CASCADING VARIANTS OF INTERNAL APPROACHABILITY

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Abstract. We construct models in which there are stationarily many structures that exhibit different variants of internal approachability at different levels. This answers a question of Foreman. We also show that the approachability property at μ is consistent with having a distinction of variants of internal approachability for stationarily many $N \in [H(\mu^+)]^\mu$, answering a question of Levine. This is obtained using a new version of Mitchell Forcing.

Introduction

The notions of variants of internal approachability were introduced by Foreman and Todorcevic in [4]. We say that a set N of size μ is

(1) *internally unbounded* if $[N]^{<\mu} \cap N$ is unbounded in $[N]^{<\mu}$
(2) *internally stationary* if $[N]^{<\mu} \cap N$ is stationary in $[N]^{<\mu}$
(3) *internally club* if $[N]^{<\mu} \cap N$ contains a club in $[N]^{<\mu}$
(4) *internally approachable* if there is an increasing and continuous sequence $(a_i)_{i\in\mu}$ of elements of $[N]^{<\mu}$ such that $\bigcup_{i\in\mu} a_i = N$ and $(a_i)_{i<j} \in N$ for every $j < \mu$.

Clearly, every such property implies all the properties above it. Much research has been focused on showing that none of these implications can be reversed: Krueger showed in [9] and [10] that, consistently, there can be stationarily many $N \in [H(\Theta)]^\mu$ which have one of the properties but not the one below it. Later, Levine and the author showed in [6], [12] and [7] that it is even possible to obtain a distinction between internal stationarity and clubness as well as internal clubness and approachability for infinitely many successive cardinals.

In [3], Question 4.6, Foreman asked if it was consistent to have a model which is internally approachable of different variants at different levels. More specifically, he asked: "Suppose κ is regular, $N \prec H(\Theta)$ and $N \cap [N \cap \kappa]^{\aleph_0}$ is stationary. Is $N \cap [N \cap \kappa^+]^{\aleph_0}$ stationary?". We note that if M is e.g. internally stationary, so is $M \cap H(\Theta)$ for any $\Theta \in M$, so the question is asking if the converse of this fact is also true. We will answer his question in the negative by showing the following:

Theorem 1. *Assume* MM. *Then there exist stationarily many $N \in [H(\omega_3)]^{\omega_1}$ such that $N \cap H(\omega_2)$ is internally approachable and $N \cap H(\omega_3)$ is not internally stationary.*

A similar situation can be forced outright at cardinals larger than ω_1. We note that it is unknown whether one can separate the principles of internal unboundedness and stationarity for sets of size above and including ω_2, so the following theorem is the best we can hope for with current techniques:

Theorem 2. *Assume κ is κ^{++}-ineffable and $\mu < \kappa$ is regular. There is a forcing extension where $\kappa = \mu^+$ and the following holds:*

(1) $\mu^+ \in I[\mu^+]$,
(2) $2^\mu = \mu^+$,

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(3) *There are stationarily many* $N \in [H(\mu^{+++})]^\mu$ *such that* $N \cap H(\mu^+)$ *is internally approachable,* $N \cap H(\mu^{++})$ *is internally club but not internally approachable and* $N \cap H(\mu^{+++})$ *is internally stationary but not internally club.*

The construction to obtain Theorem 2 can be modified to obtain a different result which solves a number of open questions: Under the assumption $2^\mu = \mu^+$, the distinction between variants of internal approachability is connected to certain combinatorial principles: Krueger showed that the existence of stationarily many $N \in [H(\mu^+)]^\mu$ which are internally stationary but not internally club implies the existence of a disjoint stationary sequence on μ^+ and it is a folklore result that the existence of stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable (that means having one of the three possible distinctions) is equivalent to the failure of the approachability property at μ. We will show that the assumption $2^\mu = \mu^+$ cannot be relaxed by proving the following theorem:

Theorem 3. *Assume* $\tau < \mu < \kappa$ *are such that* $\tau^{<\tau} = \tau$, μ *is regular and* κ *is* κ^+*-ineffable. There is a forcing extension where* $\kappa = \mu^+$, $2^\mu = \mu^{++}$ *and the following holds:*

(1) $\mu^+ \in I[\mu^+]$,
(2) *There does not exist a disjoint stationary sequence on* κ,
(3) *There are stationarily many* $N \in [H(\mu^+)]^\mu$ *which are internally unbounded but not internally approachable.*

The paper is structured as follows: In the first section, we introduce known definitions and results. In the second section, we analyze the situation under Martin's Maximum and prove Theorem 1. In the third section, we define a new variant of Mitchell's forcing to prove Theorem 2. In the fourth section, we modify the construction of the third section to prove Theorem 3. In the last section, we introduce a variant of internal approachability which gives an "iff-criterion" for the existence of a disjoint stationary sequence without any cardinal arithmetic assumptions.

Preliminaries. We will assume the reader is familiar with the basics of forcing. Good introductory material can be found in [8] and [11]. Familiarity with our earlier papers [6] and [7] is helpful but not necessary.

We will now introduce known preservation results to be used throughout the paper. We first introduce a weakening of $<\mu$-closure.

Definition 0.1. Let $\mathbb{P}$ be a forcing order and δ an ordinal. The *completeness game* $G(\mathbb{P}, \delta)$ is a game of length δ played on $\mathbb{P}$ as follows: COM plays at even steps (including limit ordinals) and INC at odd steps. COM has to start by playing $1_{\mathbb{P}}$. If $(p_\gamma)_{\gamma<\alpha}$ has been played, the player whose turn it is has to play a lower bound of $(p_\gamma)_{\gamma<\alpha}$. If COM cannot play at some $\alpha < \delta$, INC wins. Otherwise, COM wins.

The forcing order is μ-strategically closed if COM has a winning strategy in $G(\mathbb{P}, \mu)$. It is $<\mu$-strategically closed if COM has a winning strategy in $G(\mathbb{P}, \delta)$ for every $\delta < \mu$.

We also give a weakening of internal approachability:

Definition 0.2. Let Θ be a cardinal and $N \prec H(\Theta)$. *N is internally approachable of length τ*, written $N \in \mathrm{IA}(\tau)$, if there is a sequence $(N_i)_{i\in\tau}$ with $N = \bigcup_{i\in\tau} N_i$ and $(N_i)_{i<j} \in N$ for every $j < \tau$.

The weakening aspect comes from the fact that we allow the N_i to have arbitrary size. Still, this definition is strong enough for the following preservation result:

Fact 0.3. *Assume $S \subseteq [H(\Theta)]^{<\mu}$ is stationary and $S \subseteq \mathrm{IA}(\tau)$ for some τ. If $\mathbb{P}$ is a $<\mu$-strategically closed forcing order, where $\tau < \mu$, $\mathbb{P}$ forces that S is stationary in $[H(\Theta)^V]^{<\kappa}$.*

The result immediately implies the following (which we just state for convenience):

Fact 0.4. *Assume $\mathbb{P}$ is $<\mu$-strategically closed. Then $\mathbb{P}$ forces that $([H(\Theta)]^{<\mu})^V$ is stationary in $[H^V(\Theta)]^{<\mu}$.*

The following result is due to Menas:

Fact 0.5. *Assume $X \subseteq Y \subseteq Z$ and κ is a cardinal.*

(1) *If $C \subseteq [Y]^{<\kappa}$ is club, there is a function $F\colon [X]^{<\omega} \to [X]^{<\kappa}$ such that*
$$\mathrm{cl}_F := \{a \in [X]^{<\kappa} \mid \forall x \in [a]^{<\omega}\ F(x) \subseteq a\} \subseteq C$$
noting that cl_F is club in $[X]^{<\kappa}$.

(2) *If $C \subseteq [Y]^{<\kappa}$ is club, then*
$$C \restriction X := \{a \cap X \mid a \in C\}$$
contains a club in $[X]^{<\kappa}$ and
$$C \uparrow Z := \{a \in [Z]^{<\kappa} \mid a \cap Y \in C\}$$
is club in $[Z]^{<\kappa}$.

(3) *If $S \subseteq [Y]^{<\kappa}$ is stationary, then the sets*
$$S \restriction X := \{a \cap X \mid a \in C\} \text{ and } S \uparrow Y := \{a \in [Z]^{<\kappa} \mid a \cap Y \in S\}$$
are stationary in $[X]^{<\kappa}$ and $[Z]^{<\kappa}$ respectively.

We also use the following forcing to shoot a club through a stationary subset of $[X]^{<\kappa}$, due to Jech:

Definition 0.6. Let $S \subseteq [X]^{<\kappa}$ be stationary. $\mathbb{P}(S)$ consists of functions $p\colon \alpha \to S$, such that $\alpha < \kappa$ is a successor ordinal and p is increasing and continuous.

The poset $\mathbb{P}(S)$ collapses $|X|$ by shooting a club through S. In general, it can be very badly behaved. However, we will only use the following special cases which have better behaviour: Both of these statements were first shown by Krueger (see [5] and [10]).

Fact 0.7. *Let $\tau < \mu < \kappa$ be cardinals such that $\tau^{<\tau} = \tau$ and μ is regular.*

(1) *The term ordering on $\mathrm{Add}(\tau) * \mathbb{P}([\kappa]^{<\mu} \cap V)$ is μ-strategically closed.*
(2) *The poset $\mathrm{Add}(\omega) * \mathbb{P}([\omega_2]^{<\omega_1} \smallsetminus V)$ preserves stationary subsets of ω_1.*

The notion of non-internal approachability is connected to the approximation property for forcings, which states that no "fresh" sequences of a given length are added:

Definition 0.8. Let $V \subseteq W$ be models of set theory with the same ordinals and μ a cardinal in W. The pair (V, W) has the $<\mu$-approximation property if whenever $x \in W$ is such that $x \cap y \in V$ for every $y \in [x]^{<\mu} \cap V$, $x \in V$.

A forcing order $\mathbb{P}$ has the $<\mu$-approximation property if $(V, V[G])$ has it whenever G is $\mathbb{P}$-generic.

We now introduce the remaining properties that relate to this paper:

Definition 0.9. Let μ be a cardinal.

(1) The approachability ideal on μ^+, denoted by $I[\mu^+]$, is defined as follows: $A \in I[\mu^+]$ if there exists a sequence $(a_\alpha)_{\alpha<\mu^+}$ of elements of $[\mu^+]^{<\mu}$ and a club $C \subseteq \mu^+$ such that whenever $\gamma \in A \cap C$, there exists $E \subseteq \gamma$ with $\mathrm{otp}(E) = \mathrm{cf}(\gamma)$ such that E is unbounded in γ and $\{E \cap \alpha \mid \alpha < \gamma\} \subseteq \{a_\alpha \mid \alpha < \gamma\}$.
(2) The approachability property at μ, denoted by AP_μ, is the statement that $\mu^+ \in I[\mu^+]$.
(3) $(\mathcal{S}_\alpha)_{\alpha\in S}$ is a disjoint stationary sequence on μ^+ if the following holds:
  (a) $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary.
  (b) For all $\alpha \in S$, $\mathcal{S}_\alpha$ is stationary in $[\alpha]^{<\mu}$ and for all $\alpha \neq \beta$, both in S, $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \emptyset$.

These definitions relate to our material as follows:

Theorem 0.10 (Folklore, see [1])**.** *Let μ be a cardinal with $2^\mu = \mu^+$. Then AP_μ fails if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable.*

And Krueger related the existence of a disjoint stationary sequence to the previous properties:

Theorem 0.11 (Krueger)**.** *Let μ be a cardinal.*
(1) *If there exists a disjoint stationary sequence on μ^+, AP_μ fails.*
(2) *If $2^\mu = \mu^+$, then there exists a disjoint stationary sequence on μ^+ if and only if there are stationarily many $N \in [H(\mu^+)]^{<\mu}$ which are internally unbounded but not internally club.*

We will show that in both theorems the assumption $2^\mu = \mu^+$ cannot be relaxed.

1. Martin's Maximum

In this section, we show that Martin's Maximum implies that there are stationarily many structures which are "cascadingly internally approachable". We use the following formulation due to Woodin: Recall that, for a poset $\mathbb{P}$ and a set N, a filter G is $\mathbb{P}$-generic over N if for every $D \subseteq \mathbb{P}$ which is dense in $\mathbb{P}$ and in N, there is $p \in G \cap D \cap N$.

Definition 1.1. MM states that whenever $\mathbb{P}$ preserves stationary subsets of ω_1 and $\Theta \geq \omega_2$ is such that $\mathbb{P} \in H(\Theta)$, there are stationarily many $N \in [H(\Theta)]^{\omega_1}$ such that there exists a filter $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over N.

We assume that GCH holds above ω_2 (this is possible as MM is preserved by $<\omega_2$-directed closed forcing) and define the forcing notion

$$\mathbb{P} := \mathrm{Coll}(\omega_1, \omega_2) * \mathrm{Add}(\omega) * \mathbb{P}([\check{\omega_3}]^{<\omega_1} \setminus V)$$

$\mathrm{Coll}(\omega_1, \omega_2)$ is countably closed and hence even proper. Because $\omega_2^\omega = \omega_2$, $\mathrm{Coll}(\omega_1, \omega_2)$ is ω_3-cc. and thus forces $\check{\omega_3} = \omega_2$. The forcing $\mathrm{Add}(\omega) * \mathbb{P}([\omega_2]^{<\omega_1} \setminus V)$ preserves stationary subsets of ω_1 by Lemma 0.7. In summary, $\mathbb{P}$ does not destroy stationary subsets of ω_1.

The next Lemma follows easily:

Lemma 1.2. *$\mathbb{P}$ forces the existence of continuous sequences $(a_i)_{i\in\omega_1}$ and $(b_i)_{i\in\omega_1}$ with the following properties:*
(1) $\bigcup_{i\in\omega_1} a_i = H(\omega_2)^V$ *and for all $j < \omega_1$, $(a_i)_{i<j} \in [H(\omega_2)^V]^{<\omega_1} \cap H(\omega_2)^V$.*
(2) $\bigcup_{i\in\omega_1} b_i = H(\omega_3)^V$ *and for all $i < \omega_1$, $b_i \in [H(\omega_3)^V]^{<\omega_1} \setminus V$.*

We can now restate and prove Theorem 1.

Theorem 1.3. *Assume* MM. *Then there exist stationarily many $N \in [H(\omega_3)]^{\omega_1}$ such that $N \cap H(\omega_2)$ is internally approachable and $N \cap H(\omega_3)$ is not internally stationary.*

Proof. Fix names $(\dot{a}_i)_{i\in\omega_1}$ and $(\dot{b}_i)_{i\in\omega_1}$ for sequences with the properties mentioned in Lemma 1.2. Let C be club in $[H(\omega_3)]^{\omega_1}$ and define

$$D := \{N \in [H(\omega_4)]^{\omega_1} \mid N \cap H(\omega_3) \in C\}$$

D contains a club by fact 0.5. In particular, there is $N \in [H(\omega_4)]^{\omega_1}$ such that $(\dot{a}_i)_{i\in\omega_1}, (\dot{b}_i)_{i\in\omega_1} \in N$, there exists a $\mathbb{P}$-generic filter G over N and $N \cap H(\omega_3) \in C$. We want to show that $N \cap H(\omega_2)$ is internally approachable and $N \cap H(\omega_3)$ is not internally stationary.

To this end, we define the following partial evaluations for $i \in \omega_1$:

$$a_i := \{x \in N \cap H(\omega_2) \mid \exists p \in G\, p \Vdash \check{x} \in \dot{a}_i\}$$
$$b_i := \{x \in N \cap H(\omega_3) \mid \exists p \in G\, p \Vdash \check{x} \in \dot{b}_i\}$$

We will show that $(a_i)_{i\in\omega_1}$ witnesses that $N \cap H(\omega_2)$ is internally approachable and $(b_i)_{i\in\omega_1}$ witnesses that $N \cap H(\omega_3)$ is not internally stationary. We do this in a series of claims.

Claim. $\bigcup_{i\in\omega_1} a_i = N \cap H(\omega_2)$

Proof. $\subseteq$ is clear. On the other hand, let $x \in N \cap H(\omega_2)$. The set $D := \{p \in \mathbb{P} \mid \exists i(p \Vdash \check{x} \in \dot{a}_i)\}$ which is in N is open dense in $\mathbb{P}$ by Lemma 1.2 and so there is $p \in D \cap G \cap N$. Ergo there exists $i \in \omega_1$ with $x \in a_i$. □

Claim. *For every $j < \omega_1$, $(a_i)_{i<j} \in N$.*

Proof. For every $j < \omega_1$, the sequence $(\dot{a}_i)_{i<j}$ is in N and therefore so is the set $D := \{p \in \mathbb{P} \mid \exists x(p \Vdash \check{x} = (\dot{a}_i)_{i<j})\}$ which is open dense in $\mathbb{P}$ by Lemma 1.2. So there is $p \in D \cap G \cap N$. By elementarity, the x with $p \Vdash \check{x} = (\dot{a}_i)_{i<j}$ is in N as well. We are done after showing $x = (a_i)_{i<j}$. To this end, let $i < j$. Assume $y \in x(i)$. Then $p \Vdash \check{y} \in \dot{a}_i$, so $y \in a_i$. On the other hand, let $y \in a_i$. By the definition, there is $q \in G$ with $q \Vdash \check{y} \in \dot{a}_i$. Since G is a filter, there is $r \leq q, p$. This r forces $\check{y} \in \check{x}(i)$, so $y \in x(i)$. □

Now we will show that $(b_i)_{i\in\omega_1}$ witnesses that $N \cap H(\omega_3)$ is not internally stationary. Just as before, we have:

Claim. $\bigcup_{i\in\omega_1} b_i = N \cap H(\omega_3)$.

So in particular, the collection $\{b_i \mid i \in \omega_1\}$ is club in $[N \cap H(\omega_3)]^{<\omega_1}$. We are done after showing:

Claim. *$b_i \notin N \cap H(\omega_3)$ for every $i \in \omega_1$.*

Proof. Let $i \in \omega_1$. We show that $b_i \neq x$ for every $x \in N \cap H(\omega_3)$. To this end, let $x \in N \cap H(\omega_3)$ be arbitrary. The set $D := \{p \in \mathbb{P} \mid \exists y(p \Vdash y \in \check{x} \setminus \dot{b}_i \vee p \Vdash \dot{b}_i \setminus \check{x})\}$ is in N and open dense in $\mathbb{P}$ by Lemma 1.2 (since $\dot{b}_i$ is forced to not be in V, it is forced to be different from $\check{x}$ for every $x \in V$). Ergo there exists $p \in G \cap D \cap N$, witnessed by some $y \in N$. Then either $p \Vdash \check{y} \in \check{x} \setminus \dot{b}_i$, in which case $\check{y} \in x \setminus b_i$ (since no element of G can force $y \in \dot{b}_i$), or $p \Vdash \check{y} \in \dot{b}_i \setminus \check{x}$, in which case $y \in b_i \setminus x$. □

□

2. Different Levels of Approachability

In this section, we show that it is possible to have a model with different variations of internal approachability at different levels. We will make use of the following large cardinal property:

Definition 2.1. Let $\kappa \leq \lambda$ be cardinals. $\Pr(\kappa, \lambda)$ states that for every $\Theta \geq \lambda$ there exist stationarily many $N \in [H(\Theta)]^{<\kappa}$ with the following properties:

(1) $\nu := N \cap \kappa$ is an inaccessible cardinal.
(2) $[N \cap \lambda]^{<\nu} \subseteq N$.
(3) For every $\mu \in [\kappa^+, \lambda]$, $\operatorname{otp}(N \cap \mu)$ is a cardinal.

We do not know where $\Pr(\kappa, \lambda)$ fits into the large cardinal hierarchy, but we have the following partial results: $\Pr(\kappa, \kappa)$ is equivalent to κ being Mahlo. $\Pr(\kappa, \kappa^+)$ already implies the existence of $0^\#$ (see Lemma 38.11 in [8]). An upper bound for $\Pr(\kappa, \lambda)$ is the λ-supercompactness of κ: Let $j: V \to M$ be an embedding with critical point κ such that $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$. Let $\Theta \geq \lambda$ and $C \subseteq [H(\Theta)]^{<\kappa}$ be any club. By standard methods, there exists $M \in j(C)$ such that $M \cap j(\lambda) = j[\lambda]$. It follows that $M \in j(C)$ has the stated properties with respect to $j(\kappa)$ and $j(\lambda)$, so in V there exists an M as required with respect to κ and λ. A better upper bound is the $\lambda^{<\kappa}$-ineffability of κ which follows from work in the author's PhD thesis (forthcoming).

We note that it seems likely that Mahlo cardinals are not sufficient to obtain the models we will construct, since the "type of internal approachability" of N depends only on how we collapsed $|N|$ while the models given by Mahlo cardinals have the property $|N| = N \cap \kappa$ (so we collapsed every level of N in the same way).

We now define our variant of Mitchell forcing. For reasons which will become apparent later, for X a set of ordinals, we let $\operatorname{Add}^\oplus(\tau, X)$ consist of finite functions p on the successors of inaccessible cardinals in X such that $p(\alpha) \in \operatorname{Add}(\omega)$ for every $\alpha \in \operatorname{dom}(p)$.

Definition 2.2. Let $\tau < \mu < \kappa$ be cardinals such that μ is regular and $\tau^{<\tau} = \tau$. $\mathbb{M}(\tau, \mu, \kappa)$ consists of pairs (p, q) with the following properties:

(1) $p \in \operatorname{Add}^\oplus(\tau, \kappa)$.
(2) q is a $<\mu$-sized partial function on κ such that for all $\alpha \in \operatorname{dom}(q)$, the following holds:
    (a) If $\alpha = \delta^+$ for an inaccessible cardinal δ, $q(\alpha)$ is an $\operatorname{Add}^\oplus(\tau, \alpha)$-name for an element in $\mathbb{P}([\alpha]^{<\mu} \cap V[\operatorname{Add}(\tau, \delta)])$.
    (b) Otherwise, $q(\alpha)$ is an $\operatorname{Add}^\oplus(\tau, \alpha)$-name for an element in $\dot{\operatorname{Coll}}(\mu, \alpha)$.

We let $(p', q') \leq (p, q)$ if and only if

(1) $p' \leq p$ in $\operatorname{Add}^\oplus(\tau, \kappa)$.
(2) $\operatorname{dom}(q') \supseteq \operatorname{dom}(q)$ and for all $\alpha \in \operatorname{dom}(q)$,
$$p' \upharpoonright \alpha \Vdash q'(\alpha) \leq q(\alpha)$$

As is standard, we explicitely define the term ordering $\mathbb{T}(\tau, \mu, \kappa)$ to consist of all q such that $(\emptyset, q) \in \mathbb{M}(\tau, \mu, \kappa)$, ordered in the usual way.

We also have an explicit description of the quotient poset $\mathbb{M}(\tau, \mu, \kappa)/\mathbb{M}(\tau, \mu, \nu)$:

Definition 2.3. Let $\mu < \nu < \kappa$ be cardinals such that μ is regular. $\mathbb{M}(\tau, \mu, \kappa \smallsetminus \nu)$ consists of pairs (p, q) with the following properties:

(1) $p \in \operatorname{Add}^\oplus(\tau, \kappa \smallsetminus \nu)$.
(2) q is a $<\mu$-sized partial function on κ such that for all $\alpha \in \operatorname{dom}(q)$, the following holds:
    (a) If $\alpha = \delta^+$ for an inaccessible cardinal δ, $q(\alpha)$ is an $\operatorname{Add}^\oplus(\tau, \alpha)$-name for an element in $\mathbb{P}([\alpha]^{<\mu} \cap V[\operatorname{Add}(\tau, \delta)])$

(b) Otherwise, $q(\alpha)$ is an $\mathrm{Add}^{\oplus}(\tau,\alpha)$-name for an element in $\dot{\mathrm{Coll}}(\mu,\alpha)$.

We let $q' \leq q$ if and only if $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$ and for all $\alpha \in \mathrm{dom}(q)$,

$$\Vdash q'(\alpha) \leq q(\alpha)$$

As above, we define $\mathbb{T}(\tau,\mu,\kappa\setminus\nu)$ to consist of all those q with $(\emptyset,q) \in \mathbb{M}(\tau,\mu,\kappa\setminus\nu)$.

The following facts are standard for variants of Mitchell forcing. They can be obtained by applying a general framework from the author's PhD thesis (forthcoming). For the rest of this section, fix cardinals $\mu<\kappa$ such that μ is regular and κ is inaccessible.

Lemma 2.4. *Let $\mu<\nu<\kappa$ be cardinals.*

1. *$\mathbb{M}(\tau,\mu,\kappa)$ is κ-Knaster.*
2. *The poset $\mathbb{T}(\tau,\mu,\kappa)$ is μ-strategically closed.*
3. *There exists a projection from $\mathrm{Add}^{\oplus}(\tau,\kappa)\times\mathbb{T}(\tau,\mu,\kappa)$ onto $\mathbb{M}(\tau,\mu,\kappa)$.*
4. *There exists a dense embedding from $\mathbb{M}(\tau,\mu,\kappa)$ into $\mathbb{M}(\tau,\mu,\nu)*\mathbb{M}(\tau,\mu,\kappa\setminus\nu)$.*
5. *If ν is not of the form δ^+ for an inaccessible cardinal δ, the poset $\mathbb{T}(\tau,\mu,\kappa\setminus\nu)$ is μ-strategically closed.*
6. *If $\nu=\delta+1$ for an inaccessible cardinal δ, $\mathbb{M}(\tau,\mu,\kappa\setminus\nu)$ has the $<\tau^+$-approximation property.*

We can move on to the proof of Theorem 2. For simplicity, we let $\mathbb{M}(\mu,\kappa) := \mathbb{M}(\omega,\mu,\kappa)$.

Theorem 2.5. *Assume $\mathrm{Pr}(\kappa,\kappa^{++})$ holds as well as GCH above κ and $\mu<\kappa$ is regular. After forcing with $\mathbb{M}(\mu,\kappa)$, the following holds:*

1. *$\mu^+ \in I[\mu^+]$.*
2. *There are stationarily many $N \in [H(\mu^{+++})]^{\mu}$ such that $N\cap H(\mu^+)$ is internally approachable, $N\cap H(\mu^{++})$ is internally club but not internally approachable and $N\cap H(\mu^{+++})$ is internally stationary but not internally club.*

Proof. For simplicity, we define $\mathbb{M} := \mathbb{M}(\mu,\kappa)$. Let G be $\mathbb{M}$-generic and work in $V[G]$. Given any ordinal $\gamma<\kappa$, let $G(\gamma)$ be the $\mathbb{M}(\mu,\gamma)$-generic filter induced by G.

We first show $\mu^+\in I[\mu^+]$. This follows similarly to [2]. By a result of Shelah, $\mu^+\cap\mathrm{cof}(<\mu)\in I[\mu^+]$, so we only need to worry about $\mu^+\cap\mathrm{cof}(\mu)$. To this end, let $(a_\alpha)_{\alpha\in\mu^+}$ enumerate all elements of $[\mu^+]^{<\mu}$. Let $C\subseteq\mu^+$ be the set of all former limit cardinals $\beta\in\mu^+$ such that

$$\{a_\alpha \mid \alpha\in\beta\} \supseteq [\mu^+]^{<\mu} \cap \bigcup_{\alpha<\beta} V[G\cap\mathrm{Add}^{\oplus}(\omega,\alpha)]$$

C is club in μ^+ since every a_α is in $V[G\cap\mathrm{Add}^{\oplus}(\omega,\kappa)]$ by the projection analysis and hence in some $V[G\cap\mathrm{Add}^{\oplus}(\omega,\beta)]$ by the ccc. of $\mathrm{Add}^{\oplus}(\omega,\beta)$. Again because of the chain condition, we note that $\bigcup_{\alpha<\beta} V[G\cap\mathrm{Add}^{\oplus}(\omega,\alpha)] = V[G\cap\mathrm{Add}^{\oplus}(\omega,\beta)]$ whenever β has cofinality $>\tau$ (in this case, $\tau=\omega$ but we note this for the proof of theorem 3) which is true in particular if $\mathrm{cof}(\beta)=\mu$.

Now we show that every point in $C\cap\mathrm{cof}(\mu)$ is approachable with respect to $(a_\alpha)_{\alpha<\mu^+}$, showing $\mu^+\cap\mathrm{cof}(\mu)\in I[\mu^+]$ and thus the statement. To this end, let $\beta\in C$ have cofinality μ. Forcing with $\mathbb{M}(\mu,\beta+1)$ adds a set $E\subseteq\beta$ with ordertype $\mu=\mathrm{cof}(\beta)$ such that $E\cap\gamma\in V[\mathrm{Add}^{\oplus}(\omega,\beta)]$ for every $\gamma\in\beta$. By our previous remarks this implies that β is approachable with respect to $(a_\alpha)_{\alpha<\mu^+}$.

Now we show the second statement. To this end, in V, let $\dot{F}$ be an $\mathbb{M}$-name for a function from $[H(\mu^{+++})]^{<\omega}$ into $[H(\mu^{+++})]^{\mu}$. By GCH, we have $|H(\mu^+)|=\mu^+$, $|H(\mu^{++})|=\mu^{++}$ and $|H(\mu^{+++})|=\mu^{+++}$ in $V[G]$, so let $\dot{H}$ be an $\mathbb{M}$-name for a bijection between μ^{+++} and $H(\mu^{+++})$ that also witnesses the other equalities. Let Θ be large enough to contain all relevant objects and, by $\Pr(\kappa,\kappa^{++})$, let $M\prec H(\Theta)$ have the following properties:

(1) $\dot{F},\dot{H},\mathbb{M},\mu,\kappa\in M$
(2) $\nu:=M\cap\kappa$ is inaccessible
(3) $[M\cap\kappa^{++}]^{<\nu}\subseteq M$
(4) $\operatorname{otp}(M\cap\kappa^+)$ and $\operatorname{otp}(M\cap\kappa^{++})$ are cardinals.

Let $\pi\colon M\to N$ denote the Mostowski-Collapse of M. The following facts are standard:

(1) $\pi(\kappa)=\nu$
(2) $\pi(\kappa^+)=\operatorname{otp}(M\cap\kappa^+)=\nu^+$
(3) $\pi(\kappa^{++})=\operatorname{otp}(M\cap\kappa^{++})=\nu^{++}$

The last equalities in (2) and (3) follow from the fact that no ordinal in $(\operatorname{otp}(M\cap\kappa),\operatorname{otp}(M\cap\kappa^+))$ or $(\operatorname{otp}(M\cap\kappa^+),\operatorname{otp}(M\cap\kappa^{++}))$ can be a cardinal by elementarity.

Again by standard facts, $\pi\colon M\to N$ extends to $\pi\colon M[G]\to N[G']$, where $G':=G(\nu)$. Because $M[G]$ contains $\dot{F}^G$, $M[G]\cap H(\mu^{+++})=M[G]\cap H(\kappa^{++})$ is closed under $\dot{F}^G$. We will show that the set is as required.

We note that by the ν-cc. of $\mathbb{M}(\mu,\nu)$, $N[G']$ is closed under $<\nu$-sequences in $V[G']$.

Claim. *$M[G]\cap H(\kappa)$ is internally approachable in $V[G]$.*

Proof. In $V[G(\nu+1)]$ there is an increasing and continuous sequence $(a_i)_{i\in\mu}$ of $<\mu$-sized subsets of ν such that $\bigcup_{i\in\mu}a_i=\nu$ and every initial segment of the sequence lies in $V[G(\nu)]$. It follows that every initial segment of the sequence lies in $N[G']$, so for every $j\in\mu$, there is $b_j\in M[G]$ with $\pi(b_j)=(a_i)_{i<j}$. But since π is the identity on ν, $\pi(b_i)=b_i$. Hence $M[G]$ is internally approachable because $M[G]$ contains a bijection between $H(\kappa)$ and κ. □

Claim. *$M[G]\cap H(\kappa^+)$ is not internally approachable in $V[G]$.*

Proof. Assume the statement fails. In particular there exists, in $V[G]$, an increasing and continuous sequence $(a_i)_{i\in\mu}$ of $<\mu$-sized subsets of $M[G]\cap\kappa^+$ such that $\bigcup_{i\in\mu}a_i=M[G]\cap\kappa^+=M\cap\kappa^+$ and $(a_i)_{i\in j}\in M[G]\cap H(\kappa^+)$ for every $j<\mu$ (since $M[G]$ contains a bijection between $H(\kappa^+)$ and κ^+). Consider the sequence $(b_i)_{i\in\mu}:=(\pi[a_i])_{i\in\mu}$. Because $\mu<\nu$, we have $(b_i)_{i<j}=\pi((a_i)_{i<j})\in N[G']\subseteq V[G(\nu+1)]$ for every $j<\mu$. However, the pair $(V[G(\nu+1)],V[G])$ has the $<\mu$-approximation property, so $(b_i)_{i\in\mu}\in V[G(\nu+1)]$. This implies that $\pi(M\cap\kappa^+)=(\nu^+)^V$ has size μ in $V[G(\nu+1)]$, a contradiction, as $G(\nu+1)$ is generic for the forcing $\mathbb{M}(\mu,\nu)*\dot{\operatorname{Coll}}(\mu,\check{\nu})$ which is ν^+-cc. ($\mathbb{M}(\mu,\nu)$ is ν-cc. and $\dot{\operatorname{Coll}}(\mu,\check{\nu})$ is forced to be ν^+-cc.). □

Claim. *$M[G]\cap H(\kappa^+)$ is internally club in $V[G]$.*

Proof. $\mathbb{M}(\mu,\kappa)$ projects to $\mathbb{M}(\mu,\nu^++1)$ which is isomorphic to $\mathbb{M}(\mu,\nu+1)*\operatorname{Add}(\omega)*\mathbb{P}([\check{\nu^+}]^{<\mu}\cap V[\operatorname{Add}(\mu,\nu)])$. Ergo there exists, in $V[G(\nu^++1)]$, a sequence $(a_i)_{i<\mu}$ of elements of $V[\operatorname{Add}(\mu,\nu)]$ with union ν^+. By the closure of $N[G']$ in $V[G']$, every such a_i is in $N[G']$ and such of the form $\pi(b_i)$ for some $b_i\in M[G]$. Ergo the sequence $(b_i)_{i<\mu}$ has union $M[G]\cap\kappa^+$ be elementarity which implies that $M[G]\cap H(\kappa^+)$ is internally club since it contains a bijection between κ^+ and $H(\kappa^+)$. □

Claim. *$M[G]\cap H(\kappa^{++})$ is not internally club in $V[G]$.*

Proof. Assume that the statement fails. In particular there exists, in $V[G]$, a club $\{a_i \mid i \in \mu\}$ of $<\mu$-sized subsets of $M[G] \cap \kappa^{++} = M \cap \kappa^{++}$ such that $a_i \in M[G] \cap H(\kappa^{++})$ for every $i \in \mu$. Ergo, letting $b_i := \pi(a_i)$ (which equals $\pi[a_i]$, as $a_i \subseteq M[G]$), the collection $\{b_i \mid i \in \mu\}$ is club in $[(\nu^{++})^V]^{<\mu}$ and any b_i is in $N[G'] \subseteq V[G(\nu)]$.

Consider the pair $(V[G(\nu)], V[G(\nu^+ + 1)])$. $V[G(\nu^+ + 1)] \setminus V[G(\nu)]$ contains a real, μ is a regular cardinal in $V[G(\nu^+ + 1)]$ and $(\mu^+)^{V[G(\nu^+ + 1)]} = (\nu^{++})^V$ (since ν and ν^+ were both collapsed and ν^{++} was preserved). Ergo in $V[G(\nu^+ + 1)]$, the set

$$S := \{N \in [H(\nu^{++})]^{<\mu} \cap \mathrm{IA}(\omega) \mid N \cap \nu^{++} \notin V[G(\nu)]\}$$

is stationary in $[H(\nu^{++})]^{<\mu}$. In $V[G]$, the same set (which is now a subset of $[H(\nu^{++})^{V[G(\nu^+ + 1)]}]^{<\mu}$) is still stationary by Lemma 0.3 and therefore so is

$$S' := \{N \cap (\nu^{++})^V \mid N \in S\}$$

Ergo there exists $i \in \mu$ and $N \in S$ with $N \cap (\nu^{++})^V = b_i$, an obvious contradiction. $\square$

Claim. *$M[G] \cap H(\kappa^{++})$ is internally stationary in $V[G]$.*

Proof. In $V[G(\nu)]$, $[\pi[M[G] \cap H(\kappa^{++})]]^{<\mu} \subseteq \pi[M[G] \cap H(\kappa^{++})]$. By Lemma 0.4, $[\pi[M[G] \cap H(\kappa^{++})]]^{<\mu} \cap V[G(\nu)]$ is still stationary in $[\pi[M[G] \cap H(\kappa^{++})]]^{<\mu}$ in $V[G]$ because of the projection analysis, so $\pi[M[G] \cap H(\kappa^{++})]$ is internally stationary in $V[G]$. $\square$

So we have produced a model as required. $\square$

3. Approachability Together with the Distinction at $H(\mu^+)$

In this section, we will construct a model in which the approachability property holds at μ^+ but there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable (we can arrange for a distinction between internal stationarity and clubness as well as internal clubness and approachability). This shows that the cardinal arithmetic assumptions in Theorem 0.10 and Theorem 0.11 cannot be relaxed and answers question 1 raised by Levine in [12].

We use the same forcing we used to obtain a model in which AP_μ holds but the distinction holds for stationarily many $N \in [H(\mu^{++})]^\mu$. If we force with $\mathrm{Add}(\mu, \kappa^+)^V$ afterwards, the model $N \cap H(\mu^+)^W$ inherits its "approachability type" not from $N \cap H(\mu^+)^V$ but from $N \cap H(\mu^{++})^V$ (since $H(\mu^+)[G] \neq H(\mu^{++})^{V[G]}$). So it is reasonable to expect that one obtains the desired model by forcing with $\mathbb{M}(\mu, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$ which is indeed the case. In this case, we can force with $\mathbb{M}(\tau, \mu, \kappa)$ since we are only aiming for a distinction between internal clubness and approachability.

Theorem 3.1. *Assume $\tau < \mu < \kappa$ are cardinals such that $\tau^{<\tau} = \tau$, μ is regular and $\Pr(\kappa, \kappa^+)$ holds. Assume GCH holds above κ. After forcing with $\mathbb{M}(\tau, \mu, \kappa) \times \mathrm{Add}(\mu, \kappa^+)$, the following holds:*

1. *$\mu^+ \in I[\mu^+]$ (so there does not exist a disjoint stationary sequence on μ^+).*
2. *There are stationarily many $N \in [H(\mu^+)]^\mu$ such that N is internally club but not internally approachable.*

Proof. By the proof of Theorem 2.5, $\mu^+ \in I[\mu^+]$ after forcing with $\mathbb{M}(\tau, \mu, \kappa)$. So there exists a club $C \subseteq \mu^+$ and a sequence $(a_\alpha)_{\alpha \in \mu^+}$ such that any $\gamma \in C$ is approachable with respect to $(a_\alpha)_{\alpha < \mu^+}$. This is of course preserved by further forcing which does not collapse μ^+. By Theorem 0.11, this implies that there does not exist a disjoint stationary sequence on μ^+.

We write $\mathbb{Q} := \mathbb{M}(\tau,\mu,\kappa) \times \mathrm{Add}(\mu,\kappa^+)$. Let $G = H \times K$ be a $\mathbb{Q}$-generic filter. Given $\nu < \kappa$ and $A \subseteq \kappa^+$, let $G(\nu, A) = H(\nu) \times K(A)$ be the $\mathbb{M}(\tau,\mu,\nu) \times \mathrm{Add}(\mu, A)$-generic filter induced by G.

Let $\dot{F}$ be a $\mathbb{Q}$-name for a function from $[H(\mu^+)]^{<\omega}$ into $[H(\mu^+)]^\mu$. In $V[G]$, $H(\mu^+)$ has size κ^+, so we can fix a $\mathbb{Q}$-name $\dot{I}$ for a bijection between κ^+ and $H(\mu^+)$. Let Θ be large enough to contain all relevant objects and, by $\mathrm{Pr}(\kappa,\kappa^+)$, let $M \prec H(\Theta)$ have the following properties:

(1) $\dot{F}, \dot{I}, \mathbb{Q}, \tau, \mu, \kappa \in M$,
(2) $\nu := M \cap \kappa$ is inaccessible,
(3) $[M \cap \kappa^+]^{<\nu} \subseteq M$
(4) $\mathrm{otp}(M \cap \kappa^+)$ is a cardinal.

Let $\pi \colon M \to N$ denote the Mostowski-Collapse of M. The following facts are standard:

(1) $\pi(\kappa) = \nu$
(2) $\pi(\kappa^+) = \mathrm{otp}(M \cap \kappa^+) = \nu^+$
(3) $[N]^{<\nu} \subseteq N$

Again by standard facts, $\pi \colon M \to N$ extends to $\pi \colon M[G] \to N[G']$ where $G' = \pi[G \cap M] = H(\nu) \times \pi[H(M \cap \kappa^+)]$. Because $M[G]$ contains $\dot{F}^G$, $M[G] \cap H(\mu^+) = M[G] \cap H(\kappa)$ is closed under $\dot{F}^G$. We will show that the set is as required.

We note that, since $\mathrm{Add}(\mu,\kappa^+)$ is $<\mu$-closed, $N[G']$ is closed under $<\mu$-sequences in $V[H(\nu) \times K]$. We also note that, by elementarity, $\pi(\dot{I}^G) = (\pi(\dot{I}))^{G'} \in V[G(\nu) \times K]$ is a bijection between ν^+ and $\pi[M[G] \cap H(\mu^+)]$.

Claim. *$M[G] \cap H(\mu^+)$ is not internally approachable in $V[G]$.*

Proof. Assume that in $V[G]$ there is an increasing and continuous sequence $(a_i)_{i\in\mu}$ of $<\mu$-sized subsets of $M[G] \cap H(\mu^+)$ such that $\bigcup_{i\in\mu} a_i = M[G] \cap H(\mu^+)$ and every initial segment of the sequence lies in $M[G] \cap H(\mu^+)$. Consider the sequence $(b_i)_{i\in\mu} := (\pi[a_i])_{i\in\mu}$. For any $j < \mu$, $(b_i)_{i<j} = \pi((a_i)_{i<j})$ and therefore in $N[G'] \subseteq V[G(\nu+1) \times K]$. Because the pair $(V[H(\nu+1) \times K], V[G])$ has the $<\mu$-approximation property (as in [7], Lemma 2.9), the whole sequence $(b_i)_{i\in\mu}$ is in $V[H(\nu+1) \times K]$. Because $\bigcup_{i\in\mu} b_i = \pi[M[G] \cap H(\kappa)]$, we have a bijection between μ and ν^+ in $V[H(\nu+1) \times K]$. However, this leads to a contradiction as $H(\nu+1) \times K$ is a generic filter for the ν^+-cc. forcing $\mathbb{M}(\tau,\mu,\nu) * \dot{\mathrm{Coll}}(\mu,\nu) \times \mathrm{Add}(\mu,\kappa^+)$. □

We are finished after showing:

Claim. *$M[G] \cap H(\mu^+)$ is internally club in $V[G]$.*

Proof. Forcing with $\mathbb{M}(\tau,\mu,\nu^+ + 1)$ adds a sequence $(a_i)_{i\in\mu}$ of elements of $[\nu^+]^{<\mu} \cap V[\mathrm{Add}(\tau,\nu)]$ with $\bigcup_{i\in\mu} a_i = \nu^+$. We want to show that $\{J[a_i] \mid i \in \mu\}$ witnesses that $\pi[M[G] \cap H(\kappa)]$ is internally club (from this it is easy to show the same statement for $M[G] \cap H(\kappa)$). Clearly the sequence $(J[a_i])_{i\in\mu}$ is increasing and continuous. Let $i \in \mu$. Then $J[a_i] \subseteq N[G']$ and has size $<\mu$. By our previous remarks, $J[a_i] \in N[G']$ and by its size, it is in $\pi[M[G] \cap H(\kappa)]$. □

So we have produced a model as required.

□

Remark 3.2. By slightly modifying the construction (forcing with $\mathrm{Add}(\omega)$ instead of $\mathrm{Add}(\tau)$ and collapsing using the Levy collapse instead of shooting a club), we could have also instead obtained a distinction between internal stationarity and clubness.

4. Ordinal-Approachability

As we have seen, if $2^\mu \neq \mu^+$, the existence of a disjoint stationary sequence might not be equivalent to a distinction between internal unboundedness and clubness. Most of this is due to the fact that a disjoint stationary sequence is only concerned with ordinals while the distinction is related to the whole set $H(\mu^+)$. In this small section, we will introduce a concept that gives an "iff-criterion" for the existence of a disjoint stationary sequence without any cardinal arithmetic assumptions.

Definition 4.1. Let $N \in [X]^\mu$. We say that N is

(1) *ordinal-internally unbounded* if $[N \cap \mathrm{On}]^{<\mu} \cap N$ is unbounded in $[N \cap \mathrm{On}]^{<\mu}$.
(2) *ordinal-internally stationary* if $[N \cap \mathrm{On}]^{<\mu} \cap N$ is stationary in $[N \cap \mathrm{On}]^{<\mu}$.
(3) *ordinal-internally club* if $[N \cap \mathrm{On}]^{<\mu} \cap N$ contains a club in $[N \cap \mathrm{On}]^{<\mu}$.
(4) *ordinal-internally approachable* if there is a sequence $(a_i)_{i\in\mu}$ of elements of $[N \cap \mathrm{On}]^{<\mu}$ such that $\bigcup_{i\in\mu} a_i = N \cap \mathrm{On}$ and $(a_i)_{i<j} \in N$ for every $j < \mu$.

Clearly if $F\colon \Theta \to H(\Theta)$ is a bijection and $N \prec (H(\Theta), \in, F)$ has size μ, N is ordinal-internally unbounded (stationary; club; approachable) if and only if N is internally unbounded (stationary; club; approachable).

A small modification of Krueger's proof of Theorem 0.11 gives the following:

Theorem 4.2. *Let μ be a regular uncountable cardinal. The following are equivalent:*

(1) *There exists a disjoint stationary sequence on μ^+.*
(2) *There are stationarily many $N \in [H(\mu^+)]^\mu$ such that N is internally unbounded but not ordinal-internally club.*

Our previous results give us the following consistency result regarding ordinal-approachability:

Theorem 4.3. *It is consistent that there exist stationarily many $N \in [H(\mu^+)]^\mu$ which are ordinal-internally approachable but not internally stationary.*

5. Open Questions

We finish with two open questions: First we are concerned if the use of the property $\mathrm{Pr}(\kappa, \kappa^+)$ was necessary to obtain our consistency results. For concreteness, we ask:

Question 5.1. What is the consistency strength of the assertion that there are stationarily many $N \in [H(\mu^+)]^\mu$ which are ordinal-internally approachable but not internally stationary?

We are also interested in if the other directions of Theorems 0.11 and 0.10 can be obtained without the cardinal arithmetic assumption.

Question 5.2. Is it consistent that AP_μ fails (or that there exists a disjoint stationary sequence on μ^+) but there do not exist stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable?

References

[1] Sean D. Cox. "Forcing Axioms, Approachability, and Stationary Set Reflection". In: *The Journal of Symbolic Logic* 86.2 (2021), pp. 499–530.

[2] James Cummings et al. "The Eightfold Way". In: *The Journal of Symbolic Logic* 83.1 (2018), pp. 349–371. ISSN: 00224812, 19435886.

[3] Matthew Foreman. "Some Problems in Singular Cardinals Combinatorics". In: *Notre Dame Journal of Formal Logic* 46.3 (2005), pp. 309–322.

[4] Matthew Foreman and Stevo Todorcevic. "A New Löwenheim-Skolem Theorem". In: *Transactions of the American Mathematical Society* 357.5 (2005), pp. 1693–1715. ISSN: 00029947.

[5] Sy-David Friedman and John Krueger. "Thin Stationary Sets and Disjoint Club Sequences". In: *Transactions of the American Mathematical Society* 359.5 (2007), pp. 2407–2420. ISSN: 00029947.

[6] Hannes Jakob. "Disjoint Stationary Sequences on an Interval of Cardinals". 2023. arXiv: `2309.01986 [math.LO]`.

[7] Hannes Jakob and Maxwell Levine. *Distinguishing Internally Club and Approachable on an Infinite Interval.* 2024. arXiv: `2404.15230 [math.LO]`.

[8] Thomas Jech. *Set Theory: The Third Millennium Edition.* Springer, 2003.

[9] John Krueger. "Internal Approachability and Reflection". In: *Journal of Mathematical Logic* 08.01 (2008), pp. 23–39. eprint: `https://doi.org/10.1142/S0219061308000701`.

[10] John Krueger. "Some applications of mixed support iterations". In: *Annals of Pure and Applied Logic* 158.1 (2009), pp. 40–57.

[11] K. Kunen. *Set Theory.* Studies in Logic: Mathematical. College Publications, 2011.

[12] Maxwell Levine. "On disjoint stationary sequences". 2023. arXiv: `2301.02634 [math.LO]`.