RESEARCH STATEMENT

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I am a researcher in set theory. Inside set theory, I am most interested in the study of large cardinals, in particular in obtaining consistency results at small cardinals from large cardinal assumptions and in investigating the effect certain large cardinal axioms have on the structure of the universe.

Set theory as a discipline differs from other mathematical areas because we are in many cases not interested in what *is* but in what *can be*, i.e. where other researchers want to show that a particular statement follows from the commonly accepted axioms of mathematics, set theorists want to show that a particular statement cannot be refuted from these laws. One example of such a statement is the *Continuum Hypothesis*, the statement that any infinite set of real numbers is either in bijection with $\mathbb{N}$ or with $\mathbb{R}$. Paul Cohen famously invented the method of forcing to show, together with known results of Kurt Gödel, that one can neither prove nor refute the Continuum Hypothesis from the axioms of ZFC.

A different set of such statements are *large cardinal axioms*. These assert the existence of certain sets which are so massive that, while we can prove that their nonexistence is consistent with our axioms, we cannot show that their existence is also consistent (however, for most of these axioms we have not yet found a refutation). This allows us to measure set-theoretical statements ϕ according to their *consistency strength* by showing that the theories $\mathrm{ZFC}+\phi$ and $\mathrm{ZFC}+A$ (where A asserts the existence of a certain large cardinal) are equiconsistent (i.e. assuming the existence of a model of one theory one can provide a model of the other theory). An example of such a statement is the *tree property*, which holds at a cardinal κ if any thin tree on κ, i.e. a set $T \subseteq 2^{<\kappa}$ closed under restriction such that for any α there is at least one but fewer than κ many elements of T with domain α, has a cofinal branch, a function $b\colon \kappa \to 2$ such that for any $\alpha < \kappa$, $b \restriction \alpha \in T$. William Mitchell famously proved in his PhD thesis (see [21]) that the tree property at a successor cardinal is *equiconsistent* with the existence of a weakly compact cardinal. He did so by constructing a forcing order (now known as *Mitchell Forcing*) which turns a weakly compact cardinal κ into a successor cardinal which has the tree property. Another such statement is *Martin's Maximum*, a so-called forcing axiom, which asserts that for any forcing order $\mathbb{P}$ which preserves stationary subsets of ω_1 and any collection $\mathcal{D}$ of open dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$ there exists a filter $G \subseteq \mathbb{P}$ that meets every element of $\mathcal{D}$. The consistency strength of Martin's Maximum is a famous open problem in set theory. While we know that its consistency follows from the consistency of the existence of a supercompact cardinal, there are no known techniques which are able to show the consistency of the existence of a supercompact cardinal from the consistency of Martin's Maximum.

During the course of my PhD, I constructed many variants of Mitchell Forcing in order to solve questions regarding the variations of internal approachability

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(introduced by Foreman and Todorcevic) and the ineffable slender list property (introduced by Christoph Weiß). Later on I worked on Friedman's principle, a strong version of stationary reflection which follows from Martin's Maximum. In that project, I focused on obtaining distinctions of instances of this property (both between particular instances at the same cardinal as well as equal instances at different cardinals).

1. Variants of Mitchell Forcing

In Mitchell's famous work proving the relative consistency of the tree property, he (implicitly) introduced a nowadays crucial property known as the "$<\mu$-approximation property" in order to obtain the result that certain quotients of his forcing do not add branches to already existing trees. We say that a forcing $\mathbb{P}$ has the *$<\mu$-approximation property* if it does not add a set x such that $x \cap z$ is in the ground model for every $<\mu$-sized subset of V that already lies in V.

In my thesis, I introduced a framework for working with arbitrary orders on products of sets and obtained simple criteria which explain the behavior of Mitchell's original poset. I also used those results in order to construct many variants of Mitchell Forcing to obtain consistency results regarding the variants of internal approachability and the ineffable slender list property.

Variants of Internal Approachability and the Approachability Ideal. The variants of internal approachability were introduced by Foreman and Todorcevic in [5]. Given a set X with $|X| = \mu$ (a regular uncountable cardinal), we say that X is internally unbounded (stationary; club) if $[X]^{<\mu} \cap X$ is unbounded (is stationary; contains a club) in $[X]^{<\mu}$ and that X is internally approachable if there is a sequence $(x_i)_{i<\mu}$ of elements of $[X]^{<\mu}$ such that $\bigcup_{i<\mu} x_i = X$ and $(x_i)_{i<j} \in X$ for any $j < \mu$. Krueger showed in a series of papers (see [16], [17], [18] and [19]) that these properties can consistently be non-equivalent on a stationary subset of $[H(\Theta)]^{\mu}$. E.g. in any model of Martin's Maximum any two properties are nonequivalent on a stationary subset of $[H(\Theta)]^{\omega_1}$ for any $\Theta > \omega_1$. Krueger asked in [19] whether these separations can occur for many cardinals simultaneously.

I answered this question in the affirmative (partially joint with Maxwell Levine) in [12] and [15] by proving the following:

Theorem. *Assume $(\kappa_n)_{n\in\omega}$ is an increasing sequence of Mahlo cardinals. There are (separate) forcing extensions where $\kappa_n = \omega_{n+2}$ for all $n \in \omega$ and the following holds for any $n \geq 1$:*

1. *For any $\Theta > \omega_n$ there are stationarily many $N \in [H(\Theta)]^{\omega_n}$ which are internally stationary but not internally club.*
2. *For any $\Theta > \omega_n$ there are stationarily many $N \in [H(\Theta)]^{\omega_n}$ which are internally club but not internally approachable.*

This result bears some similarities to a result by Cummings and Foreman (see [2]) showing that the tree property can consistently hold at every ω_{n+2}. However, there are two crucial differences: The first difference is that obtaining the separation for many cardinals simultaneously does not dramatically increase the consistency strength: The consistency for one cardinal needs a single Mahlo cardinal while the consistency for ω many cardinals simply needs ω many Mahlos. On the other hand, having the tree property at one cardinal requires just a weakly compact cardinal while the consistency strength of having the tree property at two consecutive cardinals is much larger than two weakly compact cardinals. The second difference is that obtaining the property at successive cardinals is much easier than in the case of the tree property: In the latter construction, an iteration of Mitchell forcings with guessing components is used while in the former construction we simply used an iteration of simple Mitchell forcings (and I showed in my thesis that the same result can be obtained even more easily by just using a product forcing).

Related to the variants of internal approachability is Shelah's Approachability ideal (see [25]), defined as follows: For a cardinal μ we let $S \in I[\mu^+]$ if there exists a sequence $(x_i)_{i<\mu^+}$ where each $x_i \in [\mu^+]^{<\mu}$ and a club $C \subseteq \mu^+$ such that for any $\gamma \in S \cap C$ there is $A \subseteq \gamma$ unbounded with minimal ordertype such that $\{A \cap \beta \mid \beta < \gamma\}$ is a subset of $\{x_\beta \mid \beta < \gamma\}$. Then, assuming $2^\mu = \mu^+$, the *approachability property at* μ (stating that $\mu^+ \in I[\mu^+]$) fails if and only if there are stationarily many elements of $[H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable (see [1], where the author attributes this result to the folklore).

I showed in [11] that this relies on the assumption $2^\mu = \mu^+$:

Theorem. *Assume κ is κ^+-supercompact and $\mu < \kappa$ is regular. There is a forcing extension where $\kappa = \mu^+$, $\mu^+ \in I[\mu^+]$ and there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally stationary but not internally club.*

In the same paper, I defined the variants of On-internal approachability which provide an improved version of the equivalence: Having a stationary set $S \subseteq [H(\mu^+)]^\mu$ of structures which are On-internally unbounded but not On-internally approachable always implies the failure of the approachability property while the failure of the approachability property together with $|[\mu^+]^{<\mu}| = \mu^+$ (i.e. having a single sequence which enumerates all of $[\mu^+]^{<\mu}$) implies that there are stationarily many $N \in [H(\mu^+)]^\mu$ which are On-internally unbounded but not On-internally approachable.

Answering a question of Shelah, Mitchell famously obtained the consistency of the statement "No stationary $S \subseteq E^{\omega_2}_{\omega_1}$ is in $I[\omega_2]$" using a form of side condition forcing. An important question in this area is whether this result can be obtained simultaneously for many successive cardinals. As iterating side condition forcing (especially under the required cardinal arithmetic) is very difficult, the following two approaches seem fruitful: One such approach is obtaining Mitchell's result using a variant of Mitchell Forcing which collapses cardinals in many different ways and incorporates club-shooting forcings and then iterating that construction. Another possible approach is using side conditions as in [22] and [23] and taking a product of these forcings. This approach seems fruitful because the tree property is much more fragile when compared to statements about the approachability ideal and the approach of using product forcing did in fact work in my PhD thesis to obtain the distinctions between internal stationarity and clubness as well as internal clubness and approachability on an interval (where I used that the statements are sufficiently downwards absolute).

A very different problem is obtaining a distinction between internal clubness and approachability at more cardinals than just the ω_{n+1}. Here we can use the "easy forcibility" of the distinction (i.e. that it is possible using product forcing) and the fact that this property does not make sense for singular cardinals (no $M \prec H(\Theta)$

with $\mu := |M|$ singular can be internally unbounded because $\mathrm{cf}(M \cap \mu^+) < \mu$) in order to try to obtain an answer to the following question:

Question. Is it consistent that for any regular cardinal μ and any $\Theta > \mu$ there are stationarily many $N \in [H(\Theta)]^\mu$ which are internally club but not internally approachable?

Again different is the question of whether or not the distinctions between internal unboundedness and stationarity as well as internal stationarity and clubness can be obtained at larger cardinals or with a more favourable cardinal arithmetic. All of the results regarding those separations rely on a result by Gitik (from [9], later refined by Krueger in [19]) stating that adding a Cohen real adds stationarily many new subsets of $[X]^{<\omega_1}$ whenever $|X| > \omega_1$ and that this stationarity is preserved by countably closed forcing. Dobrinen and Friedman showed in [3] that a similar situation can consistently occur when adding a Cohen subset of ω_1 but without the forcing indestructibility. The non-availability of analogues of the result by Gitik and Krueger means that presently there is no known way to obtain a distinction between internal unboundedness and stationarity at cardinals above ω_1 and no known way to obtain a distinction between internal stationarity and clubness at some cardinal μ without also forcing $2^\omega > \mu$.

The Ineffable Slender List Property.

The notion of a slender list was introduced by Christoph Weiß in his PhD thesis (see [26]). A function $f\colon [\lambda]^{<\kappa} \to [\lambda]^{<\kappa}$ is a *(κ, λ)-list* if $f(x) \subseteq x$ for all $x \in [\lambda]^{<\kappa}$. It is moreover *$\delta$-slender* if for any sufficiently large Θ there is a club $C \subseteq [H(\Theta)]^{<\kappa}$ such that for any $M \in C$ and $x \in [\lambda]^{<\delta} \cap M$, $f(M \cap \lambda) \cap x \in M$ (notice the similarities to the $<\delta$-approximation property). A set $b \subseteq \lambda$ is an *ineffable branch* if $\{x \in [\lambda]^{<\kappa} \mid f(x) = x \cap b\}$ is stationary. It was shown by Weiß in his thesis that for any $\lambda > \omega_1$ the *ineffable $<\omega_1$-slender (ω_2, λ)-list property* $\mathrm{ISP}(\omega_1, \omega_2, \lambda)$ (the statement that every $<\omega_1$-slender (ω_2, λ)-list has an ineffable branch) follows from PFA and is thus consistent from a supercompact cardinal. While the statement ISP is similar to the tree property, it has stronger implications. E.g. $\mathrm{ISP}(\omega_1, \omega_2, \omega_2)$ implies that $2^\omega = 2^{\omega_1}$ (unless $\mathrm{cf}(2^\omega) = \omega_1$ in which case $2^{\omega_1} = (2^\omega)^+$; see [20]) and that the approachability property fails at ω_1 (this is part of the folklore).

It was shown by Holy, Lücke and Njegomir in [10] that $\mathrm{ISP}(\omega_1, \kappa, \lambda)$ is consistent from a λ-ineffable cardinal. In [14], I proved a theorem characterizing when a variant of Mitchell forcing has a suitable approximation property and used this to show that most forcings intended to force the tree property can be adapted to force ISP (by simply shifting the collapsing forcing to occur at successor ordinals). Moreover, many statements compatible with TP are also compatible with ISP (e.g. the notion of *club stationary reflection*).

In this area, one well-known open problem is the following:

Question. Is it consistent that $\mathrm{ISP}(\omega_1, \omega_n, \geq \omega_n)$ holds for any $n \in \omega$, $n \geq 2$?

Is it consistent that $\mathrm{ISP}(\omega_n, \omega_n, \geq \omega_n)$ holds for any $n \in \omega$, $n \geq 2$?

The two cases differ slightly because the first case would require constantly adding Cohen subsets of ω and thus making $2^\omega \geq \aleph_{\omega+1}$ while the second case could conceivably be achieved using the same cardinal arithmetic as for the tree property (i.e. $2^{\omega_n} = \omega_{n+2}$).

The best result in this direction comes from Mohammadpour and Velickovic in [23] who showed that the conjunction $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2) \wedge \mathrm{ISP}(\omega_1, \omega_3, \geq \omega_3)$ is consistent from two supercompact cardinals. They actually show the consistency of a stronger statement which also implies the previously mentioned result of Mitchell regarding the approachability ideal. However, their approach uses a variant of side condition forcing which has the aforementioned problems regarding iterability. It stands to reason therefore that Mitchell Forcing again presents a fruitful approach.

2. Friedman's Principle

The principle $F(\kappa)$ was introduced by Harvey Friedman in [7]. It states that any subset of κ either contains or is disjoint from a closed subset of κ with ordertype ω_1. In later work of Shelah (see [24], chapter XI) and Foreman, Magidor and Shelah (see [6]), stronger variants of this property were investigated. Collecting all of these (and introducing some new variants), we have the following:

(1) For $\kappa \geq \omega_2$ regular and $\lambda \leq \kappa$, $F(\lambda, \kappa)$ states that any regressive function $f\colon \kappa \to \lambda$ is constant on a closed subset of κ with ordertype ω_1.
(2) For a partition $(D_i)_{i\in\omega_1}$ of ω_1 and a regular cardinal $\kappa \geq \omega_2$, $F^+((D_i)_{i\in\omega_1}, \kappa)$ states that for any sequence $(A_i)_{i\in\omega_1}$ of stationary subsets of E^κ_ω there is a normal function $g\colon \omega_1 \to \kappa$ such that $g[D_i] \subseteq A_i$ for all $i \in \omega_1$.

It was shown by Silver, as stated in [7], that simply collapsing ω_1 to ω with finite conditions forces the failure of $F(2, \kappa)$ for every regular $\kappa \geq \omega_2$. On the other hand, Shelah showed in [24], chapter XI, that from a weakly compact cardinal the strongest form of $F^+((D_i)_{i\in\omega_1}, \omega_2)$ is consistent (where $D_i = \{i\}$). Moreover, it was shown by Feng and Jech in [4] that $F^+((D_i)_{i\in\omega_1}, \kappa)$ follows from SRP, a well-known consequence of Martin's Maximum, whenever $(D_i)_{i\in\omega_1}$ is a maximal partition of ω_1 into stationary sets (i.e. for any stationary $A \subseteq \omega_1$, there is $i \in \omega_1$ such that $A \cap D_i$ is stationary). Lastly, Gunter Fuchs showed in [8] that $F^+((D_i)_{i\in\omega_1}, \kappa)$ follows from the *subcomplete forcing axiom* and its corresponding variant of SRP whenever $\kappa > 2^\omega$ and $(D_i)_{i\in\omega_1}$ is an arbitrary partition of ω_1 into stationary sets.

In [13], I extended the previous results and showed that $F^+((D_i)_{i\in\omega_1}, \kappa)$ follows from SRP for any partition $(D_i)_{i\in\omega_1}$ and $\kappa \geq \omega_2$ and moreover has a canonical fragment of SRP associated to it (which we will call $\mathrm{SRP}((D_i)_{i\in\omega_1}, \kappa)$). I also introduced posets which add witnesses to the failure of F and F^+ in a more gentle manner. This allowed me to obtain separation results regarding the previous properties. For the principle F, I showed the following:

Theorem. *Assume κ is supercompact and $\lambda \in [0, \omega_1] \cup \{\kappa\}$ is a cardinal. There is a forcing extension where $\kappa = \omega_2$, $F(\lambda, \kappa)$ fails and $F(\lambda', \kappa)$ holds for all $\lambda' < \lambda$.*

In particular, it is e.g. relatively consistent that for any partition of ω_2 into 42 pieces one of those pieces contains a closed copy of ω_1 while there is a partition of ω_2 into 43 pieces such that no piece contains a closed copy of ω_1.

For the principle F^+ and the corresponding variant SRP, there is a partial order $\leq^*$ on the set of partitions (defined by letting $\overline{E} \leq^* \overline{D}$ if $\overline{E}$ refines $\overline{D}$ on a club) which has the property that, if $\overline{E} \leq^* \overline{D}$ and $\mathrm{SRP}((E_i)_{i\in\omega_1}, \kappa)$ holds, $F^+((D_i)_{i\in\omega_1}, \kappa)$ holds as well. Moreover, it turns out that $\leq^*$ characterizes the implication perfectly:

Theorem. *Assume V is the standard model of Martin's Maximum and $(D_i)_{i\in\omega_1}$ is a partition of ω_1. There is a forcing extension in which $F^+((D_i)_{i\in\omega_1}, \omega_2)$ fails and $\mathrm{SRP}((E_i)_{i\in\omega_1}, \omega_2)$ holds for any partition $(E_i)_{i\in\omega_1}$ of ω_1 with $\overline{E} \not\leq^* \overline{D}$.*

In a different direction, I obtained results regarding the effect of certain large cardinal properties on the possible patterns of F and F^+. Here there are different "meta-properties" of F and F^+ which come from the added complexity of the requirement of stationarity in F^+ (which is also reflected in the properties of the corresponding forcing notions: The forcing adding a witness to $\neg F^+((D_i)_{i\in\omega_1},\kappa)$ is always δ-strategically closed while the forcing adding a witness to $\neg F(\lambda,\kappa)$ is at most $<\kappa$-strategically closed unless $F(\lambda,\kappa)$ already fails). As an example, I showed the following:

Theorem. *If κ is weakly compact and $F(\delta)$ fails for all $\delta<\kappa$, then $F(\kappa)$ fails.*

On the other hand, if λ is supercompact and $\kappa>\lambda$ is weakly compact, there is a forcing extension in which $F^+(\kappa)$ holds but $F^+(\delta)$ fails for all $\delta<\kappa$.

Some time ago, Monroe Eskew asked on Mathoverflow whether higher analogues of F and F^+ can consistently hold. One such analogue would be the following statement:

Question. Is it consistent that for any stationary $S\subseteq E^{\omega_3}_\omega$ there is a normal function $f\colon\omega_2\to\omega_3$ such that $f[E^{\omega_2}_\omega]\subseteq S$?

Of course, one can also ask about other configurations, e.g. replacing ω by ω_1 in the preceding statement.

I expect that answering this question would necessarily lead to the development of novel and interesting forcing techniques which could be used to obtain higher analogues of consequences of Martin's Maximum: Presently, the only ways of obtaining the consistency of F and F^+ are through Martin's Maximum (which has no known analogue at ω_3) or through Shelah's S-condition (which relies on the nice behavior of trees of height ω which cannot be replicated for larger heights).

Another interesting open question which relates more to the previous material is the exact degree of F^+ that is required by the singular cardinal hypothesis. Famously, it was shown by Foreman, Magidor and Shelah in [6] that Martin's Maximum implies the singular cardinal hypothesis, by showing it implies $\kappa^{\omega_1}=\kappa$ for every regular $\kappa\geq\omega_2$. Feng and Jech showed in [4] that this follows in particular from $F^+((D_i)_{i\in\omega_1},\kappa)$ where $(D_i)_{i\in\omega_1}$ is a maximal partition of ω_1 into stationary sets. An interesting open problem would thus be the following:

Question. For which types of partitions $(D_i)_{i\in\omega_1}$ of ω_1 is it consistent that the statement $F^+((D_i)_{i\in\omega_1},\omega_2)$ holds but $\omega_2^{\omega_1}>\omega_2$?

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