gamma+1}, q_\gamma)$, $(p_{\gamma+1}, q_{\gamma+1}) \in D$. Let INC play $q_{\gamma+1}$.

This strategy is a winning strategy, because a play of length κ would give us a κ-sized antichain in $\mathbb{P}$. This contradicts our assumptions. □

Now assume $\dot{f}$ and τ are $\mathbb{P}$-names such that $\dot{f}$ is forced by some p to map $\check{\kappa}$ to open dense subsets of $\mathbb{Q}$ and τ to be an elemnent of $\mathbb{Q}$. Strengthening p if necessary, we can assume $p \Vdash \tau = \check{q}$ for some $q \in \mathbb{Q}$.

Claim. *The set* $D_\alpha := \{(p',q') \in \mathbb{P} \restriction p \times \mathbb{Q} \mid p' \Vdash \check{q}' \in \dot{f}(\check{\alpha})\}$ *is open dense.*

Proof. Openness in both coordinates follows either from the properties of the forcing relation or from $\dot{f}(\check{\alpha})$ being forced by p to be open.

For density, let $(p',q') \in \mathbb{P} \restriction p \times \mathbb{Q}$ be arbitrary. Thus $p' \Vdash \exists \tau(\tau \in \dot{f}(\check{\alpha}) \wedge \tau \leq \check{q}')$. Because τ is in particular forced to be in V, there exists $p'' \leq p'$ and q'' such that

$$p'' \Vdash (\check{q}'' \in \dot{f}(\check{\alpha}) \wedge \check{q}'' \leq \check{q}')$$

Thus, $(p'',q'') \leq (p',q')$ and $(p'',q'') \in D_\alpha$ □

Combining the two claims, for each α, the set D'_α, consisting of all $q' \in \mathbb{Q}$ such that for some $A \subseteq \mathbb{P}$ such that A is a maximal antichain below p and $A \times \{q'\} \subseteq D_\alpha$, is open dense in $\mathbb{Q}$. If $q' \in D'_\alpha$, there exists a maximal antichain A below p such that for every $p' \in A$, $p' \Vdash \check{q}' \in \dot{f}(\check{\alpha})$. By maximality, $p \Vdash \check{q}' \in \dot{f}(\check{\alpha})$.

Let $(q_\alpha)_{\alpha<\kappa}$ be a thread through $(D'_\alpha)_{\alpha<\kappa}$ below q. Then p forces $(\check{q}_\alpha)_{\alpha<\kappa}$ to be a thread through $\dot{f}$ below $\check{q}$.

□

In particular, if $\mathbb{Q}$ is $<\kappa$-closed and $\mathbb{P}$ is κ-cc., $\mathbb{Q}$ is strongly $<\kappa$-distributive after forcing with $\mathbb{P}$.

As stated before, strong $<\kappa$-distributivity has much better preservation properties than $<\kappa$-distributivity: As shown in the previous Lemma, strongly $<\kappa$-distributive forcings preserve the κ-cc., in contrast with the fact that it is consistent that T^2 collapses ω_1, where T is a suslin tree (thus, in $V[T]$, T is neither ω_1-cc. not $<\omega_1$-distributive). Furthermore, strongly $<\kappa$-distributive forcing notions preserve stationary subsets of κ (in contrast with the fact that we can destroy stationary subsets of ω_1 with $<\omega_1$-distributive forcing notions).

Lemma 3.8. *If* $\mathbb{P}$ *is strongly* $<\kappa$*-distributive and* $S \subseteq \kappa$ *is stationary,* $1_{\mathbb{P}} \Vdash \check{S}$ *is stationary.*

Proof. Assume that some $p \in \mathbb{P}$ forces $\dot{C} \subseteq \check{\kappa}$ to be a club and $\dot{f}$ to be its strictly increasing enumeration. Thus, $p \Vdash \dot{f} : \check{\kappa} \longrightarrow V$. Hence, there exists a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{\gamma}_\alpha$ for some $\gamma_\alpha \in \kappa$. Let

$$C' := \{\gamma_\alpha \mid \alpha < \kappa\}$$

Claim. $C' \subseteq \kappa$ *is club.*

Proof. If $\beta < \alpha$, $p_\alpha \Vdash \check{\gamma}_\beta = \dot{f}(\check{\beta}) \wedge \check{\gamma}_\alpha = \dot{f}(\check{\alpha})$. Because $\dot{f}$ is forced to be strictly increasing, $p_\alpha \Vdash \check{\gamma}_\beta < \check{\gamma}_\alpha$. Thus $(\gamma_\alpha)_{\alpha<\kappa}$ is a strictly increasing sequence in κ of length κ and thus unbounded.

Let $\gamma \in \kappa$ be a limit and assume $C' \cap \gamma = \{\gamma_\alpha \mid \alpha < \delta\}$ is unbounded in γ. This implies that δ is a limit ordinal (because the sequence is strictly increasing). Thus, p_δ forces that

$(\dot{f}(\check{\alpha}))_{\alpha<\delta}$ is unbounded in γ. Because p_δ forces $\dot{f}(\check{\delta}) = \gamma_\delta$ and $\dot{f}$ to be continuous, $\gamma_\delta = \gamma \in C'$. □

Since $C' \subseteq \kappa$ is club and in V, there exists $\alpha < \kappa$ such that $\gamma_\alpha \in C' \cap S$. Then $p_\alpha \Vdash \check{\gamma}_\alpha \in \check{S} \cap \dot{C}$. □

Since clubs in κ are basically the same as clubs in $[\kappa]^{<\kappa}$, we have the following corollary:

Corollary 3.9. *If $\mathbb{P}$ is strongly $<\kappa$-distributive, $\mathbb{P}$ preserves stationary subsets of $[X]^{<\kappa}$ whenever $|X|= \kappa$.*

Proof. We will show that for every set $S \subseteq [X]^{<\kappa}$, there exists a set $S' \subseteq \kappa$ such that S is stationary if and only if S' is.

Let $F : X \longrightarrow \kappa$ be a bijection. Then $a \mapsto F[a]$ is a continuous and order-preserving bijection from $[X]^{<\kappa}$ into $[\kappa]^{<\kappa}$. Hence, S is stationary if and only if $F[S]$ is. Let $S' := F[S] \cap \kappa$.

Claim. *$S' \subseteq \kappa$ is stationary if and only if $F[S] \subseteq [\kappa]^{<\kappa}$ is.*

Proof. Assume S' is nonstationary and let $C \subseteq \kappa$ be a club with empty intersection with S'. By standard arguments, $C \subseteq [\kappa]^{<\kappa}$ is also club. Since $C \cap F[S] = \emptyset$, $F[S]$ is nonstationary.

Assume $F[S]$ is nonstationary and let $C \subseteq [\kappa]^{<\kappa}$ be club with empty intersection with $F[S]$. It follows that $C \cap \kappa \subseteq \kappa$ is also club. Since $(C \cap \kappa) \cap S' = \emptyset$, we are done. □

□

4. Orders on products

Some of the difficulty in working with Mitchell Forcing stems from the fact that it is neither an iteration (as we are not using full names) nor a product (as the first component is relevant for the ordering on the second component). Therefore, to help in the later sections, this section will introduce a general way of working with arbitrary orders on products of sets.

Definition 4.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be sets and R a partial order on $\mathbb{P} \times \mathbb{Q}$ (not necessarily a product ordering). We will only consider orderings where for all p, p'

$$\exists q((p,q)R(p',q)) \longleftrightarrow \forall q((p,q)R(p',q))$$

If we want to reference this property, we will say that $\mathbb{P} \times \mathbb{Q}$ is *basic*. We define the following partial orders:

(1) The *base ordering* $b(R)$ is an ordering on $\mathbb{P}$ given by $p(b(R))p'$ if for one (equivalently, all) $q \in \mathbb{Q}$, $(p,q)R(p',q)$.

(2) The *term ordering* $t(R)$ is an ordering on $\mathbb{P} \times \mathbb{Q}$ given by $(p,q)t(R)(p',q')$ if $(p,q)R(p',q')$ and $p = p'$.

(3) For $p \in \mathbb{P}$, the *section ordering* $s(R,p)$ is an ordering on $\mathbb{Q}$ given by $q(s(R,p))q'$ if $(p,q)R(p,q')$.

We also fix the following properties:

(1) $(\mathbb{P} \times \mathbb{Q}, R)$ has *property (A)* if whenever $(p',q')R(p,q)$, there is q'' such that $(p,q'')R(p,q)$ and $(p',q'')R(p',q')R(p',q'')$.

(2) $(\mathbb{P} \times \mathbb{Q}, R)$ has *property (B)* if $p'(b(R))p$ implies that $s(R,p')$ refines $s(R,p)$, i.e. whenever $(p,q')R(p,q)$ and $p'(b(R))p$, also $(p',q')R(p',q)$.

Properties (A) and (B) hold in almost all cases, and always for iterations and products. They are necessary for most of the relevant techniques.

Lemma 4.2. *If* $(\mathbb{P} \times \mathbb{Q}, R)$ *has properties (A) and (B), there is a projection from* $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R, 1_{\mathbb{P}}))$ *onto* $(\mathbb{P} \times \mathbb{Q}, R)$.

Proof. Let $i : (\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R, 1_{\mathbb{P}})) \longrightarrow (\mathbb{P} \times \mathbb{Q}, R)$ simply be the identity. It is clear that $i(1_{\mathbb{P}}, 1_{\mathbb{Q}}) = 1_{\mathbb{P}\times\mathbb{Q}}$. To simplify notation, let R' denote the ordering on $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R, 1_{\mathbb{P}}))$.

If $(p',q')R'(p,q)$, $p'(b(R))p$ and $(1,q')R(1,q)$. It follows that $(p',q')R(p,q')R(p,q)$ by property (B). Assume $(p',q')R(p,q)$. Because $(p,q)R(1_{\mathbb{P}},q)$, there is q'' such that $(1_{\mathbb{P}},q'')R(1_{\mathbb{P}},q)$ and $(p',q'')R(p',q')$. It follows that $(p',q'')R'(p',q)R'(p,q)$. □

Cones in orders on products can be regarded again as orders on products if property (A) holds.

Lemma 4.3. *If* $(\mathbb{P}\times\mathbb{Q}, R)$ *has property (A) and* $(p,q) \in \mathbb{P}\times\mathbb{Q}$*,* $\{p' \in \mathbb{P} \mid p'(b(R))p\} \times \{q' \in \mathbb{Q} \mid (p,q')R(p,q)\}$ *is dense in* $\{(p',q') \in \mathbb{P} \times \mathbb{Q} \mid (p',q')R(p,q)\}$.

We will now generalise both the Product and the Factor Lemma, showing how we can view forcing with $\mathbb{P} \times \mathbb{Q}$ as successive forcing.

Lemma 4.4. *There exists a projection from* $\mathbb{P} \times \mathbb{Q}$ *onto* $(\mathbb{P}, b(R))$.

Proof. The projection is simply given by $\pi(p,q) = p$. Basicness and property (A) imply that π is actually a projection. □

For the rest of the section, $\mathbb{P}$ refers to $(\mathbb{P}, b(R))$. By the definitions, whenever $G \subseteq \mathbb{P}$ is generic, $(\mathbb{P} \times \mathbb{Q})/G = \{(p,q) \in \mathbb{P} \times \mathbb{Q} \mid p \in G\} = G \times \mathbb{Q}$. We will now show that $G \times \mathbb{Q}$ is forcing equivalent to a particular ordering on $\mathbb{Q}$.

Definition 4.5. Let $(\mathbb{P} \times \mathbb{Q}, R)$ be a partial order with properties (A) and (B). Let G be $\mathbb{P}$-generic. In $V[G]$, define the *generic ordering* $g(R,G)$ on $\mathbb{Q}$ by $q(g(R,G))q'$ if for some $p \in G$, $(p,q)R(p,q')$

Remark 4.6. $g(R,G)$ actually is a partial order: Reflexivity is clear. For transitivity, assume that $q_0(g(R,G))q_1(g(R,G))q_2$, i.e. for $p,p' \in G$, $(p,q_0)R(p,q_1)$ and $(p',q_1)R(p',q_2)$. Assume $p''(b(R))p,p'$ is in G. Then by property (B),

$$(p'',q_0)R(p'',q_1)R(p'',q_2)$$

i.e. $q_0(g(R,G))q_2$.

Lemma 4.7. *Let $\mathbb{P}\times\mathbb{Q}$ be a partial order with properties (A) and (B). Let G be $\mathbb{P}$-generic. In $V[G]$, the posets $G\times\mathbb{Q}$ and $(\mathbb{Q},g(R,G))$ are forcing equivalent.*

Proof. Let $\pi:(G\times\mathbb{Q},R\restriction(G\times\mathbb{Q}))\longrightarrow(\mathbb{Q},g(R,G))$ be given by $\pi(p,q)=q$. We will verify that π is a trivial projection.

$\pi((1_G,1_\mathbb{Q}))=1_\mathbb{Q}$. Let $(p',q')R(p,q)$. By property (A), there is q'' with $(p,q'')R(p,q)$ and $(p',q'')R(p',q')R(p',q'')$. Thus p witnesses $q''(g(R,G))q$ and p' witnesses $q'(g(R,G))q''$. By transitivity, $q'(g(R,G))q$.

Assume $(p,q)\in G\times\mathbb{Q}$ and $q'(g(R,G))q$. Let $p'\in G$ witness this, i.e. $(p',q')R(p',q)$. Let $p''(b(R))p',p$. Thus, by property (B), $(p'',q')R(p'',q)R(p,q)$ and (p'',q') is as required.

Lastly, if $\pi(p,q)=q=\pi(p',q)$, then let $G\ni p''(b(R))p,p'$. Thus, $(p'',q)R(p',q),(p,q)$. □

Thus, by Fact 2.6, forcing with $(\mathbb{P}\times\mathbb{Q},R)$ can be regarded as first forcing with $(\mathbb{P},b(R))$ and then with $(\mathbb{Q},g(R,G))$, where G is $\mathbb{P}$-generic. In the case of a product, $g(R,G)$ is simply equal to the original ordering on $\mathbb{Q}$. In the case of an iteration $\mathbb{P}*\dot{\mathbb{Q}}$, $(\mathbb{Q},b(R,G))$ is forcing equivalent to $\dot{\mathbb{Q}}^G$ (the only difference being that we are not identifying equivalent names).

The following results are especially important for Mitchell forcing: In most cases, the term ordering on $\mathbb{P}\times\mathbb{Q}$ is $<\kappa$-closed because $\mathbb{P}$ forces $\mathbb{Q}$ to be $<\kappa$-closed. In these cases, one obtains that $(\mathbb{Q},g(R,G))$ is forcing equivalent to a $<\kappa$-closed forcing and thus has nice regularity properties. However, as we will later see, there are cases where the term ordering on $\mathbb{P}\times\mathbb{Q}$ is $<\kappa$-closed but $(\mathbb{Q},g(R,G))$ fails to be. We will now see that, if $\mathbb{P}$ has a good enough chain condition, $(\mathbb{Q},g(R,G))$ is at least strongly $<\kappa$-distributive.

Lemma 4.8. *Assume $(\mathbb{P}\times\mathbb{Q},R)$ has properties (A) and (B), the base ordering is κ-cc. and the term ordering is strongly $<\kappa$-distributive. If $G\subseteq\mathbb{P}$ is generic, in $V[G]$, the ordering $(\mathbb{Q},g(R,G))$ is strongly $<\kappa$-distributive.*

The proof consists of two simple Lemmas.

Lemma 4.9. *If $\pi:\mathbb{P}\longrightarrow\mathbb{Q}$ is a projection and $\mathbb{P}$ is strongly $<\kappa$-distributive, so is $\mathbb{Q}$.*

Proof. If $D\subseteq\mathbb{Q}$ is open dense, so is $\pi^{-1}[D]$. Given a sequence $(D_\alpha)_{\alpha<\kappa}$ of open dense sets, find a thread through $(\pi^{-1}[D_\alpha])_{\alpha<\kappa}$ and apply π to it. □

Lemma 4.10. *Assume $(\mathbb{P}\times\mathbb{Q},R)$ has properties (A) and (B) and let G be $\mathbb{P}$-generic. In $V[G]$, there exists a projection from $(\mathbb{Q},s(R,1_{\mathbb{P}}))$ onto $(\mathbb{Q},g(R,G))$.*

Proof. The projection once again is just the identity. $g(R,G)$ is of course finer than $s(R,1_{\mathbb{P}})$, since $1_{\mathbb{P}}\in G$. If $q'(g(R,G))q$, then $(p,q')R(p,q)R(1,q)$ for some $p\in G$, so there exists q'' such that $(1,q'')R(1,q)$ and $(p,q'')R(p,q')R(p,q'')$. Hence $q''(s(R,1_{\mathbb{P}}))q$ and $q''(g(R,G))q'$, witnessed by p. □

Examples of strongly distributive forcings.

We will give examples to show some limitations of strong distributivity. Namely, we will show the following:

(1) There is no provable relationship between weak $(\lambda+1)$-strategic closure and strong $<\lambda^+$-distributivity. There can consistently be a forcing which is weakly $(\lambda+1)$-strategically closed but not strongly $<\lambda^+$-distributive and there can consistently be a forcing which is strongly $<\lambda^+$-distributive but not weakly $(\lambda+1)$-strategically closed.
(2) Strong $<\lambda^+$-distributivity need not be downwards absolute.
(3) Strongly $<\kappa$-distributive forcings can destroy the stationarity of subsets of $[\lambda]^{<\kappa}$. In particular, strongly $<\omega_1$-distributive forcings need not be proper.
(4) There can be forcings $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P}\times\mathbb{Q}$ is strongly $<\kappa$-distributive but $\mathbb{Q}$ is no longer strongly $<\kappa$-distributive after forcing with $\mathbb{P}$. Furthermore, there can be forcings $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P}$ is $<\kappa$-closed and $\mathbb{Q}$ is strongly $<\kappa$-distributive but $\mathbb{Q}$ is no longer strongly $<\kappa$-distributive after forcing with $\mathbb{P}$.

Example 4.11. Let $\mathbb{P}$ be the forcing to add a $\square_\lambda$-sequence. Conditions are functions p such that

(1) $\mathrm{dom}(p)=\{\beta\leq\alpha\mid\lim(\beta)\}$ for some limit ordinal $\alpha\in\lambda^+$.
(2) For all $\alpha\in\mathrm{dom}(p)$, $p(\alpha)$ is club in α of ordertype $\leq\lambda$.
(3) Whenever β is a limit point of $p(\alpha)$, $p(\beta)=p(\alpha)\cap\beta$.

ordered by extension.

This forcing is weakly $(\lambda+1)$-strategically closed. However, if $\square_\lambda$ fails, the forcing is not strongly $<\lambda^+$-distributive: Let $D_\alpha:=\{p\in\mathbb{P}\mid\alpha\in\mathrm{dom}(p)\}$ for limit α and $\mathbb{P}$ otherwise. If $(p_\alpha)_{\alpha<\lambda^+}$ is a thread through $(D_\alpha)_{\alpha<\lambda^+}$, $\bigcup_{\alpha<\lambda^+}p_\alpha$ is a $\square_\lambda$-sequence, a contradiction.

The above example also shows (2): After forcing with $\mathbb{P}$, $\square_\lambda$ holds, so by [3], Theorem 3.3, every weakly $(\lambda+1)$-strategically closed poset (and in particular, $\mathbb{P}$) is weakly λ^+-strategically closed and thus strongly $<\lambda^+$-distributive.

Example 4.12. Let $\mathbb{P}$ be $\mathrm{Add}(\omega_1)$. Again, let G be $\mathrm{Add}(\omega)$-generic. In $V[G]$, $\mathbb{P}$ is still strongly $< \omega_1$-distributive. Assume $\mathbb{P}$ is weakly $(\omega+1)$-strategically closed. Thus $\check{\mathbb{P}}$ is an $\mathrm{Add}(\omega)$-name for an $(\omega+1)$-strategically closed forcing, so by Lemma 1.3 in [9], $\mathrm{Add}(\omega) * \check{\mathbb{P}}$ has the $< \omega_1$-approximation property. This is obviously not the case, as the forcing is equivalent to $\mathrm{Add}(\omega) \times \mathrm{Add}(\omega_1)$.

Example 4.13. Assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Let $\mathbb{P}$ be the forcing to collapse ω_2 by adding a cofinal, continuous sequence of length ω_1 to $[\omega_2]^{<\omega_1}$. Conditions are functions p such that

(1) $\mathrm{dom}(p)$ is a successor ordinal.
(2) for every $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [\omega_2]^{<\omega_1}$. For every limit $\delta \in \mathrm{dom}(p)$, $p(\delta) = \bigcup_{\alpha<\delta} p(\alpha)$

ordered by end-extension. This forcing is $< \omega_1$-closed.

Now let G be $\mathrm{Add}(\omega) * \dot{\mathrm{Coll}}(\check{\omega_1}, \check{\omega_2})$-generic and let H be the induced $\mathrm{Add}(\omega)$-generic filter. In $V[G]$, ω_2^V has size ω_1. Furthermore, in this extension, $\mathbb{P}$ still adds a cofinal and continuous sequence through $([\omega_2]^{<\omega_1})^V$, which is forced to be a club by the $< \omega_1$-covering property. By the later discussion, $[\omega_2^V]^{<\omega_1} \smallsetminus V$ is still stationary inside $[\omega_2^V]^{<\omega_1}$ in $V[G]$ (using the internal approachability), so $\mathbb{P}$ (in $V[G]$) destroys a stationary subset of $[\omega_2^V]^{<\omega_1}$. Since ω_2^V is of size ω_1 in $V[G]$, $\mathbb{P}$ cannot be strongly $< \omega_1$-distributive in this model. However, $\mathbb{P}$ is strongly $< \omega_1$-distributive in $V[H]$, so its strong distributivity is destroyed by the $< \omega_1$-closed forcing $\mathrm{Coll}(\omega_1, \omega_2)$. Furthermore, in $V[H]$, $\mathrm{Coll}(\omega_1, \omega_2) \times \mathbb{P}$ is (forcing equivalent to) a strongly $< \omega_1$-distributive forcing, because the term ordering on $\mathrm{Add}(\omega) \times (\dot{\mathrm{Coll}}(\check{\omega_1}, \check{\omega_2}) \times \mathbb{P})$ is $< \omega_1$-closed.

5. Proving the main theorems

In this section, we will first introduce the forcing we intend to use and then state as well as prove our main theorems. For this section, fix an increasing sequence $(\kappa_n)_{n\in\omega}$ of Mahlo cardinals and their supremum κ. For simplicity, denote $\kappa_{-1} := \omega_1$.

Definition 5.1. We will define $\mathbb{M}((\kappa_n)_{n\in\omega}, \delta)$ by induction on $\delta \leq \kappa$.

$\mathbb{M}((\kappa_n)_{n\in\omega}, 0) := \{\emptyset\}$. If $\mathbb{M}((\kappa_n)_{n\in\omega}, \beta)$ has been defined for all $\beta < \delta$, let $\mathbb{M}((\kappa_n)_{n\in\omega}, \delta)$ consist of all (p, q) such that

(1) $p \in \mathrm{Add}(\omega, \delta)$
(2) q is a partial function on the successor ordinals in $\delta \smallsetminus \omega_1$ with the following properties:
  (a) for all $n \in \omega$, $|\mathrm{dom}(q) \cap \kappa_n| < \kappa_{n-1}$
  (b) for all $n \geq -1$ and $\beta \in \delta \cap [\kappa_n, \kappa_{n+1})$, $q(\beta)$ is an $\mathbb{M}((\kappa_n)_{n\in\omega}, \beta)$-name forced by $1_{\mathbb{M}((\kappa_n)_{n\in\omega},\beta)}$ to be in $\dot{\mathrm{Coll}}(\check{\kappa_n}, \check{\beta})$

We let $(p', q') R_\delta (p, q)$ if

(1) $p' \leq p$
(2) $\operatorname{dom}(q') \supseteq \operatorname{dom}(q)$ and for all $\beta \in \operatorname{dom}(q)$, $(p' \restriction \beta, q' \restriction \beta) \Vdash q'(\beta) \leq q(\beta)$.

For simplicity, if δ is an ordinal, we will write $\mathbb{M}(\delta) := \mathbb{M}((\kappa_n)_{n\in\omega}, \delta)$ and $\mathbb{M} := \mathbb{M}(\kappa)$. Let $R := R_\kappa$.

This version of Mitchell forcing can be viewed as a kind of mixed support iteration, where instead of two kinds of supports, we are using infinitely many. The reason for that is that we need increasing closure in the collapses to preserve the κ_n's and the stationary sets we are adding, but we also need our Cohen reals to stay ω_1-Knaster.

As we will later see, we need our forcing to be decomposable as $\mathbb{P} * \operatorname{Add}(\omega) * \mathbb{Q}$, where $\mathbb{Q}$ preserves certain stationary sets. Levine obtained this decomposition in [7] by making $q(\delta)$ depend on the first $\delta + 1$ Cohen Reals (thus ensuring that the collapse will still be closed after adding $\delta + 1$ Cohen Reals). We obtain the decomposition by making q only defined on successor ordinals.

As with most versions of Mitchell forcing, we get nice decompositions, either vertically or horizontally:

Definition 5.2. Let $\delta \leq \kappa$ be an ordinal.

(1) $\mathbb{T}(\delta)$ consists of all q such that $(\emptyset, q) \in \mathbb{M}(\delta)$.
(2) $\mathbb{M}(\kappa \setminus \delta)$ consists of all $(p, q) \in \mathbb{M}$ such that $p \in \operatorname{Add}(\omega, \kappa \setminus \delta)$ and $\operatorname{dom}(q) \subseteq \kappa \setminus \delta$.
(3) $\mathbb{T}(\kappa \setminus \delta)$ consists of all q such that $(\emptyset, q) \in \mathbb{M}(\kappa \setminus \delta)$.

We have bijections from $\mathbb{M}(\delta)$ to $\operatorname{Add}(\omega, \delta) \times \mathbb{T}(\delta)$, from $\mathbb{M}(\kappa)$ to $\mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$ and from $\mathbb{M}(\kappa)$ to $(\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta)) \times \mathbb{T}(\kappa \setminus \delta)$. These isomorphisms of course induce orderings on the given products, all of which are basic. We will refer to all of these orderings as follows: $R(\delta)$ is the ordering on $\mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$ and $R_s(\delta)$ (read"R split") is the ordering on $(\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta)) \times \mathbb{T}(\kappa \setminus \delta)$.

Note the following: Unlike the more sophisticated way of splitting the Mitchell forcing (which we will define later) at some point δ, the second component does still use $\mathbb{M}(\alpha)$-names (and not $\mathbb{M}(\alpha \setminus \delta)$-names). So we are not obtaining an ordering on $\mathbb{M}(\kappa \setminus \delta)$ (apart from the section oderings), only one on $\mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$.

For illustration purposes, we will explicitely write down what $R(\delta)$ looks like. Let $\iota : \mathbb{M} \longrightarrow \mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$ be the isomorphism (whose inverse is given by $((p, q), (p', q')) \mapsto (p \cup p', q \cup q')$). Given $((p_0, q_0), (p_0', q_0')), ((p_1, q_1), (p_1', q_1')) \in \mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$,

$$((p_0, q_0), (p_0', q_0'))(R(\delta))((p_1, q_1), (p_1', q_1'))$$

holds if and only if

$$(p_0 \cup p_1, q_0 \cup q_0')R(p_1 \cup p_1', q_1 \cup q_1')$$

which holds if and only if

(1) $p_0 \cup p_0' \supseteq p_1 \cup p_1'$, i.e. $p_0 \supseteq p_1$ and $p_0' \supseteq p_1'$
(2) $\mathrm{dom}(q_0 \cup q_0') \supseteq \mathrm{dom}(q_1 \cup q_1')$, i.e. $\mathrm{dom}(q_0) \supseteq \mathrm{dom}(q_1)$ and $\mathrm{dom}(q_0') \supseteq \mathrm{dom}(q_1')$ and for all $\alpha \in \mathrm{dom}(q_1 \cup q_1')$:
   (a) If $\alpha \in \mathrm{dom}(q_1)$,
$$(p_0 \upharpoonright \alpha, q_0 \upharpoonright \alpha) \Vdash_{\mathbb{M}} q_0(\alpha) \leq q_1(\alpha)$$
   (since $\alpha \leq \delta$, so $(p_0 \cup p_1) \upharpoonright \alpha = p_0 \upharpoonright \alpha$ and $(q_0 \cup q_1) \upharpoonright \alpha = q_0 \upharpoonright \alpha$)
   (b) If $\alpha \in \mathrm{dom}(q_1')$,
$$((p_0 \cup p_0') \upharpoonright \alpha, (q_0 \cup q_0') \upharpoonright \alpha) \Vdash_{\mathbb{M}} q_0'(\alpha) \leq q_1'(\alpha)$$

We immediately obtain the following Lemma:

Lemma 5.3. *The inductively defined order R_δ and the order $b(R(\delta))$ induced by R_δ (both of which are orders on $\mathbb{M}(\delta)$) coincide. Moreover, the order $b(R_s(\delta))$ on $\mathbb{M}(\delta) \times \mathrm{Add}(\kappa \setminus \delta)$ is equal to the product ordering of R_δ ($= R(\delta)$) and the standard ordering on $\mathrm{Add}(\kappa \setminus \delta)$.*

For the section orderings, we also obtain a nice description: Given either $(p, q) \in \mathbb{M}(\delta)$ (or $((p, q), p') \in \mathbb{M}(\delta) \times \mathrm{Add}(\kappa \setminus \delta)$), the section ordering $s(R, (p, q))$ (or $s(R, ((p, q), p')))$ is isomorphic (via the same isomorphism) to the ordering on $\{(p_0, q_0) \in \mathbb{M}(\kappa) \mid p_0 \upharpoonright \delta = p, q_0 \upharpoonright \delta = q\}$ (or $\{(p_0, q_0) \in \mathbb{M}(\kappa) \mid p_0 = p, q_0 \upharpoonright \delta = q\}$) induced by R. For the following Lemma, remember that for any $(\mathbb{P} \times \mathbb{Q}, R')$, the term ordering $t(R')$ is $< \nu$-closed if and only if for all $p \in \mathbb{P}$, the section ordering $s(R', p)$ is $< \nu$-closed (since the term ordering is isomorphic to the disjoint union of all the section orderings).

Lemma 5.4. *Let $\delta \leq \kappa$ be an ordinal.*

(1) *The term ordering $t(R(\delta))$ on $\mathrm{Add}(\omega, \delta) \times \mathbb{T}(\delta)$ is $< \omega_1$-closed.*
(2) *If δ is inaccessible, the base ordering $b(R_s(\delta))$ on $\mathbb{M}(\delta) \times \mathrm{Add}(\omega, \kappa \setminus \delta)$ is δ-Knaster.*
(3) *If $n \geq -1$ and $\delta \in [\kappa_n, \kappa_{n+1})$, the term ordering $t(R_s(\delta))$ on $(\mathbb{M}(\delta) \times \mathrm{Add}(\omega, \kappa \setminus \delta)) \times \mathbb{T}(\kappa \setminus \delta)$ is $< \kappa_n$-closed.*

Proof. We prove the statements one by one.

(1) Let $(p, q_\alpha)_{\alpha < \omega}$ be a descending sequence in $\mathrm{Add}(\omega, \delta) \times \mathbb{T}(\delta)$.
   Let $x := \bigcup_{\alpha < \omega} \mathrm{dom}(q_\alpha)$. By regularity of every κ_n (recall $\kappa_{-1} = \omega_1$), $|x \cap \kappa_n| < \kappa_{n-1}$ for all n. We will define, by induction on $\beta < \delta$, a function q on x such that $(p \upharpoonright \beta + 1, q \upharpoonright \beta + 1)$ is a lower bound of $(p \upharpoonright \beta + 1, q_\alpha \upharpoonright \beta + 1)_{\alpha < \omega}$ in $\mathbb{M}(\beta + 1)$.
   Assume $q \upharpoonright \beta$ has been defined.

Case 1: Assume $\beta \notin x$. Leave $q(\beta)$ undefined. We will verify that $(p \upharpoonright \beta+1, q \upharpoonright \beta+1)$ is a lower bound of $(p \upharpoonright \beta + 1, q_\alpha \upharpoonright \beta + 1)_{\alpha<\omega}$. To this end, let $\alpha < \omega$ be arbitrary. $p \upharpoonright \beta + 1 \leq p \upharpoonright \beta + 1$ as well as $\mathrm{dom}(q \upharpoonright \beta + 1) \supseteq \mathrm{dom}(q_\alpha \upharpoonright \beta + 1)$ is clear, so let $\gamma \in \mathrm{dom}(q_\alpha \upharpoonright \beta + 1)$. Because $\beta \notin x$, $\gamma \in \mathrm{dom}(q_\alpha \upharpoonright \beta)$ and by the inductive hypothesis, $(p \upharpoonright \gamma, q \upharpoonright \gamma) \Vdash q(\gamma) \leq q_\alpha(\gamma)$.

Case 2: Assume $\beta \in x$. Then there exists $\alpha_0 \in \omega$ such that for all $\alpha \geq \alpha_0$, $\beta \in \mathrm{dom}(q_\alpha)$. By the inductive hypothesis, $(p \upharpoonright \beta, q \upharpoonright \beta)$ forces that $(q_\alpha(\beta))_{\alpha_0 \leq \alpha < \omega}$ is a descending sequence in some $< \omega_1$-closed forcing, so by the maximum principle we can find a $\mathbb{M}(\beta)$-name $q(\beta)$ that is forced to be a lower bound. Now let $\alpha \in \omega$. Again, $p \upharpoonright \beta + 1 \leq p \upharpoonright \beta + 1$ and $\mathrm{dom}(q \upharpoonright \beta + 1) \supseteq \mathrm{dom}(q_\alpha \upharpoonright \beta + 1)$ is clear, so let $\gamma \in \mathrm{dom}(q_\alpha \upharpoonright \beta + 1)$. If $\gamma < \beta$, we argue as in Case 1. If $\gamma = \beta$, $(p \upharpoonright \beta, q \upharpoonright \beta) \Vdash q(\beta) \leq q_\alpha(\beta)$ by assumption.

(2) By Lemma 5.3, it suffices to show that $\mathbb{M}(\delta)$ is δ-Knaster. This follows from a standard application of the Δ-System Lemma.

(3) Let $\lambda < \kappa_n$ and assume $(((p, q), r), s_\alpha)_{\alpha < \lambda}$ is a descending sequence in $(\mathbb{M}(\delta) \times \mathrm{Add}(\omega, \kappa \setminus \delta)) \times \mathbb{T}(\kappa \setminus \delta)$. Let $x := \bigcup_{\alpha < \lambda} \mathrm{dom}(s_\alpha)$. Given $k \in \omega$, if $k \leq n$, then $x \cap \kappa_k = \emptyset$ and if $k > n$, then $|x \cap \kappa_k| < \kappa_{k-1}$, since $\kappa_{k-1} \geq \kappa_n > \lambda$ is regular. By induction on $\beta \in [\delta, \kappa)$ we will define a function s on x such that $(((p, q), r \upharpoonright \beta + 1), s \upharpoonright \beta + 1)$ is a lower bound of $(((p, q), r \upharpoonright \beta + 1), s_\alpha \upharpoonright \beta + 1)_{\alpha < \lambda}$. Assume $s \upharpoonright \beta$ has been defined.

Case 1: Assume $\beta \notin x$. Leave $r(\beta)$ undefined. We will verify that $(((p, q), r \upharpoonright \beta + 1), s \upharpoonright \beta + 1)$ is a lower bound of $(((p, q), r \upharpoonright \beta + 1), s_\alpha \upharpoonright \beta + 1)_{\alpha < \lambda}$ Using the isomorphism, this amounts to showing that, for any arbitrary α, $(p \cup (r \upharpoonright \beta + 1), q \cup (s \upharpoonright \beta + 1)) \leq ((p \cup (r \upharpoonright \beta + 1)), q \cup (s_\alpha \upharpoonright \beta + 1))$. This however follows very similarly to (1).

Case 2: Assume $\beta \in x$. Then there exists $\alpha_0 \in \lambda$ such that $\beta \in \mathrm{dom}(s_\alpha)$ for any $\alpha \in [\alpha_0, \lambda)$. By the inductive hypothesis and again using the isomorphism, $((p \cup (q \upharpoonright \beta)), (r \cup (s \upharpoonright \beta)))$ forces that $(s_\alpha(\beta))_{\alpha < \lambda}$ is a descending sequence in some $< \kappa_n$-closed forcing (since after δ, we are collapsing to some κ_k with $k \geq n$), so by the maximum principle, we can fix a lower bound $s(\beta)$. Now proceed as in (1).

□

We obtain the fact that $\mathbb{M}$ preserves κ_n for every $n \geq -1$:

Lemma 5.5. *For $n \geq -1$, $\mathbb{M}$ has the $< \kappa_n$-covering property.*

Proof. There is an isomorphism

$$\mathbb{M} \cong ((\mathbb{M}(\delta) \times \mathrm{Add}(\omega, \kappa \setminus \delta)) \times \mathbb{T}(\kappa \setminus \delta), R_s(\delta))$$

Now let $\delta = \kappa_n$. $b(R_s(\kappa_n))$ is κ_n-Knaster by Lemma 5.4. Additionally, by the same lemma, the term ordering $t(R_s(\delta))$ is $< \kappa_n$-closed.

Lastly, the product has properties (A) and (B): (B) is easy to verify.

Let $(((p_0, q_0), p'_0), q'_0)R(((p_1, q_1), p'_1), q'_1))$. By induction on $\alpha \geq \delta$, we will define $q''(\alpha)$ such that

$$(p_1 \cup p'_1, q_1 \cup q'' \restriction \alpha + 1)R(p_1 \cup p'_1, q_1 \cup q'_1 \restriction \alpha + 1)$$

and

$$(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha + 1)R(p_0 \cup p'_0, q_0 \cup q'_0 \restriction \alpha + 1)R(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha + 1)$$

which immediately implies the required statements given how the ordering on $\mathbb{M}(\delta) \times \mathbb{M}(\kappa \setminus \delta)$ is defined.

Assume $q'' \restriction (\alpha \setminus \delta)$ has been defined. By the inductive hypothesis (since the ordering is "continuous"),

$$(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha)R(p_0 \cup p'_0, q_0 \cup q'_0 \restriction \alpha)R(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha)$$

and

$$(p_1 \cup p'_1, q_1 \cup q'' \restriction \alpha)R(p_1 \cup p'_1, q_1 \cup q'_1 \restriction \alpha)$$

In particular, $(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha)$ and $(p_0 \cup p'_0, q_0 \cup q'_0 \restriction \alpha)$ force the same statements and thus

$$(p_0 \cup p'_0 \restriction \alpha, q_0 \cup q'' \restriction \alpha) \Vdash q'_0(\alpha) \leq q'_1(\alpha)$$

by assumption. Using standard name arguments, there exists a name $q''(\alpha)$ such that conditions below $(p_0 \cup p'_0 \restriction \alpha, q_0 \cup q'' \restriction \alpha)$ force $q''(\alpha) = q'_0(\alpha)$ and conditions incompatible with $(p_0 \cup p'_0 \restriction \alpha, q_0 \cup q'' \restriction \alpha)$ force $q''(\alpha) = q'_1(\alpha)$. In particular, using the inductive hypothesis,

$$(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha + 1)R(p_0 \cup p'_0, q_0 \cup q'_0 \restriction \alpha + 1)R(p_0 \cup p'_0, q_0 \cup q'' \restriction \alpha + 1)$$

and since in any case $q''(\alpha)$ is forced to be below $q'_1(\alpha)$,

$$(p_1 \cup p'_1, q_1 \cup q'' \restriction \alpha + 1)R(p_1 \cup p'_1, q_1 \cup q'_1 \restriction \alpha + 1)$$

Let G be $\mathbb{M}$-generic. Forcing with $\mathbb{M}$ can be regarded as forcing first with $(\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta))$ and then with $(\mathbb{T}(\kappa \setminus \delta), g(R_s(\delta), G_\delta))$, where G_δ is $(\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta))$-generic. By Lemma 4.8 (with $\mathbb{P} = (\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta))$, $\mathbb{Q} = \mathbb{T}(\kappa \setminus \delta)$ and $R = R_s(\delta)$), $g(R_s(\delta), G_\delta)$ is strongly $<\kappa_n$-distributive in $V[G_\delta]$. Hence every set of size $<\kappa_n$ in $V[G]$ is also in $V[G_\delta]$ and can thus be covered by a set of size $<\kappa_n$ by the κ_n-cc. of $(\mathbb{M}(\delta) \times \operatorname{Add}(\omega, \kappa \setminus \delta))$. □

Note that $\mathbb{M}$ does not have the κ-Knaster property: Since κ is singular, this would imply that $\mathbb{M}$ is δ-cc. for some $\delta < \kappa$, which is impossible since cofinally many $\delta < \kappa$ are collapsed. However, $\mathbb{M}$ does preserve κ, since also cofinally many $\kappa_n < \kappa$ are not collapsed.

We will also make use of the following, more sophisticated way, of splitting the Mitchell Forcing: This is done to obtain a forcing which has a closed term ordering in an intermediate model (as opposed to the ground model) and works similarly to the proof of Lemma 22 in [7].

Definition 5.6. Let $\delta \in [\kappa_{n-1}, \kappa_n)$ and let G be $\mathbb{M}(\delta)$-generic. In $V[G]$, define the poset $\mathbb{M}(\kappa_n \setminus \delta, G, \beta)$ by induction on $\beta \geq \delta$. $\mathbb{M}(\kappa_n \setminus \delta, G, \delta) = \{\emptyset\}$. If $\mathbb{M}(\kappa_n \setminus \delta, G, \gamma)$ has been defined for all $\gamma < \beta$, $\mathbb{M}(\kappa_n \setminus \delta, G, \beta)$ consists of pairs (p, q) such that

(1) $p \in \mathrm{Add}(\omega, \beta \setminus \delta)$
(2) q is a partial function on the successor ordinals in $\beta \setminus \delta$ of size $< \kappa_{n-1}$ such that for all $\gamma \in \mathrm{dom}(q)$, $q(\gamma)$ is an $\mathbb{M}(\kappa_n \setminus \delta, G, \gamma)$-name for an element of $\dot{\mathrm{Coll}}(\check{\kappa}_n, \check{\gamma})$.

We order $\mathbb{M}(\kappa_n \setminus \delta, G, \beta)$ similarly to $\mathbb{M}$. Lastly, set $\mathbb{M}(\kappa_n \setminus \delta, G) := \mathbb{M}(\kappa_n \setminus \delta, G, \kappa_n)$.

We obtain that forcing with $\mathbb{M}(\kappa_n)$ is equivalent to forcing with $\mathbb{M}(\delta)$ and then with $\mathbb{M}(\kappa_n \setminus \delta, G, \kappa_n)$:

Lemma 5.7. *Let $\delta \in [\kappa_{n-1}, \kappa_n)$. The forcings $\mathbb{M}(\kappa_n)$ and $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \kappa_n)$ are equivalent.*

Proof. We will show by induction on $\beta \geq \delta$ that the forcings $\mathbb{M}(\beta)$ and $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \beta)$ are equivalent, namely, we will define a dense embedding from $\mathbb{M}(\beta)$ into $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \beta)$.

Beginning: $\beta = \delta$. Then obviously $(p, q) \mapsto ((p, q), (\check{\emptyset}, \check{\emptyset}))$ is a dense embedding.

Assume the embedding has been defined for $\gamma < \beta$. Let $(p, q) \in \mathbb{M}(\gamma)$. We define the $\mathbb{M}(\delta)$-name $\iota(q)$ as follows: $\iota(q)$ is forced to be a function with domain $\mathrm{dom}(\check{q}) \setminus \delta$ and for each $\gamma \in \mathrm{dom}(q) \setminus \delta$, $\iota(q)(\check{\gamma})$ is forced to be equal to the $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \gamma)$-name corresponding to $q(\gamma)$ (using the inductive hypothesis). Now we let $\pi : (p, q) \mapsto ((p \upharpoonright \delta, q \upharpoonright \delta), (p \upharpoonright (\beta \setminus \delta), \iota(q)))$ and verify that π is a dense embedding. The only difficult part is to show that the image of π is dense. Let $((p_0, q_0), (\dot{p}, \dot{q})) \in \mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \gamma)$. $\dot{p}$ is forced to be an element of V (by finiteness), so we can find (p_1, q_1) and $p \in \mathrm{Add}(\omega, \beta \setminus \delta)$ such that $(p_1, q_1) \Vdash \dot{p} = \check{p}$. Since $\mathrm{dom}(\dot{q})$ is forced to have size $< \kappa_{n-1}$ and $\mathbb{M}(\delta)$ has the $< \kappa_{n-1}$-covering property, there is $x \in V$ (assume that $x \subseteq \beta \setminus \delta$) of size $< \kappa_{n-1}$ and $(p_2, q_2) \leq (p_1, q_1)$ such that $(p_2, q_2) \Vdash \mathrm{dom}(\dot{q}) \subseteq \check{x}$. Now let $\overline{q}$ be a function such that for any $\gamma \in x$, $\overline{q}(\gamma)$ is equal to the $\mathbb{M}(\gamma)$-name corresponding to the $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \gamma)$-name

$$q'(\gamma) := \{(\tau, r) \mid r \in \mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \gamma) \wedge r \Vdash \tau \in \dot{q}(\check{\gamma})\}$$

Lastly, let $(p', q') := (p_2 \cup p, q_2 \cup \overline{q})$. We will show $\pi(p', q') \leq ((p_0, q_0), (\dot{p}, \dot{q}))$. $\pi(p', q') = ((p_2, q_2), (p, \iota(\overline{q})))$. By assumption $(p_2, q_2) \leq (p_0, q_0)$ and $(p_2, q_2) \Vdash \check{p} = \dot{p}$. Furthermore, $\Vdash \check{x} = \mathrm{dom}(\iota(\overline{q}))$ and thus $(p_2, q_2) \Vdash \mathrm{dom}(\iota(\overline{q})) \supseteq \mathrm{dom}(\dot{q})$. So the only way $((p_2, q_2), (p, \iota(\overline{q}))) \leq ((p_0, q_0), (\dot{p}, \dot{q}))$ can fail is if (p', q') does not force that for every $\gamma \in \mathrm{dom}(\dot{q})$, $(p \upharpoonright \gamma, \iota(\overline{q}) \upharpoonright \gamma) \Vdash \iota(\overline{q})(\gamma) \leq \dot{q}(\gamma)$, Thus let $(p_3, q_3) \leq (p_2, q_2)$ and γ be such that $(p_3, q_3) \Vdash \check{\gamma} \in \mathrm{dom}(\dot{q})$. Assume $(p_3, q_3) \Vdash (p \upharpoonright \gamma, \iota(\overline{q} \upharpoonright \gamma) \not\Vdash \iota(\overline{q})(\gamma) \leq \dot{q}(\gamma)$, i.e.

$$((p_3, q_3), (p \upharpoonright \gamma, \iota(\overline{q} \upharpoonright \gamma))) \not\Vdash \iota(\overline{q})(\gamma) \leq \dot{q}(\gamma)$$

We aim to show that this is not the case by proving $((p_3, q_3), p \upharpoonright \gamma, \iota(\overline{q}) \upharpoonright \gamma)) \Vdash \iota(\overline{q})(\check{\gamma}) = \dot{q}(\check{\gamma})$:

$\iota(\overline{q})(\check{\gamma})$ is forced to be equal to the $\mathbb{M}(\delta) * \mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \gamma)$-name corresponding to $\overline{q}(\gamma)$ which itself is forced to be equal to the $\mathbb{M}(\gamma)$-name corresponding to $q'(\gamma)$ which by definition is equivalent to $\dot{q}(\check{\gamma})$. □

The main part is that we can factor $\mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \kappa_n)$ similarly to $\mathbb{M}(\kappa_n)$.

Lemma 5.8. $\mathbb{M}(\kappa_n \setminus \delta, \dot{G}, \kappa_n)$ *is forced to be the projection of the product of a* κ_{n-1}*-cc. and a* $<\kappa_{n-1}$*-closed poset.*

Proof. Let G be $\mathbb{M}(\delta)$-generic and work in $V[G]$. As in the case of $\mathbb{M}(\kappa_n)$, write $\mathbb{M}(\kappa_n \setminus \delta, G, \kappa_n) = \mathrm{Add}(\omega, \kappa_n \setminus \delta) \times \mathbb{T}(\kappa_n \setminus \delta)$. It follows as before that the base ordering is even ω_1-cc., the term ordering is $<\kappa_{n-1}$-closed and the ordering has properties (A) and (B). □

The following supplemental Lemmas will help later:

Lemma 5.9.
(1) *For any limit ordinal* δ, $\mathbb{M}(\delta+1) \cong \mathbb{M}(\delta) * \mathrm{Add}(\omega)$
(2) *For any successor ordinal* $\delta \in [\kappa_n, \kappa_{n+1})$. $\mathbb{M}(\delta+1) \cong \mathbb{M}(\delta) * (\mathrm{Add}(\omega) \times \mathrm{Coll}(\dot{\kappa}_n, \delta))$

Proof. It is easy to verify that, for the relevant δ, the functions $(p,q) \mapsto ((p \upharpoonright \delta, q \upharpoonright \delta), p(\delta))$ and $(p,q) \mapsto ((p \upharpoonright \delta, q \upharpoonright \delta), (p(\delta), q(\delta)))$ are dense embeddings. □

Lemma 5.10. *For* $n \in \{-1\} \cup \omega$ *and* $\delta \in (\kappa_n, \kappa_{n+1}]$, $\mathbb{M}(\delta)$ *forces* $\delta = \kappa_n^+$.

Proof. This follows easily, as $\mathbb{M}(\delta)$ is δ-Knaster and for any $\mu \in [\kappa_n, \delta)$, $\mathbb{M}(\mu+2)$ collapses $\mu+1$ (and thus μ) to κ_n. □

Thus, $\mathbb{M}$ turns the sequence $(\kappa_n)_{n\in\omega}$ into $(\aleph_{n+2})_{n\in\omega}$:

Corollary 5.11. *For any* $n \in \omega$, $\mathbb{M}$ *forces* $\kappa_n = \aleph_{n+2}$.

Proof. This follows easily by induction on n, using the previous Lemma and the fact that $\mathbb{M}$ preserves $\aleph_1$ as well as every κ_n. □

We will now apply all of our results to prove the main theorems one by one.

5.1. **Disjoint stationary sequences.**

Definition 5.12. If κ is a regular cardinal, a disjoint stationary sequence on κ^+ is a sequence $(S_\alpha)_{\alpha\in S}$ such that the following holds:

(1) $S \subseteq \kappa^+ \cap \mathrm{cof}(\kappa)$ is stationary,
(2) $S_\alpha \subseteq [\alpha]^{<\kappa}$ is stationary for $\alpha \in S$,
(3) $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$.

We say that $\mathrm{DSS}(\kappa^+)$ holds if there is a disjoint stationary sequence on κ^+.

Krueger asked in [5] if it is consistent that DSS holds for successive cardinals or even an infinite interval of cardinals. Levine in [7] partially answered this question by showing that a two-step iteration of a certain variant of Mitchell forcing forces $\mathrm{DSS}(\aleph_2)\wedge\mathrm{DSS}(\aleph_3)$. We will use our tall version of Mitchell forcing to produce a model in which $\mathrm{DSS}(\aleph_n)$ holds for any $n\in\omega$, $n\geq 2$. Sadly, the naive approach of simply taking an iteration of ω many Mitchell orders cannot work: For the preservation of the κ_n's, we need our parts to be either closed or cc enough. While full support would preserve closure of the collapses, it would not preserve ccc-ness of the Cohen Forcings. On the other hand, finite support would preserve ccc-ness of the Cohen Forcings, but not closure of the collapses. That is why we are using a version of Mitchell's forcing which takes increasingly large support on the collapses and finite support for the Cohen Reals.

We will use the following facts for the addition and preservation of a particular stationary set.

Definition 5.13. A stationary set $S\subseteq[H(\Theta)]^{<\kappa}$ is internally approachable of length τ if for all $N\in S$ with $N\prec H(\Theta)$ there is a continuous chain of elementary submodels $(M_i)_{i<\tau}$ such that $N=\bigcup_{i<\tau}M_i$ and for all $j<\tau$, $(M_i)_{i<j}\in M_{j+1}$. If S is internally approachable of length τ, we write $S\subseteq IA(\tau)$.

If S is internally approachable by small sequences, S is still preserved by sufficiently closed forcing, even if S is large.

Fact 5.14. *If $S\subseteq[H(\Theta)]^{<\kappa}\cap IA(\tau)$, $\tau<\kappa$ and $\mathbb{P}$ is $<\kappa$-closed, $\mathbb{P}$ forces that S is stationary in $[H(\Theta)^V]^{<\kappa}$.*

Gitik showed in [2] that under some circumstances, adding a real also adds a new stationary set. Krueger refined this to the following:

Fact 5.15 ([5]). *Suppose $V\subseteq W$ are models of* ZFC *with the same ordinals, $W\smallsetminus V$ contains a real, κ is a regular cardinal in W, $X\in V$ is such that $(\kappa^+)^W\subseteq X$ and Θ is regular in W with $X\subseteq H(\Theta)$. Then in W the set $\{N\in[H(\Theta)]^{<\kappa}\cap IA(\omega)\mid N\cap X\notin V\}$ is stationary.*

Now we can show that consistency of $\mathrm{DSS}(\aleph_n)$ for every $n\in\omega, n\geq 2$.

Theorem 5.16. *After forcing with $\mathbb{M}$, there is a disjoint stationary sequence on every κ_n.*

Proof. First of all, the proof of Lemma 5.5 actually shows that $\mathbb{M}$ preserves all stationary subsets of κ_n for each n, since neither κ_n-cc. nor $<\kappa_n$-strongly distributive forcings can destroy stationary subsets of κ_n.

Let G be $\mathbb{M}$-generic over V and let $n\geq -1$ be arbitrary. Let S denote the set of all cardinals in the interval $[\kappa_n,\kappa_{n+1})$ that are inaccessible in V. Take $\delta\in S$ and fix the following generics:

(1) $\mathbb{M}(\delta)$, $\mathbb{M}(\delta+1)$ and $\mathbb{M}(\kappa_{n+1})$-generic filters $G(\delta)$, $G(\delta+1)$ and $G(\kappa_{n+1})$.
(2) An $(\mathbb{M}(\kappa_{n+1}) \times \mathrm{Add}(\omega, \kappa \smallsetminus \kappa_{n+1}))$-generic filter $G_A(\kappa_{n+1})$.

Work in the extension by $G(\delta+1)$. Define

$$N_\delta := \{N \in [H(\delta)]^{<\kappa_n} \cap IA(\omega) \mid N \cap \delta \notin V[G(\delta)]\}$$

We want to apply fact 5.15 to show that $N_\delta \subseteq [H(\delta)]^{<\kappa_n}$ is stationary in $V[G(\delta+1)]$. We can write $\mathbb{M}(\delta+1) \cong \mathbb{M}(\delta) * \mathrm{Add}(\omega)$. Hence, $V[G(\delta+1)] \smallsetminus V[G(\delta)]$ contains a real. Furthermore, $\mathbb{M}(\delta)$ forces $\kappa_n^+ = \delta$. The same holds for $\mathbb{M}(\delta+1)$ (as we are only adding a Cohen real). Therefore, $(\kappa_n^+)^{V[G(\delta+1)]} = \delta$. Lastly, of course $\delta \subseteq H(\delta)$.

Since for any club $C \subseteq [\delta]^{<\kappa}$, the set $\{x \in [H(\delta)]^{<\kappa} \mid x \cap \delta \in C\}$ is club, we obtain that $S_\delta := \{N \cap \delta \mid N \in N_\delta\}$ is stationary in $[\delta]^{<\kappa_n}$. Now we show that this stationarity still holds in the final extension.

Claim. *$S_\delta \subseteq [\delta]^{<\kappa_n}$ is stationary in $V[G]$.*

Proof. By Lemma 5.7, the extension $V[G(\kappa_{n+1})]$ can be viewed as an extension of $V[G(\delta+1)]$ by $\mathbb{M}(\kappa_{n+1} \smallsetminus (\delta+1), G(\delta+1))$. $\mathbb{M}(\kappa_{n+1} \smallsetminus (\delta+1), G(\delta+1))$ can be projected onto by the product of a κ_n-cc. and a κ_n-closed poset, neither of which can destroy stationary subsets of $[\delta]^{<\kappa_n}$. Therefore, in $V[G(\kappa_{n+1})]$, $S_\delta \subseteq [\delta]^{<\kappa_n}$ is still stationary. Moreover, in this extension, $|\delta| = \kappa_n$. The extension $V[G]$ can be viewed as an extension of $V[G(\kappa_{n+1})]$ by first forcing with $\mathrm{Add}(\omega, \kappa \smallsetminus \kappa_{n+1})$ and then $(\mathbb{T}(\kappa \smallsetminus \kappa_{n+1}), g(R, G_A(\kappa_{n+1})))$. $\mathrm{Add}(\omega, \kappa \smallsetminus \kappa_{n+1})$ is ω_1-Knaster and does not destroy stationary subsets of $[\delta]^{<\kappa_n}$. The base ordering $b(R)$ on $\mathbb{M}(\kappa_{n+1}) \times \mathrm{Add}(\omega, \kappa \smallsetminus \kappa_{n+1})$ is κ_{n+1}-cc. and the term ordering $t(R)$ on $(\mathbb{M}(\kappa_{n+1}) \times \mathrm{Add}(\omega, \kappa \smallsetminus \kappa_{n+1})) \times \mathbb{T}(\kappa \smallsetminus \kappa_{n+1})$ is $<\kappa_{n+1}$-closed. Hence, by Lemma 4.8, $(\mathbb{T}(\kappa \smallsetminus \kappa_{n+1}), g(R, G_A(\kappa_{n+1})))$ is strongly $<\kappa_{n+1}$-distributive in $V[G_A(\kappa_{n+1})]$. In summary, S_δ is still stationary in $V[G]$. □

Now we are done since $(S_\delta)_{\delta \in S}$ is as desired: S is still stationary in κ_{n+1} and for $\delta \in S$, $\mathbb{M}(\delta+2)$ forces $|\delta| = \mathrm{cof}(\delta) = \kappa_n$, which still holds in $V[G]$ using the same analysis as above. By construction, $S_\delta \subseteq V[G(\delta+1)] \smallsetminus V[G(\delta)]$, so the S_δ are disjoint. □

5.2. **Distinguishing internal clubness and stationarity.** Similarly to Levine's paper, the model we used for DSS at every $\aleph_{n+2}$ also has the property that internal clubness and stationarity are distinct at every $\aleph_{n+2}$. This answers another question of Krueger.

Definition 5.17. Let $\lambda \geq \kappa$ be cardinals and $N \in [\lambda]^\kappa$.

(1) N is internally unbounded if for all $x \in [N]^{<\kappa}$, there exists $M \in N$ with $x \subseteq N$.
(2) N is internally stationary if $[N]^{<\kappa} \cap N$ is stationary in $[N]^{<\kappa}$
(3) N is internally club if $[N]^{<\kappa} \cap N$ contains a club subset of $[N]^{<\kappa}$

Because $\mathbb{M}(\kappa_n)$ preserves many stationary sets but also adds many new stationary sets, we have that it forces a distinction of internal stationarity and clubness: We follow the proof of Lemma 29 from [7]. Just like in that paper, we need a concept from Harrington and Shelah:

Definition 5.18. Let $\mathcal{N}$ be a model of a fragment of ZFC. We say that $\mathcal{M} \prec \mathcal{N}$ is rich if the following hold:

(1) $\lambda \in \mathcal{M}$
(2) $\overline{\lambda} = \mathcal{M} \cap \lambda \in \lambda$
(3) $\overline{\lambda}$ is an inaccessible cardinal in $\mathcal{N}$
(4) $|\mathcal{M}| = \overline{\lambda}$
(5) $\mathcal{M}^{<\overline{\lambda}} \subseteq \mathcal{M}$

Lemma 5.19. *$\mathbb{M}(\kappa_n)$ forces that there are stationarily many $Z \in [H(\kappa_n)]^{\kappa_{n-1}}$ which are internally stationary but not internally club.*

Proof. Denote $\mathbb{M}' := \mathbb{M}(\kappa_n)$. Let $\dot{C}$ be an $\mathbb{M}'$-name for a club in $[H(\kappa_n)]^{\kappa_{n-1}}$ and $\dot{F}$ an $\mathbb{M}'$-name for a function $(\kappa_n)^{<\omega} \longrightarrow H(\kappa_n)$ such that all of its closure points are in $\dot{C}$. Denote $\mathcal{N} := (H(\Theta), \in, <, \mathbb{M}, \dot{F}, \kappa_n, \kappa_{n-1})$ (where Θ is large enough) and find a rich submodel $\mathcal{M} \prec \mathcal{N}$ with $\kappa_{n-1} \subseteq \mathcal{M}$. Denote $\overline{\kappa_n} := \kappa_n \cap \mathcal{M}$. Let $G \subseteq \mathbb{M}$ be generic. Let $\pi_{\mathcal{M}} : \mathcal{M} \longrightarrow X$ be the transitive collapse and $h := \pi_{\mathcal{M}}(\mathcal{M} \cap H(\kappa_n))$. h is transitive and $\pi_{\mathcal{M}}(\mathbb{M}') = \{\pi_{\mathcal{M}}(p) \mid p \in \mathcal{M} \cap H(\kappa_n)\} \subseteq h$, so $\pi_{\mathcal{M}}[G]$ is $\pi_{\mathcal{M}}(\mathbb{M}')$-generic over h. We can extend $\pi_{\mathcal{M}}^{-1} : h \longrightarrow \mathcal{M} \cap H(\kappa_n)$ to $\pi_{\mathcal{M}}^{-1} : h[\overline{G}] \longrightarrow \{\tau_G \mid \tau \in \mathcal{M} \cap H(\kappa_n)\} =: Z$. We shall show that Z is as required.

Claim. *Z is a closure point of $\dot{F}^G$, so $Z \in \dot{C}^G$.*

Proof. Let $\tau_0, ..., \tau_{n-1} \in \mathcal{M} \cap H(\kappa_n)$. By elementarity, the $<$-least antichain A deciding $\dot{F}(\tau_0, ..., \tau_{n-1})$ is in $\mathcal{M}$. By the κ_n-cc. and $\mathbb{M}' \subseteq H(\kappa_n)$, it is also in $H(\kappa_n)$. Thus $\pi_{\mathcal{M}}(A) \in h$ and by transitivity, $\pi_{\mathcal{M}}(A) \subseteq h$. So the value of $\pi_{\mathcal{M}}(\dot{F}^{\overline{G}}(\pi_{\mathcal{M}}(\tau_0)^{\overline{G}}, ..., \pi_{\mathcal{M}}(\tau_{n-1})^{\overline{G}})$, which is decided by $\pi_{\mathcal{M}}(A)$, is in h and thus $\dot{F}^G(\tau_0^G, ..., \tau_{n-1}^G)$ is in Z. □

The rest of the proof is devoted to showing that Z is internally stationary but not internally club.

Claim. *If $x \in h[\overline{G}]^{<\overline{\kappa_n}}$ is in $V[\overline{G}]$, $x \in h[\overline{G}]$.*

Proof. Since $\overline{\kappa_n} \subseteq h$, it suffices to show that every subset of h of size $< \overline{\kappa_n}$ that is in $V[\overline{G}]$, is in h. Let x be such a set. Since $\mathbb{M}(\overline{\kappa_n})$ has the $\overline{\kappa_n}$-cc., there exists $y \in V$, $y \subseteq h$ of size $< \overline{\kappa_n}$ with $x \subseteq y$. Hence we can find a nice name $\dot{x}$ for a subset of y with $\dot{x}^{\overline{G}} = x$. Thus $\dot{x}$ is a subset of h of size $< \overline{\kappa_n}$ in V and $\dot{x} \in h$. It follows that $\dot{x}^{\overline{G}} = x \in h[\overline{G}]$. □

Claim. *Z is internally stationary.*

Proof. In $\mathcal{N}[\overline{G}]$, $[h[\overline{G}]]^{<\kappa_{n-1}} \cap IA(\omega)$ is of course stationary in $[h[\overline{G}]]^{<\kappa_{n-1}}$. By Lemma 5.7, $\mathcal{N}[G]$ is an extension of $\mathcal{N}[\overline{G}]$ by a forcing which can be projected onto from the product of a $<\kappa_{n-1}$-closed and a κ_{n-1}-cc. forcing, both of which cannot destroy the stationarity of $[h[\overline{G}]]^{<\kappa_{n-1}} \cap IA(\omega)$. Thus, $([h[\overline{G}]]^{<\kappa_{n-1}})^{\mathcal{N}[\overline{G}]} \cap IA(\omega)$ (and in particular, $([h[\overline{G}]]^{<\kappa_{n-1}})^{\mathcal{N}[\overline{G}]}$) is stationary in $[h[\overline{G}]]^{<\kappa_{n-1}}$ in $\mathcal{N}[G]$.

In $\mathcal{N}[G]$, $|h[\overline{G}]|= |\overline{\kappa_n}|= \kappa_{n-1}$. Thus, we can write $h[\overline{G}] = \bigcup_{i<\kappa_{n-1}} x_i$, where the sequence $(x_i)_{i<\kappa_{n-1}}$ is increasing and continuous. In particular, $\{x_i \mid i < \kappa_{n-1}\}$ is club in $[h[\overline{G}]]^{<\kappa_{n-1}}$ so $\{x_i \mid i < \kappa_{n-1}\} \cap ([h[\overline{G}]]^{<\kappa_{n-1}})^{\mathcal{N}[\overline{G}]} := \{x_i \mid i \in X\}$ is stationary. Since $\pi_{\mathcal{M}}$ is a bijection, $\{\pi_{\mathcal{M}}^{-1}[x_i] \mid i \in X\}$ is stationary in $[Z]^{<\kappa_{n-1}}$, but if $i \in X$, then by the previous claim $x_i \in h[\overline{G}]$, so $\pi_{\mathcal{M}}^{-1}[x_i] = \pi_{\mathcal{M}}^{-1}(x_i) \in Z$. □

Claim. *Z is not internally club.*

Proof. Assume toward a contradiction that Z is internally club. Because $|Z|= \kappa_{n-1}$, This means that we can write $Z = \bigcup_{i<\kappa_{n-1}} z_i$, where $(z_i)_{i<\kappa_{n-1}}$ is an increasing and continuous chain and $z_i \in Z$ for any $i < \kappa_{n-1}$. Letting $w_i := \pi_{\mathcal{M}}[z_i]$, we see that since $|w_i|< \kappa_{n-1}$, $w_i = \pi_{\mathcal{M}}(z_i)$, so $(w_i)_{i<\kappa_{n-1}}$ is an increasing and continuous chain of sets in $[h[\overline{G}]]^{<\kappa_{n-1}}$ with union $h[\overline{G}]$. Work in the extension by $G(\overline{\kappa_n}+1)$. Similarly to the proof of Theorem 5.16, for some large enough χ, the set

$$U := \{A \in [H(\chi)]^{<\kappa_{n-1}} \cap IA(\omega) \mid A \cap h \notin \mathcal{N}[\overline{G}]\}$$

is stationary in $[H(\chi)]^{<\kappa_{n-1}}$ in $\mathcal{N}[G(\overline{\kappa_n}+1)]$ (as $(\kappa_{n-1}^+)^{\mathcal{N}[G(\overline{\kappa_n}+1)]} = \overline{\kappa_n} \subseteq h$). This is furthermore preserved into the extension $\mathcal{N}[G]$. Thus, the set

$$\{A \cap h[\overline{G}] \mid A \in U\}$$

is stationary in $[h[\overline{G}]]^{<\kappa_{n-1}}$ and there exists $i < \kappa_{n-1}$ with $w_i = A \cap h[\overline{G}]$ for some $A \in U$. However, this implies $w_i \notin \mathcal{N}[\overline{G}]$ by the definition, contradicting the fact that of course, $h[\overline{G}] \subseteq \mathcal{N}[\overline{G}]$. □

□

Now we will show that this distinction is preserved when going from $V[\mathbb{M}(\kappa_n)]$ to $V[\mathbb{M}]$.

The following Lemmas are basically Propositions 26 and 27 from [7], only modified to use strong distributivity.

Lemma 5.20. *Suppose $\mathbb{P}$ is strongly $<\nu$-distributive and $S \subseteq [X]^{<\delta}$ is stationary, where $|X|^{<\delta} \leq \nu$ and $\delta \leq \nu$. Then $\mathbb{P}$ preserves the stationarity of S.*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in $[X]^{<\delta}$. Fix (in V) an enumeration $[X]^{<\delta} = (x_\alpha)_{\alpha<\overline{\nu}}$ and assume without loss of generality that $\overline{\nu} = \nu$. $\mathbb{P}$ adds a function $\dot{f}$ such that for every $\alpha < \nu$, $\Vdash x_\alpha \subseteq \dot{f}(\check{\alpha})$. By strong $<\nu$-distributivity, there exists descending sequence $(p_\alpha)_{\alpha<\nu}$ such that, for every $\alpha < \nu$, p_α decides $\dot{f}(\check{\alpha})$ to be some z_α. Let D

consist of all unions of increasing chains of elements of $\{z_\alpha \mid \alpha < \nu\}$ of length $< \delta \leq \nu$. $D \in V$ and is a club in $[X]^{<\delta}$, so there exists $w \in D \cap S$. Write $w = \bigcup_{i<\bar{\delta}} z_{\xi_i}$ and let $\xi := \sup_i \xi_i$. It follows that $p_\xi \Vdash w \in \dot{C} \cap \check{S}$. □

Lemma 5.21. *Assume $\mathbb{P}_1$ has the δ-cc., $\mathbb{P}_2$ is strongly $< \nu$-distributive and X is such that $|X|^\delta \leq \nu$ with $\delta^+ \leq \nu$. If $S \subseteq [X]^\delta$ is stationary such that every $N \in S$ is internally stationary but not internally club, the same holds in any extension by $\mathbb{P}_1 \times \mathbb{P}_2$.*

Proof. S remains stationary in the extension by $\mathbb{P}_1 \times \mathbb{P}_2$ by taking it to be an extension first by $\mathbb{P}_2$ and then by $\mathbb{P}_1$ and using Lemma 5.20. If $N \in S$, then the stationarity of $[N]^{<\delta} \cap N$ in $[N]^{<\delta}$ is preserved by the same reason. Lastly, $[N]^{<\delta} \cap N$ not being internally club is preserved by $\mathbb{P}_1 \times \mathbb{P}_2$ using the same arguments since any set A is stationary if and only if its complement does not contain a club (and thus preservation of not containing a club is equivalent to preservation of stationarity). □

Now we can prove our second theorem:

Theorem 5.22. *In the extension by $\mathbb{M}$, there exists, for any $n \in \omega$, a stationary set $S \subseteq [\kappa_n]^{\kappa_{n-1}}$ such that any $N \in S$ is internally stationary but not internally club.*

Proof. We have already established that, for a given n, the statement holds in the extension by $\mathbb{M}(\kappa_n)$. Let S witness this. Let $G \subseteq \mathbb{M}$ be generic and let $G(\kappa_n)$ be the induced filter on $\mathbb{M}(\kappa_n)$. Thus $V[G]$ is an extension of $V[G(\kappa_n)]$ by $(\text{Add}(\kappa \smallsetminus \kappa_n) \times \mathbb{T}(\kappa \smallsetminus \kappa_n), g(R,G))$. By Lemma 4.2, this forcing can be projected onto by the product of $\text{Add}(\kappa \smallsetminus \kappa_n)$ and $(\mathbb{T}(\kappa \smallsetminus \kappa_n), s(g(R, G(\kappa_n)), 1_{\text{Add}(\kappa \smallsetminus \kappa_n)}))$, i.e. we are first taking the generic ordering on $\text{Add}(\kappa \smallsetminus \kappa_n) \times \mathbb{T}(\kappa \smallsetminus \kappa_n)$ with respect to $G(\kappa_n)$ and then the section ordering on $\mathbb{T}(\kappa \smallsetminus \kappa_n)$ with respect to $1_{\text{Add}(\kappa \smallsetminus \kappa_n)}$. This however is, by a small computation, the same as the generic ordering induced by the product $(\mathbb{M}(\kappa_n) \times \{1\}) \times \mathbb{T}(\kappa \smallsetminus \kappa_n)$ (ordered as a suborder) using the $\mathbb{M}(\kappa_n) \times \{1\}$-generic filter $G(\kappa_n) \times \{1\}$. $\mathbb{M}(\kappa_n) \times \{1\}$ is κ_n-cc. and the term ordering on $(\mathbb{M}(\kappa_n) \times \{1\}) \times \mathbb{T}(\kappa \smallsetminus \kappa_n)$ is $< \kappa_n$-closed, so $(\mathbb{T}(\kappa \smallsetminus \kappa_n), s(g(R, G(\kappa_n)), 1_{\text{Add}(\kappa \smallsetminus \kappa_n)}))$ is strongly $< \kappa_n$-distributive. Because $|\kappa_n|^{\kappa_{n-1}} = \kappa_n$ in $V[G(\kappa_n)]$ (by a nice name argument), $\text{Add}(\kappa \smallsetminus \kappa_n) \times (\mathbb{T}(\kappa \smallsetminus \kappa_n), s(g(R, G(\kappa_n)), 1_{\text{Add}(\kappa \smallsetminus \kappa_n)}))$ preserves that S is stationary and any $N \in S$ is internally stationary but not internally club. This is of course also true in the intermediate extension $V[G]$. □

6. Open Questions

Our open questions concern the topic of strong $< \kappa$-distributivity. The most important is probably the existence of more "natural" examples of strongly distributive forcings. Right now all of our examples come from viewing sufficiently closed forcings in sufficiently cc. extensions. It is therefore natural to ask if there are such forcings in a model which is not a forcing extension, e.g. the model L:

Question 6.1. Do there exist in L forcing notions $\mathbb{P}$ such that $\mathbb{P}$ is strongly $<\kappa$-distributive but not weakly $<\kappa$-strategically closed? In other words, is the completeness game $G(\mathbb{P},\kappa)$ determined in L?

The previous question is connected to the next one: Since all of our examples were formerly closed forcings, it is difficult to obtain examples where productivity of strong distributivity fails (even though it should, similarly to “regular“ distributivity).

Question 6.2. If $\mathbb{P}$ and $\mathbb{Q}$ are strongly $<\kappa$-distributive, is $\mathbb{P}\times\mathbb{Q}$ as well?

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