CASCADING VARIANTS OF INTERNAL APPROACHABILITY

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Abstract. We construct models in which there are stationarily many structures that exhibit different variants of internal approachability at different levels. This answers a question of Foreman. We also show that the approachability property at μ is consistent with having a distinction between variants of internal approachability for stationarily many $N \in [H(\mu^+)]^\mu$, answering a question of Levine. Our results are obtained using a new version of Mitchell Forcing.

Introduction

The notions of variants of internal approachability were introduced by Foreman and Todorcevic in [5]. We say that a set N of size μ (μ regular uncountable) is

(1) *internally unbounded* if $[N]^{<\mu} \cap N$ is unbounded in $[N]^{<\mu}$,
(2) *internally stationary* if $[N]^{<\mu} \cap N$ is stationary in $[N]^{<\mu}$,
(3) *internally club* if $[N]^{<\mu} \cap N$ contains a club in $[N]^{<\mu}$,
(4) *internally approachable* if there is a $\subseteq$-increasing and continuous sequence $(a_i)_{i\in\mu}$ of elements of $[N]^{<\mu}$ such that $\bigcup_{i\in\mu} a_i = N$ and $(a_i)_{i<j} \in N$ for every $j < \mu$.

Clearly, these properties are ordered in ascending strength (at least for sets N which are elementary submodels of some $H(\Theta)$). Much research has been done in trying to separate these properties: Krueger showed in [15], [16] and [17] that it is relatively consistent that any two properties are distinct for stationarily many $N \in [H(\Theta)]^\mu$. For the distinction between internal unboundedness and stationarity, this is only known for $\mu = \omega_1$ while the other properties can also be separated for larger μ (as has been shown by Krueger in the cited works). In [10], [11] and [20], Jakob and Levine extended Kruegers result to obtain stationarily many $N \in [H(\Theta)]^\mu$ which are internally stationary but not internally club or internally club but not internally approachable for all $\mu = \aleph_{n+1}$ ($n \in \omega$) and $\Theta > \mu$ simultaneously.

In this paper, we will focus on a different direction of research: In [3, Question 4.6], Foreman repeated a question previously asked by himself and Todorcevic in [5]: "Suppose κ is regular, $N \prec H(\Theta)$ and $N \cap [N \cap \kappa]^{\aleph_0}$ is stationary. Is $N \cap [N \cap \kappa^+]^{\aleph_0}$ stationary?". We will answer his question in the negative by showing:

Theorem 1. *Assume* MM *and* $2^{\omega_2} = \omega_3$. *There exist stationarily many* $N \in [H(\omega_3)]^{\omega_1}$ *such that* $N \cap H(\omega_2)$ *is internally approachable and* N *is not internally stationary.*

So in particular, assuming MM and $2^{\omega_2} = \omega_3$, there exists $N \prec H(\omega_3)$ with size ω_1 such that $N \cap [N \cap \omega_2]^{<\omega_1}$ is stationary but $N \cap [N \cap \omega_3]^{<\omega_1}$ is not stationary (by Lemma 2.5).

The solution to that problem prompted an investigation into related questions. We can ask if, in general, any variants of internal approachability propagate upwards. In this paper, we show that this is in general not the case:

Date: December 6, 2024.
2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E55.

Theorem 2. *Assume κ is κ^+-supercompact and $\mu < \kappa$ is regular. There is a forcing extension where $\kappa = \mu^+$ and there are stationarily many $N \in [H(\mu^{++})]^\mu$ such that $N \cap H(\mu^+)$ is internally approachable and N is internally stationary but not internally club.*

The construction used to obtain Theorem 2 can be modified to obtain a different result which answers another open problem: Under the assumption $2^\mu = \mu^+$ (in which case $|H(\mu^+)| = \mu^+$), distinctions between some variants of internal approachability are equivalent to the existences of certain combinatorial objects: Krueger showed in [17, Theorem 6.5] that there is a *disjoint stationary sequence on μ^+* if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club and a folklore theorem (see e.g. [1, Lemma 1]) states that the *approachability property fails at μ*, i.e. $\mu^+ \notin I[\mu^+]$, if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable. We will show that in both theorems the assumption $2^\mu = \mu^+$ cannot be relaxed by showing the following, answering a question of Levine (see [20, Question 1]):

Theorem 3. *Assume κ is κ^+-supercompact and $\mu < \kappa$ is regular. There is a forcing extension where $\kappa = 2^{<\mu} = \mu^+$, $2^\mu = \mu^{++}$ and the following holds:*

(1) $\mu^+ \in I[\mu^+]$ *(so there does not exist a disjoint stationary sequence on μ^+),*
(2) *there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally stationary but not internally club.*

The paper is organized as follows: In the first section, we introduce known definitions and results that will be used throughout the paper. In the second section, we prove Theorem 1. In the third section, we define our new variant of Mitchell forcing. In the fourth section, we prove a theorem which allows us to easily obtain both Theorem 2 and Theorem 3. In the fifth section, we introduce new ordinal variants of internal approachability and give a criterion for DSS and AP with relaxed cardinal arithmetic assumptions. We close with a few open questions.

Acknowledgements. The author wants to thank the anonymous referee to an earlier version of this paper for their diligent reading and immensely helpful referee report leading to improvement of the manuscript.

1. Preliminaries

We assume the reader is familiar with the basics of forcing and the study of large cardinals. Good introductory material can be found in [13], [18] and [14].

We denote the following forcing notions: For any regular cardinal δ, $\mathrm{Add}(\delta)$ consists of $<\delta$-sized partial functions from δ to 2, ordered by $\supseteq$. $\mathrm{Coll}(\mu, \delta)$ consists of $<\mu$-sized partial functions from μ to δ, ordered by $\supseteq$. $\mathrm{Add}(\delta)$ is $<\delta$-closed, $(2^{<\delta})^+$-cc. and adds a new subset of δ, while $\mathrm{Coll}(\mu, \delta)$ is $<\mu$-closed, $(\delta^{<\mu})^+$-cc. and adds a surjection from μ onto δ.

Shelah introduced the *approachability ideal* in [23] in order to obtain results regarding the preservation of stationary subsets of $\delta \cap \mathrm{cof}(<\mu)$ by $<\mu$-closed forcing notions.

Definition 1.1. Let μ be a cardinal.

(1) The *approachability ideal on μ^+*, denoted by $I[\mu^+]$, is defined as follows: $A \in I[\mu^+]$ if there exists a sequence $(a_\alpha)_{\alpha<\mu^+}$ of elements of $[\mu^+]^{<\mu}$ and a club $C \subseteq \mu^+$ such that whenever $\gamma \in A \cap C$, there exists $E \subseteq \gamma$ with $\mathrm{otp}(E) = \mathrm{cf}(\gamma)$ such that E is unbounded in γ and $\{E \cap \alpha \mid \alpha < \gamma\} \subseteq \{a_\alpha \mid \alpha < \gamma\}$ (we say that γ *is approachable with respect to $(a_\alpha)_{\alpha<\mu^+}$*).

(2) The *approachability property at μ*, denoted by AP_μ, is the statement that $\mu^+ \in I[\mu^+]$.

As has been shown by Shelah in [23], for any regular cardinal μ, $I[\mu^+]$ is a $<\mu^+$-complete normal ideal, so it is in particular closed under $<\mu^+$-sized unions. Furthermore, Shelah showed in [23, Lemma 4.4] that $\mu^+ \cap \mathrm{cof}(<\mu) \in I[\mu^+]$ for any regular cardinal μ, so that AP_μ is equivalent to stating that $\mu^+ \cap \mathrm{cof}(\mu) \in I[\mu^+]$.

The approachability ideal relates to the variants of internal approachability as follows (see [1, Lemma 1]):

Theorem 1.2 (Folklore)**.** *Let μ be a regular uncountable cardinal with $2^\mu = \mu^+$. Then AP_μ fails if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable.*

Krueger introduced the notion of a *disjoint stationary sequence* in [17]:

Definition 1.3. Let μ be a regular uncountable cardinal. $\mathrm{DSS}(\mu^+)$ states that there exists a *disjoint stationary sequence on μ^+*, i.e. a sequence $(\mathcal{S}_\alpha)_{\alpha\in S}$ such that the following holds:

(1) $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary.
(2) For all $\alpha \in S$, $\mathcal{S}_\alpha$ is stationary in $[\alpha]^{<\mu}$ and for all $\alpha \neq \beta$, both in S, $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \emptyset$.

Krueger related the existence of a disjoint stationary sequence to the previous properties (see [17, Corollary 3.7, Theorem 6.5]):

Theorem 1.4. *Let μ be a regular cardinal.*

(1) *If there exists a disjoint stationary sequence on μ^+, AP_μ fails.*
(2) *If $2^\mu = \mu^+$, then there exists a disjoint stationary sequence on μ^+ if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club.*

For the preservation of stationarity in $[X]^{<\mu}$, we use the notion of *weak internal approachability* (see [4, Definition 2.1]):

Definition 1.5. Let Θ be a cardinal and $N \prec H(\Theta)$, we say that N is *weakly internally approachable of length τ* and write $N \in \mathrm{IA}(\tau)$ if there is an $\subseteq$-increasing and continuous sequence $(N_i)_{i<\tau}$ with $N = \bigcup_{i<\tau} N_i$ and $(N_i)_{i<j} \in N$ for every $j < \tau$.

This notion is enough to ensure stationary sets are preserved by sufficiently closed forcing notions (see the combination of Lemma 2.2 and Lemma 2.4 in [17]):

Fact 1.6. *Assume $S \subseteq [H(\Theta)]^{<\mu}$ is stationary and $S \subseteq \mathrm{IA}(\tau)$ for some regular cardinals $\tau < \mu < \Theta$. If $\mathbb{P}$ is $<\mu$-closed and G is $\mathbb{P}$-generic, S is stationary in $[H(\Theta)^V]^{<\mu}$ in $V[G]$.*

We will also make use of the following results due to Menas:

Fact 1.7. *Assume $X \subseteq Y \subseteq Z$ are sets and μ is a regular cardinal.*

(1) *If $C \subseteq [Y]^{<\mu}$ is a club, there is a function $F\colon [Y]^{<\omega} \to [Y]^{<\mu}$ such that*
$$\mathrm{cl}_F := \{a \in [Y]^{<\mu} \mid \forall y \in [a]^{<\omega}\ F(y) \subseteq a\} \subseteq C$$
and for any $F\colon [Y]^{<\omega} \to [Y]^{<\mu}$, cl_F is club in $[Y]^{<\mu}$.
(2) *If $C \subseteq [Y]^{<\mu}$ is club, then*
$$C \upharpoonright X := \{a \cap X \mid a \in C\}$$
contains a club in $[X]^{<\mu}$ and
$$C \uparrow Z := \{a \in [Z]^{<\mu} \mid a \cap Y \in C\}$$
is club in $[Z]^{<\mu}$.

(3) *If* $S \subseteq [Y]^{<\mu}$ *is stationary, then*

$$S \restriction X := \{a \cap X \mid a \in S\}$$

and

$$S \uparrow Z := \{a \in [Z]^{<\mu} \mid a \cap Y \in S\}$$

are stationary in $[X]^{<\mu}$ *and* $[Z]^{<\mu}$ *respectively.*

Proof. We prove the first part of (2), (1) is [21, Theorem 1.5] and the other parts follow easily.

Assume $C \subseteq [Y]^{<\mu}$ is club and $F\colon [Y]^{<\omega} \to [Y]^{<\mu}$ is such that $\mathrm{cl}_F \subseteq C$. Let $G\colon [X]^{<\omega} \to [X]^{<\mu}$ be such that $G(x)$ is the closure of x under F, intersected with X. If $x \in [X]^{<\mu}$ is closed under G, the closure of x under F, intersected with X, is equal to x. Ergo $x = y \cap X$ for some $y \in \mathrm{cl}_F \subseteq C$. □

In order to obtain models which are internally unbounded but not internally stationary (or internally stationary but not internally club), we use a construction due to Krueger: Refining a theorem of Gitik from [8], Krueger showed the following (see [17, Theorem 7.1]):

Fact 1.8. *Suppose* $V \subseteq W$ *are models of* ZFC *with the same ordinals and there is a real in* $W \setminus V$*. Let* μ *be a regular uncountable cardinal in* W *and let* X *be a set in* V *such that* $(\mu^+)^W \subseteq X$*. In* W *let* $\Theta \geq \mu^+$ *be a regular cardinal such that* $X \subseteq H(\Theta)^W$*. Then in* W *the set of all* $N \in [H(\Theta)^W]^{<\mu} \cap \mathrm{IA}(\omega)$ *with* $N \cap X \notin V$ *is stationary in* $[H(\Theta)^W]^{<\mu}$*.*

By combining this with Fact 1.6 and Fact 1.7 we can see that e.g. in $V[\mathrm{Add}(\omega)]$, $[\omega_2]^{<\omega_1}$ naturally splits into two disjoint stationary sets, namely $[\omega_2]^{<\omega_1} \cap V$ and $[\omega_2]^{<\omega_1} \setminus V$, whose stationarity is preserved by countably closed forcing. So if we want to collapse ω_2 to ω_1 we can either use $\mathrm{Coll}(\omega_1, \omega_2)$ and preserve the stationarity of both sets or shoot a club through one of them. This is done using the following poset which first occurred in [16, Definition 2.5]:

Definition 1.9. Let μ be a regular uncountable cardinal and X a set. Let $S \subseteq [X]^{<\mu}$ be stationary. $\mathbb{P}(S)$ consists of functions $p\colon \alpha \to S$ such that $\alpha < \mu$ is a successor ordinal and p is $\subseteq$-increasing and continuous, ordered by end-extension.

It is easy to see that $\mathbb{P}(S)$ collapses $|X|$ to $|\mu|$. Under additional assumptions, $\mathbb{P}(S)$ is $<\mu$-distributive and thus preserves all cardinals up to (and including) μ.

As is common for variants of Mitchell Forcing, we will use a projection analysis.

Definition 1.10. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be posets. Then $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *projection* if the following holds:

(1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
(2) If $p' \leq_{\mathbb{P}} p$, $\pi(p') \leq_{\mathbb{Q}} \pi(p)$.
(3) If $q \leq_{\mathbb{Q}} \pi(p)$, there is $p' \leq_{\mathbb{P}} p$ with $\pi(p') \leq_{\mathbb{Q}} q$.

Projections are used for the following result (which is part of the folklore):

Fact 1.11. *Assume* $\mathbb{P}$ *and* $\mathbb{Q}$ *are posets and* $\pi\colon \mathbb{P} \to \mathbb{Q}$ *a projection. Let* H *be* $\mathbb{Q}$*-generic. In* $V[H]$*, let* $\mathbb{P}/H$ *consist of all those* $p \in \mathbb{P}$ *such that* $\pi(p) \in H$*. Then whenever* G *is* $\mathbb{P}/H$*-generic over* $V[H]$*,* G *is* $\mathbb{P}$*-generic over* V *and* $V[H][G] = V[G]$*. Ergo,* $V[G]$ *is an extension of* $V[H]$ *using the poset* $\mathbb{P}/H$*.*

2. An Application of Martin's Maximum

In this section, we show that MM implies that there are stationarily many structures which are "cascadingly internally approachable". Recall that MM (which was introduced in [6]) states that whenever $\mathbb{P}$ preserves stationary subsets of ω_1 and $(D_\alpha)_{\alpha<\omega_1}$ is a sequence of dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ which intersects every D_α. We will use the following reformulation due to Woodin (see [24], in the proof of Theorem 2.53): Recall that for a poset $\mathbb{P}$ and a set N we say that G is $\mathbb{P}$-*generic over* N if for every $D \subseteq \mathbb{P}$ which is an element of N and dense in $\mathbb{P}$ there is $p \in G \cap D \cap N$.

Fact 2.1. *Assume* MM. *Let Θ be a regular cardinal. Whenever $\mathbb{P} \in H(\Theta)$ preserves stationary subsets of ω_1 there are stationarily many $N \in [H(\Theta)]^{<\omega_2}$ such that there exists a $\mathbb{P}$-generic filter over N.*

Define the following forcing notion:

$$\mathbb{P} := \mathrm{Coll}(\omega_1, \omega_2) * \mathrm{Add}(\omega) * \mathbb{P}([\check{\omega}_3]^{<\omega_1} \smallsetminus V[\mathrm{Coll}(\omega_1, \omega_2)])$$

$\mathrm{Coll}(\omega_1, \omega_2)$ is countably closed and thus preserves stationary subsets of ω_1. MM implies $2^\omega = \omega_2$ and thus $\omega_2^\omega = \omega_2$. Ergo $|\mathrm{Coll}(\omega_1, \omega_2)| = \omega_2$ and the poset forces $\check{\omega}_3 = \dot{\omega}_2$. Friedman and Krueger show the following in [7, Proposition 4.3, Theorem 4.4]:

Fact 2.2. $\mathrm{Add}(\omega)$ *forces that $\mathbb{P}([\omega_2]^{<\omega_1} \smallsetminus V)$ is $<\omega_1$-distributive. The two-step iteration* $\mathrm{Add}(\omega) * \mathbb{P}([\omega_2]^{<\omega_1} \smallsetminus V)$ *preserves stationary subsets of ω_1.*

Since $\omega_3^V = \omega_2^{V[\mathrm{Coll}(\omega_1,\omega_2)]}$, $\mathbb{P}$ indeed preserves stationary subsets of ω_1.

We show that $\mathbb{P}$ collapses $H(\omega_2)^V$ and $H(\omega_3)^V$ to have size ω_1 in such a way that $H(\omega_2)^V$ becomes internally approachable and $H(\omega_3)^V$ does not become internally stationary.

Lemma 2.3. *Assume $2^{\omega_2} = \omega_3$. $\mathbb{P}$ forces the existence of $\subseteq$-increasing and continuous sequences $(a_i)_{i\in\omega_1}$ and $(b_i)_{i\in\omega_1}$ with the following properties:*

(1) $\bigcup_{i\in\omega_1} a_i = H(\omega_2)^V$ *and for all $j < \omega_1$, $(a_i)_{i<j} \in [H(\omega_2)^V]^{<\omega_1} \cap H(\omega_2)^V$.*
(2) $\bigcup_{i\in\omega_1} b_i = H(\omega_3)^V$ *and for all $i < \omega_1$, $b_i \in [H(\omega_3)^V]^{<\omega_1} \smallsetminus H(\omega_3)^V$.*

Proof. Let G be $\mathbb{P}$-generic. Fix filters H, I and J such that $G = H * I * J$.

In V, $|H(\omega_2)| = |\omega_2^{\omega_1}| = \omega_2$ by MM. In $V[H]$, $|H(\omega_2)^V| = \omega_1$. Let $F\colon \omega_1 \to H(\omega_2)^V$ be a bijection in $V[H]$. By the countable distributivity of $\mathrm{Coll}(\omega_1, \omega_2)$, the sequence $(a_i)_{i\in\omega_1} := (F[i])_{i\in\omega_1}$ clearly has the required properties.

Let $F' := \bigcup J$. Since $\mathbb{P}([\omega_3^V]^{<\omega_1} \smallsetminus V[H])$ is $<\omega_1$-distributive in $V[H * I]$, for any $\alpha < \omega_1$ the set $D_\alpha := \{p \in \mathbb{P}([\omega_3^V]^{<\omega_1} \smallsetminus V[H]) \mid \alpha \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}([\omega_3^V]^{<\omega_1} \smallsetminus V[H])$. By genericity F' is a $\subseteq$-increasing, continuous and cofinal function from ω_1 into $[\omega_3^V]^{<\omega_1} \smallsetminus V[G] \subseteq [\omega_3^V]^{<\omega_1} \smallsetminus V$. In V, let $F''\colon \omega_3^V \to H(\omega_3)^V$ be a bijection. Then the sequence $(b_i)_{i\in\omega_1} := (F''[F'(i)])_{i\in\omega_1}$ is again easily seen to be as required (since $F'' \in V$, $F'(i) \notin V$ implies $F''[F'(i)] \notin V$). $\square$

This enables us to show Theorem 1. Note that the assumptions of the theorem are consistent (relative to a supercompact cardinal) because MM can be forced assuming the existence of a supercompact cardinal (see [6, Theorem 5]) and is preserved by $<\omega_2$-directed closed forcing (see [19, Theorem 4.3])which can be used to force $2^{\omega_2} = \omega_3$.

Theorem 2.4. *Assume* MM *and $2^{\omega_2} = \omega_3$. Then there exist stationarily many $N \in [H(\omega_3)]^{\omega_1}$ such that $N \cap H(\omega_2)$ is internally approachable and N is not internally stationary.*

Proof. Fix names $(\dot{a}_i)_{i\in\omega_1}$ and $(\dot{b}_i)_{i\in\omega_1}$ for sequences exemplifying Lemma 2.3. Let $C \subseteq [H(\omega_3)]^{\omega_1}$ be club and let Θ be large enough so that $\mathbb{P} \in H(\Theta)$. Then $\{N \in [H(\Theta)]^{\omega_1} \mid N \cap H(\omega_3) \in C \wedge \omega_1 \subseteq N\}$ is club in $[H(\Theta)]^{\omega_1}$ by Fact 1.7 and by Fact 2.1 we can find $N \prec H(\Theta)$ with size ω_1 such that $\mathbb{P}, (\dot{a}_i)_{i\in\omega_1}, (\dot{b}_i)_{i\in\omega_1} \in N$, there exists a $\mathbb{P}$-generic filter G over N, $\omega_1 \subseteq N$ and $N \cap H(\omega_3) \in C$. We will show that $N \cap H(\omega_2)$ is internally approachable and $N \cap H(\omega_3)$ is not internally stationary.

To this end, we let (for $i \in \omega_1$) a_i (resp. b_i) consist of all $x \in N \cap H(\omega_2)$ (resp. $x \in N \cap H(\omega_3)$) such that for some $p \in G \cap N$, $p \Vdash \check{x} \in \dot{a}_i$ (resp. $p \Vdash \check{x} \in \dot{b}_i$).

Claim. *$\bigcup_{i\in\omega_1} a_i = N \cap H(\omega_2)$ and for every $j < \omega_1$, $(a_i)_{i<j} \in N \cap H(\omega_2)$.*

Proof. Let $x \in N \cap H(\omega_2)$. By Lemma 2.3 the set D of all $p \in \mathbb{P}$ which force $\check{x} \in \dot{a}_i$ for some $i \in \omega_1$ is dense in $\mathbb{P}$ and lies in N. Since G is $\mathbb{P}$-generic over N, there is $p \in D \cap G \cap N$ which witnesses that $x \in a_i$ for some $i \in \omega_1$.

For $j < \omega_1$, the sequence $(\dot{a}_i)_{i<j}$ is in N and therefore so is the set D of all $p \in \mathbb{P}$ which force $(\dot{a}_i)_{i<j} = \check{x}$ for some $x \in H(\omega_2)$ (which is dense in $\mathbb{P}$ by Lemma 2.3). So there is $p \in G \cap D \cap N$. By elementarity the corresponding x is in N as well. We will show $x = (a_i)_{i<j}$ (clearly x is a function with domain j). If $y \in x(i)$, $p \Vdash \check{y} \in \dot{a}_i$, so $y \in a_i$ (since $x(i) \subseteq N$ as it is an element of N and countable). If $y \in a_i$, there is $q \in G \cap N$ forcing $\check{y} \in \dot{a}_i$. Because G is a filter and generic over N, there is $r \leq p, q$ in $G \cap N$. Then r forces $\check{y} \in \check{x}(\check{i})$, so $y \in x(i)$. ☐

In summary, $(a_i)_{i\in\omega_1}$ witnesses that N is internally approachable.

Claim. *$\bigcup_{i\in\omega_1} b_i = N \cap H(\omega_3)$ and for every $i \in \omega_1$, $b_i \notin N \cap H(\omega_3)$.*

Proof. $\bigcup_{i\in\omega_1} b_i = N \cap H(\omega_3)$ follows just as before.

Let $i \in \omega_1$. For any $x \in N \cap H(\omega_3)$, the set D of all $p \in \mathbb{P}$ such that for some y, $p \Vdash \check{y} \in \check{x} \triangle \dot{b}_i$ is in N and dense in $\mathbb{P}$ (since $\dot{b}_i$ is forced not to be in $H(\omega_3)^V$ by Lemma 2.3). Ergo there is $p \in G \cap D \cap N$. The corresponding y is in N as well by elementarity. Then either $p \Vdash \check{y} \in \check{x} \setminus \dot{b}_i$, in which case $y \in x \setminus b_i$ (as G is a filter, so no $q \in G$ can force $\check{y} \in \dot{b}_i$) or $p \Vdash \check{y} \in \dot{b}_i \setminus \check{x}$, in which case $y \in b_i \setminus x$. So $b_i \neq x$. As x was arbitrary, $b_i \notin N \cap H(\omega_3)$. ☐

In particular, $\{b_i \mid i \in \omega_1\}$ is club in $[N \cap H(\omega_3)]^{<\omega_1}$ and disjoint from $N \cap H(\omega_3)$, so $N \cap H(\omega_3)$ is not internally stationary. ☐

The last ingredient we need to answer Foreman's question is the following:

Lemma 2.5. *Let $M \prec H(\Theta)$ have size μ and let $X, Y \in M$ be such that $|X| = |Y|$. Then $M \cap [M \cap X]^{<\mu}$ is stationary in $[M \cap X]^{<\mu}$ if and only if $M \cap [M \cap Y]^{<\mu}$ is stationary in $[M \cap Y]^{<\mu}$ and $M \cap [M \cap X]^{<\mu}$ contains a club in $[M \cap X]^{<\mu}$ if and only if $M \cap [M \cap Y]^{<\mu}$ contains a club in $[M \cap Y]^{<\mu}$.*

Proof. By elementarity, M contains a bijection F between X and Y. We just prove one statement (the others follow with the same arguments). Assume $M \cap [M \cap X]^{<\mu}$ is stationary in $[M \cap X]^{<\mu}$ and $c \subseteq [M \cap Y]^{<\mu}$ is a club. By elementarity, F restricts to a bijection between $M \cap X$ and $M \cap Y$, so $\{F^{-1}[x] \mid x \in c\}$ is club in $[M \cap X]^{<\mu}$. Ergo there exists $x \in c$ with $F^{-1}[x] \in M$. But then $x = F[F^{-1}[x]] \in M$. ☐

Corollary 2.6. *Assume* MM *and $2^{\omega_2} = \omega_3$. Then there are stationarily many $N \in [H(\omega_3)]^{\omega_1}$ such that $N \cap [N \cap \omega_2]^{<\omega_1}$ is stationary in $[N \cap \omega_2]^{<\omega_1}$ and $N \cap [N \cap \omega_3]^{<\omega_1}$ is not stationary in $[N \cap \omega_3]^{<\omega_1}$.*

Proof. Let $C \subseteq [H(\omega_3)]^{\omega_1}$ be club. Using Fact 1.7 and Theorem 2.4, find $M \prec H(\Theta)$ for Θ large enough with $M \cap H(\omega_3) \in C$ such that $M \cap H(\omega_2)$ is internally approachable and $M \cap H(\omega_3)$ is not internally stationary. In particular, $M \cap H(\omega_2)$ is internally stationary.

It follows easily that $M \cap [M \cap H(\omega_2)]^{<\omega_1}$ is stationary in $[M \cap H(\omega_2)]^{<\omega_1}$ and $M \cap [M \cap H(\omega_3)]^{<\omega_1}$ is not stationary in $[M \cap H(\omega_2)]^{<\omega_1}$. So Lemma 2.5 implies the desired result since $|H(\omega_2)| = \omega_2$ and $|H(\omega_3)| = \omega_3$. $\square$

3. The New Mitchell Forcing

In the literature, there are (for the most part) the following two distinct versions of Mitchell's poset: The original version, used by Mitchell in [22] to obtain the consistency of the tree property at successor cardinals (and used by Levine in [20] to distinguish internal stationarity from clubness), collapses cardinals after adding a real in order to obtain that many quotients by initial segments have the so-called approximation property. A different variant, used e.g. in [2], collapses cardinals immediately at limit steps. Thanks to this, the poset forces the approachability property to hold (while the original forcing forces it to fail). In this section, we introduce a new variant of Mitchell forcing which incorporates both ideas in order to obtain a model where the approachability property holds but "higher up" there still is a distinction between internal stationarity and clubness.

We use the following version of Cohen forcing: For a set X of ordinals, let $\mathrm{Add}^{\oplus}(\omega, X)$ consist of finite functions p on X such that whenever $\alpha \in \mathrm{dom}(p)$, $\alpha = \delta + 1$ for a cardinal δ and $p(\alpha) \in \mathrm{Add}(\omega)$.

Definition 3.1. Let μ be regular uncountable and ξ an ordinal. $\mathbb{M}(\mu, \xi)$ consists of pairs (p, q) such that the following holds:

(1) $p \in \mathrm{Add}^{\oplus}(\omega, \xi)$.
(2) q is a $<\mu$-sized partial function on ξ such that for all $\alpha \in \mathrm{dom}(q)$, $\alpha = \delta$ or $\alpha = \delta + 2$ for a limit cardinal δ and $q(\alpha)$ is an $\mathrm{Add}(\omega, \alpha)$-name either for an element of $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta})$ (if $\alpha = \delta$) or $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta^+})$ (if $\alpha = \delta + 2$).

We let $(p', q') \leq (p, q)$ if

(1) $p' \leq p$ in $\mathrm{Add}^{\oplus}(\omega, \xi)$,
(2) $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$ and for all $\alpha \in \mathrm{dom}(q)$,

$$p' \upharpoonright \alpha \Vdash q'(\alpha) \leq q(\alpha)$$

For the rest of the paper, we fix a regular uncountable cardinal μ. As is common when working with variants of Mitchell Forcing, we explicitely define the quotient forcings of different instances:

Definition 3.2. Let $\xi < \kappa$ be ordinals and let G be an $\mathbb{M}(\mu, \xi)$-generic filter. In $V[G]$, let $\mathbb{M}(G, \mu, \kappa \setminus \xi)$ consist of pairs (p, q) such that the following holds:

(1) $p \in \mathrm{Add}^{\oplus}(\omega, \kappa \setminus \xi)$.
(2) q is a $<\mu$-sized partial function on $\kappa \setminus \xi$ such that for all $\alpha \in \mathrm{dom}(q)$, $\alpha = \delta$ or $\alpha = \delta + 2$ for a limit cardinal δ (in V) and $q(\alpha)$ is an $\mathrm{Add}(\omega, \alpha \setminus \xi)$-name either for an element of $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta})$ (if $\alpha = \delta$) or $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta^+})$ (if $\alpha = \delta + 2$).

We let $(p', q') \leq (p, q)$ if

(1) $p' \leq p$ in $\mathrm{Add}^{\oplus}(\omega, \kappa \setminus \xi)$,
(2) $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$ and for all $\alpha \in \mathrm{dom}(q)$,

$$p' \upharpoonright (\alpha \setminus \xi) \Vdash q'(\alpha) \leq q(\alpha)$$

We also explicitely define a term ordering which enables us to obtain a projection analysis of our forcings:

Definition 3.3. Let $\mathbb{T}(\mu,\kappa)$ (resp. $\mathbb{T}(G,\mu,\kappa\setminus\xi)$) consist of those $(p,q)\in\mathbb{M}(\mu,\kappa)$ (resp. $(p,q)\in\mathbb{M}(G,\mu,\kappa\setminus\xi)$) where $p=\emptyset$, ordered as a suborder of $\mathbb{M}(\mu,\kappa)$ (resp. $\mathbb{M}(G,\mu,\kappa\setminus\xi)$).

The following properties are standard for variants of Mitchell forcing, see e.g. Section 2 in [20]. We prove them here because the exact variant we use has not been used anywhere before.

Lemma 3.4. *Let $\xi<\kappa$ be ordinals above μ such that κ is an inaccessible cardinal.*

(1) *$\mathbb{M}(\mu,\kappa)$ is κ-Knaster.*
(2) *There is a projection from $\mathrm{Add}^{\oplus}(\omega,\kappa)\times\mathbb{T}(\mu,\kappa)$ onto $\mathbb{M}(\mu,\kappa)$.*
(3) *If G is $\mathbb{M}(\mu,\xi)$-generic, in $V[G]$ there is a projection from $\mathrm{Add}^{\oplus}(\omega,\kappa\setminus\xi)\times\mathbb{T}(G,\mu,\kappa\setminus\xi)$ onto $\mathbb{M}(G,\mu,\kappa)$.*
(4) *$\mathbb{T}(\mu,\kappa)$ is $<\mu$-closed and $\mathbb{T}(G,\mu,\kappa\setminus\xi)$ is $<\mu$-closed in $V[G]$ whenever G is $\mathbb{M}(\mu,\xi)$-generic.*
(5) *If $x\in[V]^{<\mu}\cap V[\mathbb{M}(\mu,\kappa)]$, $x\in V[\mathrm{Add}^{\oplus}(\omega,\kappa)]$.*

Proof. We prove the statements one by one.

(1) This is an easy application of the Δ-System-Lemma.
(2) We let π map $(p,(\emptyset,q))$ to (p,q). It is easy to see that π preserves the ordering. Let $(p',q')\leq(p,q)=\pi(p,(\emptyset,q))$. Using standard arguments on names, let q'' be a function with domain $\mathrm{dom}(q')$ such that for all $\alpha\in\mathrm{dom}(q')$, $q''(\alpha)$ is forced by p' to be equal to $q'(\alpha)$ and by conditions incompatible with p' to be equal to $q(\alpha)$ (or $\emptyset$, if $\alpha\notin\mathrm{dom}(q)$). Then $(p',(\emptyset,q''))\leq(p,(\emptyset,q))$ and $\pi(p',(\emptyset,q''))=(p',q'')\leq(p',q')$.
(3) Just as in (2).
(4) We do the proof for $\mathbb{T}(\mu,\kappa)$. Given a sequence $(\emptyset,q_i)_{i<\tau}$ ($\tau<\mu$), let q be a function with domain $\bigcup_{i<\tau}\mathrm{dom}(q_i)$ (which has size $<\mu$) defined as follows: For any $\alpha\in\mathrm{dom}(q)$, the sequence $(q_i(\alpha))_{i_0\leq i<\tau}$ (where $\alpha\in\mathrm{dom}(q_{i_0})$) is forced by the empty condition be descending in $\dot{\mathrm{Coll}}(\check{\mu},\check{\nu})$ for some cardinal ν, so by the maximum principle we can fix an $\mathrm{Add}^{\oplus}(\omega,\alpha)$-name $q(\alpha)$ forced by $\emptyset$ to be a lower bound. Then $(\emptyset,q)$ is a lower bound of $(\emptyset,q_i)_{i<\tau}$.
(5) By (2), $V[\mathbb{M}(\mu,\kappa)]$ is contained in an extension by $\mathrm{Add}^{\oplus}(\omega,\kappa)\times\mathbb{T}(\mu,\kappa)$. By Easton's Lemma, $\mathbb{T}(\mu,\kappa)$ is $<\mu$-distributive in $V[\mathrm{Add}^{\oplus}(\omega,\kappa)]$. So any $x\in V[\mathbb{M}(\mu,\kappa)]$ is in $V[\mathrm{Add}^{\oplus}(\omega,\kappa)\times\mathbb{T}(\mu,\kappa)]$ and thus in $V[\mathrm{Add}^{\oplus}(\omega,\kappa)]$.

□

It follows that $\mathbb{M}(\mu,\kappa)$ preserves all cardinals up to (and including) μ. Furthermore, it is easy to see that $\mathbb{M}(\mu,\kappa)$ collapses any cardinal between μ and κ, so it forces $\kappa=\mu^+$. So in particular, we have the following (by a nice name argument and since $|\mathbb{M}(\mu,\kappa)|=\kappa$):

Corollary 3.5. $\mathbb{M}(\mu,\kappa)$ *forces* $2^{\omega}=2^{\mu}=\mu^+=\kappa$ *and* $2^{\kappa}=(2^{\kappa})^V$.

We also show that $\mathbb{M}(G,\mu,\kappa\setminus\xi)$ works as intended:

Lemma 3.6. *Let $\xi<\kappa$ be ordinals above μ. There is a dense embedding from $\mathbb{M}(\mu,\kappa)$ into $\mathbb{M}(\mu,\xi)*\mathbb{M}(\Gamma,\mu,\kappa\setminus\xi)$.*

Proof. We first describe the mapping ι and then show that it works. Let $(p,q)\in\mathbb{M}(\mu,\kappa)$. Then we map (p,q) to the pair $((p\restriction\xi,q\restriction\xi),\check{p}\restriction[\check{\xi},\check{\kappa}),\overline{q}))$, where $\overline{q}$ is an $\mathbb{M}(\mu,\xi)$-name with the following properties: For any $\mathbb{M}(\mu,\xi)$-generic filter G, $\overline{q}^G$ is a function with domain $\mathrm{dom}(q)\cap[\xi,\kappa)$ and for $\alpha\in\mathrm{dom}(\overline{q}^G)$, $\overline{q}^G(\alpha)$ is the $\mathrm{Add}^{\oplus}(\omega,\alpha\setminus\xi)$-name for the partial evaluation of $q(\alpha)$ (which is an $\mathrm{Add}^{\oplus}(\omega,\alpha)\cong\mathrm{Add}^{\oplus}(\omega,\xi)*\mathrm{Add}^{\oplus}(\omega,\alpha\setminus\xi)$-name) by the $\mathrm{Add}^{\oplus}(\omega,\xi)$-generic filter induced by G. It is easy to see that ι is well-defined and preserves the ordering.

Let us show that the image of ι is dense in $\mathbb{M}(\mu,\xi) * \mathbb{M}(\mu,\kappa \setminus \xi)$. Let $((p,q),\tau)$ be arbitrary. Strengthening (p,q) to (p',q'), we can assume that τ is forced to be of the form $(\check{r},\dot{s})$ for $r \in \mathrm{Add}^{\oplus}(\omega,\kappa \setminus \xi)$. By the $<\mu$-covering property of $\mathbb{M}(\mu,\xi)$, there is $x \in V$ and $(p'',q'') \leq (p',q')$ such that $(p'',q'') \Vdash \mathrm{dom}(\dot{s}) \subseteq \check{x}$. By enlarging $\dot{s}$, we can assume that equality is forced. Furthermore, $\dot{s}$ is forced to be an element of $V[\mathrm{Add}^{\oplus}(\omega,\xi)]$ (by the argument given for Lemma 3.4 (5)), so we can once again find $(p''',q''') \leq (p'',q'')$ and an $\mathrm{Add}^{\oplus}(\omega,\xi)$-name $\dot{s}'$ such that $(p''',q''') \Vdash \dot{s}' = \dot{s}$.

We are now ready to define a preimage: Let $t := p''' \cup r$ and let u be defined as follows: u is a function with domain $\mathrm{dom}(q''') \cup x$ (which has size $<\mu$). For $\alpha \in \mathrm{dom}(q''')$, $u(\alpha) = q'''(\alpha)$. For $\alpha \in x$, $\dot{s}'(\check{\alpha})$ is an $\mathrm{Add}^{\oplus}(\omega,\xi)$-name for an $\mathrm{Add}^{\oplus}(\omega,\alpha \setminus \xi)$-name, so we can let $u(\alpha)$ be the corresponding $\mathrm{Add}^{\oplus}(\omega,\alpha)$-name. It follows that $\iota(t,u) = ((p''',q'''),(\check{r},\dot{s}'))$ which is below $((p,q),\tau)$. □

We record the following easy statements for later:

Lemma 3.7. *Assume ξ is a limit cardinal. Then ($\equiv$ denotes forcing equivalence):*

(1) $\mathbb{M}(\mu,\xi+1) \equiv \mathbb{M}(\mu,\xi) * \dot{\mathrm{Coll}}(\check{\mu},\check{\xi})$.
(2) $\mathbb{M}(\mu,\xi+2) \equiv \mathbb{M}(\mu,\xi) * \dot{\mathrm{Coll}}(\check{\mu},\check{\xi}) * \mathrm{Add}(\omega)$.

Proof. For (1), map $(p,q) \in \mathbb{M}(\mu,\xi+1)$ to $((p,q \upharpoonright \xi),q(\xi))$. Since any condition in $\dot{\mathrm{Coll}}(\check{\mu},\check{\xi})^{\mathbb{M}(\mu,\xi)}$ has size $<\mu$, it is actually in $V[\mathrm{Add}^{\oplus}(\omega,\xi)]$ which makes it easy to see that the mapping is a dense embedding. (2) follows the same way. □

4. Different Levels of Internal Approachability

In this section, we prove a theorem that will allow us to obtain Theorem 2 and Theorem 3 as corollaries. The idea is to show that (assuming κ is a large enough cardinal), any product of $\mathbb{M}(\mu,\kappa)$ with a $<\mu$-closed poset forces $\mu^+ \in I[\mu^+]$ and the existence of stationarily many $M \prec H(\Theta)$ with size $<\mu$ which are stationary, but not club, in $[M \cap \kappa^+]^{<\mu}$. Then by using Lemma 2.5 and manipulating powerset sizes we can obtain both Theorem 2 and Theorem 3.

Our last ingredient is the following new large cardinal property:

Definition 4.1. Let $\kappa \leq \lambda$ be cardinals. $\mathrm{Pr}(\kappa,\lambda)$ states that for every $\Theta \geq \lambda$ there are stationarily many $N \in [H(\Theta)]^{<\kappa}$ such that:

(1) $\nu := N \cap \kappa$ is inaccessible,
(2) $[N \cap \lambda]^{<\nu} \subseteq N$,
(3) For every cardinal $\mu \in [\kappa^+,\lambda] \cap N$, $\mathrm{otp}(N \cap \mu)$ is a cardinal.

Variants of this property have been considered by Zeman in [25] (and $\mathrm{Pr}(\kappa,\kappa)$ just states that κ is Mahlo, see the arguments below Claim 1 in [9]). For $\lambda > \kappa$ we have the following upper bound:

Lemma 4.2. *Assume κ is $|H(\lambda)|$-supercompact. Then $\mathrm{Pr}(\kappa,\lambda)$ holds.*

Proof. Let $\Theta \geq \lambda$ and $C \subseteq [H(\Theta)]^{<\kappa}$ club. Then the set $D := \{M \cap H(\lambda) \mid M \in C\}$ contains a club in $[H(\lambda)]^{<\kappa}$ by Fact 1.7.

Let $j\colon V \to M$ be a $|H(\lambda)|$-supercompact embedding, i.e. $j(\kappa) > |H(\lambda)^V|$ and ${}^{|H(\lambda)^V|}M \subseteq M$. Ergo $j[H(\lambda)^V] \in M$ and is a member of $j(D)$. So there is $N \in j(C)$ with $N \cap j(H(\lambda)^V) = j[H(\lambda)^V]$. Then the following holds:

(1) $N \cap j(\kappa) = \kappa$ is inaccessible in M,
(2) $[N \cap j(\lambda)]^{<\kappa} \subseteq N$: We have $N \cap j(\lambda) = j[\lambda]$. If $x \in [N \cap j(\lambda)]^{<\kappa}$, $x = j[y]$ for some $y \in [\lambda]^{<\kappa}$ (so $y \in H(\lambda)^V$). However, as $|y| < \kappa$, $x = j[y] = j(y) \in N$.
(3) For any cardinal $\mu \in [j(\kappa)^+, j(\lambda)] \cap N$, $\mu = j(\nu)$ for some $\nu \in H(\lambda)^V$, so ν is a cardinal by elementarity. It follows that $\mathrm{otp}(N \cap j(\nu)) = \mathrm{otp}(j[\nu]) = \nu$ which is a cardinal in V and thus in M.

So by elementarity there is $N \in C$ such that $\nu := N \cap \kappa$ is inaccessible, $[N \cap \lambda]^{<\nu} \subseteq N$ and $\operatorname{otp}(N \cap \mu)$ is a cardinal for every $\mu \in [\kappa^+, \lambda] \cap N$. $\square$

A better upper bound (however with considerably more work) is the λ-ineffability of κ, see a sketch in [12].

We can now prove our multi-purpose Theorem:

Theorem 4.3. *Assume κ is inaccessible,* $\Pr(\kappa, \kappa^+)$ *holds and $2^\kappa = \kappa^+$. Let $\mu < \kappa$ be regular and $\mathbb{P}$ any $<\mu$-closed poset which is δ-Knaster for some $\delta < \kappa$ and is a subset of $H(\kappa^+)$. After forcing with $\mathbb{M}(\mu, \kappa) \times \mathbb{P}$, the following holds:*

1. *$\mu^+ \in I[\mu^+]$.*
2. *For any sufficiently large Θ there are stationarily many $N \in [H(\Theta)]^\mu$ such that* $\operatorname{cf}(N \cap \mu^+) = \mu$ *and $N \cap [N \cap \mu^{++}]^{<\mu}$ is stationary but does not contain a club in $[N \cap \mu^{++}]^{<\mu}$.*

Proof. Denote $\mathbb{M} := \mathbb{M}(\mu, \kappa)$. Let $G \times H$ be $\mathbb{M} \times \mathbb{P}$-generic. For any $\gamma < \kappa$, we let $G(\gamma)$ be the $\mathbb{M}(\mu, \gamma)$-generic filter induced by G.

First of all, by Lemma 3.4, $V[G]$ is contained in some model $V[I \times J]$, where $I \times J$ is $\operatorname{Add}^{\oplus}(\omega, \kappa) \times \mathbb{T}(\mu, \kappa)$-generic. In $V[J]$, $\operatorname{Add}^{\oplus}(\omega, \kappa)$ is ccc. and $\mathbb{P}$ is $<\mu$-closed, so $\mathbb{P}$ is $<\mu$-distributive in $V[I \times J]$ by Easton's Lemma. Ergo $\mathbb{P}$ is $<\mu$-distributive in the smaller model $V[G]$.

Furthermore, $\mathbb{M} \times \mathbb{P}$ is κ-cc. and thus $\mathbb{P}$ is κ-cc. in $V[G]$. In summary, as $\kappa = \mu^+$ in $V[G]$, $\mathbb{P}$ does not collapse any cardinals in $V[G]$, so in $V[G \times H]$, we have $(\mu^+)^{<\mu} = \mu^+ = \kappa$ (any $x \in [\mu^+]^{<\mu}$ has already been added by $\operatorname{Add}^{\oplus}(\omega, \kappa)$ by Lemma 3.4 (5)).

We first deal with the approachability property (which is in its proof similar to [2]). Fix a sequence $(a_\alpha)_{\alpha < \mu^+}$ enumerating all elements of $[\mu^+]^{<\mu}$ in $V[G \times H]$. Let $C \subseteq \mu^+$ be the set of all ordinals $\beta \in \mu^+$ which are limit cardinals in V such that

$$\{a_\alpha \mid \alpha < \beta\} \supseteq [\mu^+]^{<\mu} \cap \bigcup_{\alpha < \beta} V[G \cap \operatorname{Add}^{\oplus}(\omega, \alpha)]$$

C is club in μ^+ since $(a_\alpha)_{\alpha < \mu^+}$ enumerates $[\mu^+]^{<\mu}$. If β has cofinality $\geq \mu$ in $V[G \times H]$, any $x \in [\mu^+]^{<\mu} \cap V[G \cap \operatorname{Add}^{\oplus}(\omega, \beta)]$ depends only on a $<\mu$-sized subset of $\operatorname{Add}^{\oplus}(\omega, \beta)$ by the ccc. of $\operatorname{Add}^{\oplus}(\omega, \beta)$ and is thus a member of $[\mu^+]^{<\mu} \cap V[G \cap \operatorname{Add}^{\oplus}(\omega, \alpha)]$ for some $\alpha \in \beta$.

Let $\beta \in C \cap \operatorname{cof}(\mu)^{V[G \times H]}$. We show that β is approachable with respect to $\overline{a}$ (which suffices as [23, Lemma 4.4] states that $\mu^+ \cap \operatorname{cof}(< \mu) \in I[\mu^+]$). By assumption $\{a_\alpha \mid \alpha < \beta\}$ contains all elements of $[\mu^+]^{<\mu} \cap V[G \cap \operatorname{Add}^{\oplus}(\omega, \beta)]$. Furthermore, $\mathbb{M}(\mu, \beta+1) \equiv \mathbb{M}(\mu, \beta) * \dot{\operatorname{Coll}}(\check{\mu}, \check{\beta})$ by Lemma 3.7. Ergo in $V[G(\beta+1)]$ there is a set $E \subseteq \beta$ with ordertype μ such that all its initial segments (which have size $<\mu$) are in $V[G(\beta)]$ and thus in $V[\operatorname{Add}^{\oplus}(\omega, \beta)]$. Since $\operatorname{cf}^{V[G \times H]}(\beta) \geq \mu$, any initial segment is in some $V[G \cap \operatorname{Add}^{\oplus}(\omega, \alpha)]$ for $\alpha < \mu$ and thus in $\{a_\alpha \mid \alpha < \beta\}$.

Now let, in $V[G \times H]$, Θ be sufficiently large and $C \subseteq [H(\Theta)^{V[G \times H]}]^\mu$ club. By Fact 1.7, find $F \colon [H(\Theta)^{V[G \times H]}]^{<\omega} \to [H(\Theta)^{V[G \times H]}]^\mu$ such that $\operatorname{cl}_F \subseteq C$. Let $\dot{F}$ be an $\mathbb{M} \times \mathbb{P}$-name such that $\dot{F}^{G \times H} = F$. Let Θ' be even larger so that $\dot{F} \in H(\Theta')^V$.

In V, apply $\Pr(\kappa, \kappa^+)$ to find $M \prec H(\Theta')^V$ containing $\dot{F}, \mathbb{M}, \mu, \kappa, \delta$ such that $\nu := M \cap \kappa$ is inaccessible, $[M \cap \kappa^+]^{<\nu} \subseteq M$ and $\operatorname{otp}(M \cap \kappa^+)$ is a cardinal.

It follows that $\operatorname{otp}(M \cap \kappa^+) = \nu^+$: $\geq$ is clear. For any $\alpha \in M \cap \kappa^+$, M contains a surjection from κ onto α which restricts to a surjection from $M \cap \kappa$ onto $M \cap \alpha$. So all initial segments of $M \cap \kappa^+$ have size $\leq \nu$. This shows $\operatorname{otp}(M \cap \kappa^+) \leq \nu^+$.

Because $2^\kappa = \kappa^+$ in V, $|H(\kappa^+)^V| = \kappa^+$. Ergo M contains a bijection between $H(\kappa^+)^V$ and κ^+. By using this bijection we see that $[M \cap H(\kappa^+)^V]^{<\nu} \subseteq M$. By the κ-cc. of $\mathbb{M} \times \mathbb{P}$ (which implies that every antichain of $\mathbb{M} \times \mathbb{P}$ which is an element of M is also a subset of M), $M[G \times H] \cap V = M$. We will show that $M[G \times H] \cap H(\Theta)^{V[G \times H]}$ is as required. It is clearly closed under $\dot{F}^{G \times H}$ and thus a member of C. Also $\mathrm{cf}(M[G \times H] \cap \mu^+) = \mathrm{cf}(M \cap \kappa) = \mathrm{cf}(\nu) = \mu$ in $V[G \times H]$.

Claim 1. *If $x \in [M \cap \kappa^+]^{\mu} \cap V[G(\nu)]$, $x \in M[G]$.*

Proof. Let $x \in [M \cap \kappa^+]^{\mu} \cap V[G(\nu)]$ and let f be an enumeration of x, so $f \in {}^{\mu}(M \cap \kappa^+)$. Let $\dot{f}$ be an $\mathbb{M}(\mu, \nu)$-name for f. For any $\alpha < \mu$, let A_α be a maximal antichain of conditions in $\mathbb{M}(\mu, \nu)$ deciding $\dot{f}(\check{\alpha})$ (by the ν-cc. of $\mathbb{M}(\mu, \nu)$, $|A_\alpha| < \nu$) and let B_α consist of pairs (p, γ_p^α), where $p \in A_\alpha$ and $p \Vdash \dot{f}(\check{\alpha}) = \check{\gamma}_p^\alpha$. Then B_α is a $<\nu$-sized subset of $M \cap \kappa^+ \times \mathbb{M}(\mu, \nu) \subseteq M \cap H(\kappa^+)^V$. Ergo $B_\alpha \in M$. By its size, $B_\alpha \in H(\kappa^+)^V$. So the sequence $(B_\alpha)_{\alpha < \mu}$ is a $<\nu$-sized subset of $M \cap H(\kappa^+)^V$ and thus in M. From $(B_\alpha)_{\alpha < \mu}$ and G we can define f, so $f \in M[G]$ and $x \in M[G]$. □

Since $\mathbb{P}$ is $<\mu$-distributive in $V[G(\nu)]$ (as this property is downwards absolute):

Claim 2. *If $x \in [M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times H]$, $x \in M[G \times H]$.* □

Now it suffices to show that $M[G \times H] \cap [M[G \times H] \cap \mu^{++}]^{<\mu}$ is stationary but does not contain a club in $[M[G \times H] \cap \mu^{++}]^{<\mu}$, since $[M[G \times H] \cap \mu^{++}]^{<\mu} \subseteq H(\Theta)^{V[G \times H]}$. We also note that $M[G \times H] \cap \mu^{++} = M \cap \mu^{++} = M \cap \kappa^+$.

Claim 3. *$M[G \times H] \cap [M \cap \kappa^+]^{<\mu}$ is stationary in $[M \cap \kappa^+]^{<\mu}$.*

Proof. By Claim 2, $[M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times H] \subseteq [M \cap \kappa^+]^{<\mu} \cap M[G \times H]$, so it suffices to show that the former set is stationary in $[M \cap \kappa^+]^{<\mu}$ in $V[G \times H]$. By Lemma 3.6, $V[G \times H]$ is an extension of $V[G(\nu) \times H]$ using $\mathbb{M}(G(\nu), \mu, \kappa \setminus \nu)^{V[G(\nu)]}$, which is the projection of the product $\mathrm{Add}^{\oplus}(\omega, \kappa \setminus \nu) \times \mathbb{T}(G(\nu), \mu, \kappa \setminus \nu)^{V[G(\nu)]}$, where $\mathbb{T}(G(\nu), \mu, \kappa \setminus \nu)^{V[G(\nu)]}$ is $<\mu$-closed in $V[G(\nu)]$ (by Lemma 3.4 (4)). By the $<\mu$-distributivity of $\mathbb{P}$ in $V[G(\nu)]$, $\mathbb{T}(\mu, \kappa \setminus \nu)^{V[G(\nu)]}$ is still $<\mu$-closed in $V[G(\nu) \times H]$.

It suffices to show that whenever $I \times J$ is $\mathrm{Add}^{\oplus}(\omega, \kappa \setminus \nu) \times \mathbb{T}(G(\nu), \mu, \kappa \setminus \nu)^{V[G(\nu)]}$-generic over $V[G(\nu) \times H]$, $[M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times H]$ is stationary in $[M \cap \kappa^+]^{<\mu}$ in the model $V[G(\nu) \times H \times I \times J]$, as $V[G \times H]$ is included in a model of this form (by Lemma 3.4 (3)). In $V[G(\nu) \times H \times J]$, $[M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times H]$ is the whole set $[M \cap \kappa^+]^{<\mu}$ (by the closure of $\mathbb{T}(G(\nu), \mu, \kappa \setminus \nu)^{V[G(\nu)]}$) and thus in particular stationary. Lastly, $V[G(\nu) \times H \times I \times J]$ is an extension of $V[G(\nu) \times H \times J]$ using the μ-cc. poset $\mathrm{Add}^{\oplus}(\omega, \kappa \setminus \nu)$ which preserves the stationarity of subsets of $[X]^{<\mu}$ for any set X. This shows the claim. □

We now show that $M[G \times H] \cap [M \cap \kappa^+]^{<\mu}$ does not contain a club in $[M \cap \kappa^+]^{<\mu}$. We first show a converse to Claim 2:

Claim 4. *If $x \in M[G \times H] \cap [M \cap \kappa^+]^{<\mu}$, $x \in V[G(\nu) \times H]$.*

Proof. Once again, we can fix an $\mathrm{Add}^{\oplus}(\omega, \kappa)$-name $\dot{x}$ for x. For any α, the antichain A_α of conditions deciding the αth element of $\dot{x}$ is an element of M. As it is countable, this implies that A_α is a subset of M. So $\dot{x}$ is equivalent to an $\mathrm{Add}^{\oplus}(\omega, \kappa) \cap M = \mathrm{Add}^{\oplus}(\omega, \nu)$-name and therefore x is in $V[G(\nu) \times H]$. □

So we can show:

Claim 5. *$M[G \times H] \cap [M \cap \kappa^+]^{<\mu}$ does not contain a club subset of $[M \cap \kappa^+]^{<\mu}$.*

Proof. Assume toward a contradiction that $c \subseteq M[G \times H] \cap [M \cap \kappa^+]^{<\mu}$ is club in $[M \cap \kappa^+]$. By Claim 4 we have $c \subseteq V[G(\nu) \times H]$. In V, there is a bijection between $M \cap \kappa^+$ and ν^+ by assumption. So, assuming the claim fails, there is in $V[G \times H]$ a club $D \subseteq [(\nu^+)^V]^{<\mu}$ such that $D \subseteq V[G(\nu) \times H]$. We will show that this is not the case.

We apply Fact 1.8 with $V[G(\nu)\times H]$ in lieu of V, $V[G(\nu+2)\times H]$ in lieu of W, μ in lieu of μ and $(\nu^+)^V$ in lieu of X and Θ. By Lemma 3.7, $V[G(\nu+2)]$ is an extension of $V[G(\nu)]$ using $\mathrm{Coll}(\mu,\nu)*\mathrm{Add}(\omega)$. Moreover, $\mathbb{P}$ is $<\mu$-distributive and ν-cc. (resp. $(\nu^+)^V$-cc.) in $V[G(\nu)]$ (resp. $V[G(\nu+2)]$). Because ν is inaccessible, $\mathrm{Coll}(\mu,\nu)$ is ν-cc. and so is $\mathrm{Coll}(\mu,\nu)*\mathrm{Add}(\omega)$. Ergo $(\nu^+)^V=(\mu^+)^{V[G(\nu+2)\times H]}$ is a regular cardinal in $V[G(\nu)\times H]$ and $(\nu^+)^V\subseteq H((\nu^+)^V)^{V[G(\nu+2)\times H]}$. So in $V[G(\nu+2)\times H]$ the set S of all $N\in[H((\nu^+)^V)^{V[G(\nu+2)\times H]}]^{<\mu}\cap\mathrm{IA}(\omega)$ with $N\cap(\nu^+)^V\notin V[G(\nu)\times H]$ is stationary in $[H((\nu^+)^V)^{V[G(\nu+2)\times H]}]^{<\mu}$.

$V[G\times H]$ is an extension of $V[G(\nu+2)\times H]$ using $\mathbb{M}(G(\nu+2),\mu,\kappa\smallsetminus(\nu+2))^{V[G(\nu+2)]}$ which is the projection of the product of a $<\mu$-closed and a ccc. poset. By Fact 1.6 the $<\mu$-closed part preserves that S is a stationary subset of $[H((\nu^+)^V)^{V[G(\nu+2)\times H]}]^{<\mu}$ and so does the ccc. part. As before, this shows that S is a stationary subset of $[H((\nu^+)^V)^{V[G(\nu+2)\times H]}]^{<\mu}$ in $V[G\times H]$. By Fact 1.7, $S':=\{N\cap(\nu^+)^V\mid N\in S\}$ is stationary in $[(\nu^+)^V]^{<\mu}$. However, S' is clearly disjoint from D, a contradiction. □

This finishes the proof. □

Now we obtain the two theorems as corollaries: The proof of Theorem 2 is a bit more involved because we also have to show that approachable points are connected to internally approachable structures (so we are showing one direction of Theorem 1.2).

Theorem 4.4. *Assume κ is an inaccessible cardinal such that $\mathrm{Pr}(\kappa,\kappa^+)$ holds and $2^\kappa=\kappa^+$. Let $\mu<\kappa$ be regular. After forcing with $\mathbb{M}(\mu,\kappa)$, the following holds:*

(1) $\mu^+\in I[\mu^+]$.
(2) *There are stationarily many $N\in[H(\mu^{++})]^\mu$ such that $N\cap H(\mu^+)$ is internally approachable and N is internally stationary but not internally club.*

Proof. We apply Theorem 3.4 with $\mathbb{P}:=\{\emptyset\}$. Let $G\times H$ be $\mathbb{M}(\mu,\kappa)\times\mathbb{P}$-generic. In $V[G\times H]$, we have $\mu^+\in I[\mu^+]$, so fix a sequence $\overline{a}$ and a club $C\subseteq\mu^+$ witnessing this fact. In $V[G\times H]$, we still have $|H(\kappa^+)^{V[G\times H]}|=|2^\kappa|=\kappa^+$ because $|\mathbb{M}(\mu,\kappa)|=\kappa$.

Work in $V[G\times H]$. Let $D\subseteq[H(\mu^{++})]^\mu$ be any club. Using standard arguments, find $M\prec H(\Theta)$ for Θ sufficiently large such that $\overline{a}\in M$, $M\cap H(\mu^{++})\in D$, $M\cap\mu^+\in C$, $\mathrm{cf}(M\cap\mu^+)=\mu$ and $M\cap[M\cap\mu^{++}]^{<\mu}$ is stationary but does not contain a club in $[M\cap\mu^{++}]^{<\mu}$. By Lemma 2.5, since $|H(\mu^{++})|=\mu^{++}$, $M\cap[M\cap H(\mu^{++})]^{<\mu}$ is stationary but does not contain a club in $[M\cap H(\mu^{++})]^{<\mu}$. This shows easily that $M\cap H(\mu^{++})$ is internally stationary but not internally club.

Since $M\cap\mu^+\in C$, there exists $A\subseteq M\cap\mu^+$ unbounded with ordertype μ (since $\mathrm{cf}(M\cap\mu^+)=\mu$) such that any initial segment of A is in $\{a_\alpha\mid\alpha<M\cap\mu^+\}$ and thus in M by elementarity.

Let $f\colon\mu\to A$ be the increasing enumeration of A. For any ordinal $\gamma\in[\mu,\mu^+)$, let $f_\gamma\colon\mu\to\gamma$ be a bijection (we can assume that $f_\gamma\in M$ for all $\gamma\in M\cap\mu^+$). Also let $F\colon\mu^+\to H(\mu^+)$ be a bijection in M. Now define the sequence $(b_i)_{i\in\mu}$ as follows: For $i\in\mu$, let

$$b_i:=F\left[\bigcup_{\gamma\in f[i]}f_\gamma[i]\right]$$

for any $j<\mu$, $(b_i)_{i<j}$ is definable from an initial segment of f (which is itself definable from an initial segment of A) and parameters in M, so it is in M. The only thing left to show is that $\bigcup_{i<\mu}b_i=M\cap H(\mu^+)$. Let $x\in M\cap H(\mu^+)$. By elementarity there is $\alpha\in M\cap\mu^+$ with $x=F(\alpha)$. Furthermore, there is $j\in\mu$ with $f(j)>\alpha$. Lastly, there is $i\in\mu$ with $\alpha=f_{f(j)}(i)$. Then $x\in b_{\max(i,j)+1}$. □

And we can obtain Theorem 3 with a less involved proof:

Theorem 4.5. *Assume κ is inaccessible,* $\Pr(\kappa,\kappa^+)$ *holds and* $2^\kappa=\kappa^+$*. Let* $\mu<\kappa$ *be regular. After forcing with* $\mathbb{M}(\mu,\kappa)\times\mathrm{Add}(\mu,\kappa^+)$*, the following holds:*

1. $\mu^+\in I[\mu^+]$.
2. *There does not exist a disjoint stationary sequence on* μ^+*.*
3. *There are stationarily many* $N\in[H(\mu^+)]^\mu$ *which are internally stationary but not internally club.*

Proof. Let $G\times H$ be $\mathbb{M}(\mu,\kappa)\times\mathrm{Add}(\mu,\kappa^+)$-generic. In $V[G\times H]$, $\kappa=\mu^+$, $2^\mu=\mu^{++}$, and so $|H(\mu^+)^{V[G\times H]}|=\mu^{++}$.

Work in $V[G\times H]$. By Theorem 4.3 we have AP_μ which implies by Theorem 1.4 that there does not exist a disjoint stationary sequence on μ^+.

Let $C\subseteq[H(\mu^+)]^\mu$ be any club. Let Θ be sufficiently large. Using standard arguments and applying Theorem 4.3, find $M\prec H(\Theta)$ such that $M\cap H(\mu^+)\in C$ and $M\cap[M\cap\mu^{++}]^{<\mu}$ is stationary but does not contain a club in $[M\cap\mu^{++}]^{<\mu}$. By Lemma 2.5 $M\cap[M\cap H(\mu^+)]^{<\mu}$ is stationary but does not contain a club in $[M\cap H(\mu^+)]^{<\mu}$. This easily shows that $M\cap H(\mu^+)$ is internally stationary but not internally club. □

5. Ordinal-Internal Approachability

In this small section, we define On-versions of the variants of internal approachability. These allow us to obtain criteria for the existence of a disjoint stationary sequence and the failure of the approachability with relaxed cardinal arithmetic assumptions.

Definition 5.1. Let M be a set with size μ (μ regular uncountable).

1. M is On-*internally unbounded* if $M\cap[M\cap\mathrm{On}]^{<\mu}$ is unbounded in $[M\cap\mathrm{On}]^{<\mu}$,
2. M is On-*internally stationary* if $M\cap[M\cap\mathrm{On}]^{<\mu}$ is stationary in $[M\cap\mathrm{On}]^{<\mu}$,
3. M is On-*internally club* if $M\cap[M\cap\mathrm{On}]^{<\mu}$ contains a club in $[M\cap\mathrm{On}]^{<\mu}$,
4. M is On-*internally approachable* if there exists an increasing and continuous sequence $(x_i)_{i<\mu}$ of elements of $[M\cap\mathrm{On}]^{<\mu}$ such that $\bigcup_{i<\mu}x_i=M\cap\mathrm{On}$ and $(x_i)_{i<j}\in M$ for every $j<\mu$.

If $|H(\mu^+)|=\mu^+$ and $M\prec H(\Theta)$ for a sufficiently large Θ, Lemma 2.5 and its proof shows that $M\cap H(\mu^+)$ is internally unbounded (stationary; club; approachable) if and only if $M\cap H(\mu^+)$ is On-internally unbounded (stationary; club; approachable). Our previous arguments show that this in general not the case if $|H(\mu^+)|>\mu^+$.

The proof of Krueger's result on the existence of a disjoint stationary sequence (see [17, Theorem 6.5]) can be readily modified to show:

Theorem 5.2. *Assume μ is a regular uncountable cardinal. Then* $\mathrm{DSS}(\mu^+)$ *holds if and only if there are stationarily many* $N\in[H(\mu^+)]^\mu$ *which are* On*-internally unbounded but not* On*-internally club.*

Also, a straightforward modification of Cox' proof of the folklore result relating the approachability to the distinction between internal unboundedness and approachability shows the following:

Theorem 5.3. *If* $\mu^+\in I[\mu^+]$*, there is a club* $C\subseteq[H(\mu^+)]^\mu$ *such that any* $N\in C$ *is either not* On*-internally unbounded or it is* On*-internally approachable.*

If $(\mu^+)^{<\mu}=\mu^+$ *and* $\mu^+\notin I[\mu^+]$*, there are stationarily many* $N\in[H(\mu^+)]^\mu$ *which are* On*-internally unbounded but not* On*-internally approachable.*

Proof. Let us give a sketch of the argument. Assume $(a_\alpha)_{\alpha<\mu^+}$ and $D \subseteq \mu^+$ witness $\mu^+ \in I[\mu^+]$. Let $C \subseteq [H(\mu^+)]^\mu$ consist of all those $M \cap H(\mu^+)$, where $M \prec H(\Theta)$ (Θ large enough), $\overline{a} \in M$, $D \in M$ and $M \cap \mu^+ \in \mu^+$. C contains a club by Fact 1.7. Assume $M \cap H(\mu^+) \in C$. If M is On-internally unbounded, $\mathrm{cf}(M \cap \mu^+) = \mu$ (otherwise $M \cap H(\mu^+)$ would need to contain a $<\mu$-sized superset of an unbounded subset of $M \cap \mu^+$ which is impossible). By elementarity, $M \cap \mu^+ \cap D$ is unbounded in $M \cap \mu^+$, so $M \cap \mu^+ \in D$. Ergo there is $A \subseteq M \cap \mu^+$ unbounded with ordertype μ such that every initial segment of A is in $M \cap \mu^+$. Let $f\colon \mu \to A$ be its increasing enumeration and for $\gamma \in [\mu, \mu^+)$, let $f_\gamma\colon \mu \to \gamma$ be a bijection (for $\gamma \in M \cap \mu^+$, we can assume $f_\gamma \in M$). Now define $(b_i)_{i\in\mu}$ as follows: For $i \in \mu$, let

$$b_i := \bigcup_{\gamma \in f[i]} f_\gamma[i]$$

It follows as in the proof of Theorem 4.4 that $(b_i)_{i\in\mu}$ witnesses the On-internal approachability of M.

Now assume $(\mu^+)^{<\mu} = \mu^+$ and $\mu^+ \notin I[\mu^+]$. Let $(a_\alpha)_{\alpha<\mu^+}$ enumerate $[\mu^+]^{<\mu}$. By assumption (and the fact that $\mu^+ \cap \mathrm{cof}(<\mu) \in I[\mu^+]$ by [23, Lemma 4.4]) the set of all $\gamma \in \mu^+ \cap \mathrm{cof}(\mu)$ which are not approachable with respect to $\overline{a}$ is stationary in μ^+. Let $D \subseteq [H(\mu^+)]^\mu$ be club. Then there exists $M \prec H(\Theta)$ for Θ large enough with $D, \overline{a} \in M$ and $M \cap \mu^+ \in \mu^+ \cap \mathrm{cof}(\mu)$ such that $M \cap \mu^+$ is not approachable with respect to $\overline{a}$. Then clearly $M \cap H(\mu^+)$ is On-internally unbounded (any $<\mu$-sized subset of $M \cap \mu^+$ is contained in an ordinal in $M \cap \mu^+$). Assume toward a contradiction that $M \cap H(\mu^+)$ is On-internally approachable, witnessed by $(x_i)_{i<\mu}$. Then the set $A := \{\sup(x_i) \mid i < \mu\}$ is unbounded in $M \cap \mu^+$ and has ordertype μ. Furthermore, any initial segment of A is definable from some $(x_i)_{i<j}$ for $j < \mu$ and thus in M. By elementarity, it is equal to some a_α for $\alpha < M \cap \mu^+$. So $M \cap \mu^+$ is approachable with respect to $\overline{a}$, a contradiction. □

6. Open Questions

We have shown in Theorem 1 that one direction of Theorem 1.2 and Theorem 1.4 can fail without the assumed cardinal arithmetic. We do not know if the same is true for the other direction:

Question 6.1. Is it consistent that AP_μ fails and there are club many $N \in [H(\mu^+)]^\mu$ which are either not internally unbounded or internally approachable? Is it consistent that $\mathrm{DSS}(\mu^+)$ holds and there are club many $N \in [H(\mu^+)]^\mu$ which are either not internally unbounded or internally club?

Our other question concerns the "improved version" of Theorem 1.2, where we do not know whether this cardinal arithmetic assumption can be relaxed:

Question 6.2. Is it consistent that $\mu^+ \notin I[\mu^+]$ and there are club many $N \in [H(\mu^+)]^\mu$ which are either not On-internally unbounded or On-internally approachable (here necessarily $(\mu^+)^{<\mu} \geq \mu^{++}$)?

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