Towards Cherlin's Conjecture for Zariski Groups

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Introduction

One of the major contemporary problems in stability theory is Cherlin's conjecture, whether a simple $\aleph_0$-stable group (of finite rank) is an algebraic group. As long as the general Cherlin conjecture is still unsolved, it is natural to consider weaker forms.

One possibility is motivated by Hrushovski's and Zil'ber's work on strongly minimal sets and Zil'ber's conjecture, which is a similar problem. The main part of Zil'ber's conjecture asked whether a non locally modular, strongly minimal set were an algebraic curve. Hrushovski constructed counter-examples to this, but Hrushovski and Zil'ber succeeded in proving the conjecture for special strongly minimal sets, so called Zariski geometries. These are structures equipped with Noetherian topologies as abstract Zariski topologies. More precisely, their result characterizes abstractly the Zariski topology of smooth algebraic curves over algebraically closed fields.

In the light of there work, it seems natural to consider "Zariski groups": $\aleph_0$-stable groups with an axiomatically given abstract Zariski topology. In this paper, I introduce a higher-dimensional generalization of Hrushovski–Zil'ber's Zariski geometries, and I define Zariski groups.

The main interest of Cherlin's conjecture, at least from an algebraic point of view, is the hope to get an abstract characterization of algebraic groups, not mentioning fields and varieties. While a positive solution of the general conjecture would characterize the constructible structure of algebraic groups, a solution for Zariski groups would provide a characterization of the Zariski topology of algebraic groups

This article gives an approach to Cherlin's conjecture for Zariski groups. In particular I show that any non nilpotent smooth Zariski group interprets an algebraically closed field. Most probably, this result follows also from Hrushovski's and Zil'ber's work [HZ2]. The problem is to show that there is a strongly minimal subset of a smooth Zariski group that satisfies the dimension formula. Anyhow, I hope my proof is of interest because my methods are more elementary and might be more easily understood.

1 Zariski geometries

The following definition of a Zariski geometry is a generalization to arbitrary finite dimensions of Hrushovski's and Zil'ber's one dimensional Zariski geometries (cf. [HZ2]). Zil'ber's notion of a Zariski–type structure (see [Z2]) is similar, but he introduces dimension axiomatically. In fact, a smooth, simple, sufficiently saturated Zariski group will be a Zariski–type structure in Zil'ber's sense.

Some topological prerequisites:
Let T be a Noetherian topological space, i.e. a space without infinite strictly descending chains of closed sets. A subset X of T is **irreducible** if it is not empty and not the union of two proper relatively closed subsets. An **irreducible component** of a set is a maximal irreducible subset. Any subset of T is the union of its finitely many irreducible components. The (topological) **dimension** of a set is the maximal length of a chain of relatively closed irreducible subsets[1], more precisely $\dim X := \sup\{n \mid \exists X_i \text{ irreducible}, X_i = \overline{X_i} \cap X \text{ and } X_0 \subset X_1 \subset \ldots \subset X_n\}$. A **hypersurface** of X is a relatively closed irreducible subset of dimension $\dim X - 1$. A **constructible** set is a Boolean combination of closed sets.

Definition 1.1 *A **Zariski geometry** is an infinite set Z and a finite dimensional Noetherian topology $\tau_n[Z]$ on Z^n for each n such that:*

irreducibility & separation: *Z is irreducible and the diagonal $\Delta(Z)$ is closed;*

quantifier elimination: *for each n, every projection $Z^{n+1} \to Z^n$ maps constructible sets onto constructible sets;*

compatibility: *for all n, k, every map $f = (f_1, \ldots, f_k) : Z^n \to Z^k$ where f_i is either a projection $(x_1, \ldots, x_n) \mapsto x_i$ or a constant map $(x_1, \ldots, x_n) \mapsto a$ for some $a \in Z$ is continuous.*

A first order language for a Zariski geometry is any relational language (with equality) such that the interpretations of the quantifier-free positive formulae with parameters are exactly the closed sets. One possibility it to take all closed sets as basic relations. But in some cases it might be more natural to consider smaller languages. E.g. for algebraically closed fields (with the Zariski topologies), it is sufficient to take the graphs of addition and multiplication as basic closed sets.

Let Z be a Zariski geometry and fix any possible language $\mathcal{L}$. Then Z eliminates quantifiers by item 3 of the definition. In other words, any definable set is constructible.

Let Y be an elementary equivalent $\mathcal{L}$-structure. There is a natural notion of closed subsets of Y^n, namely the sets defined by quantifier-free positive formulae with parameters in Y.

Fact 1.2 If for each n the closed sets on Y^n satisfy the descending chain condition, then they endow Y with a structure of a Zariski geometry, called the natural Zariski geometry on Y.

[1] This definition is not the general one, but it works well in the case of finite dimension.

Definition 1.3 *A Zariski geometry Z in a fixed language $\mathcal{L}$ is called an* ***elementary Zariski geometry*** *iff every elementary equivalent $\mathcal{L}$-structure is naturally a Zariski geometry.*

The following characterization of elementary Zariski geometries is a direct application of the compactness theorem:

Proposition 1.4 *Let Z be a Zariski geometry with language $\mathcal{L}$. The following are equivalent:*
(a) Z is an elementary Zariski geometry in $\mathcal{L}$.
(b) there are no closed sets F_i and tuples of parameters $\bar{a}_{ij}$ such that $Z \models F_0(\bar{a}_{0k}) \supset F_1(\bar{a}_{1k}) \supset \cdots \supset F_k(\bar{a}_{kk})$ for every $k \in \omega$.
(c) there is an $\aleph_0$-saturated elementary equivalent Zariski geometry.

Fact 1.5 (a) The property of being elementary is independent of the choice of the language.
(b) If $Y \preccurlyeq Z$ are models of an elementary Zariski geometry, then the inclusion maps $Y^n \to Z^n$ are continuous.

By induction on Cantor rank RC for formulae, it is easy to prove that Cantor rank is bounded by the dimension in a Zariski geometry, which yields immediately the following result:

Proposition 1.6 *An elementary Zariski geometry with a countable language is an $\aleph_0$-stable structure of finite rank.*

In general, neither Morley or Cantor rank need to equal the dimension, nor the dimensions in two models have to coincide. This is the case in $\aleph_0$-saturated models, or more generally if the dimension is definable in all models.

Lemma 1.7 *Let Z be a Zariski geometry. The following are equivalent:*
(a) $\mathrm{RC}(Q) = \dim Q$ for all constructible sets Q.
(b) $\dim Q = \dim \overline{Q}$ for all constructible set Q.
(c) The set $\mathcal{H}(\mathcal{F}) := \{\mathcal{G} \mid \mathcal{G}$ closed irreducible $\subset \mathcal{F},\ \dim \mathcal{G} + \infty = \dim \mathcal{F}\}$ is infinite for each closed irreducible infinite set F.

Note that given an irreducible set, property (c) allows to choose subsets of smaller dimension in sufficiently generic position.

2 Varieties and Morphisms

Fix any Zariski geometry Z and let V be an imaginary set in Z, i.e. a definable subset W of some Z^n divided by an definable equivalence relation E. Then there is a natural family of Noetherian topologies on the products $V \times Z^n$: first take the induced topology on $W \times Z^n$ and then its quotient topology under the natural surjection $W \times Z^n \to W/E \times Z^n$.

Definition 2.1 *An imaginary set $V = W/E$ equipped with these natural topologies on the products $V \times Z^n$ is called a* ***variety*** *iff E is closed in $W \times W$.*

Note that Z is itself a variety, hence properties of varieties or definitions for varieties apply to Z, too. If $V_1 = W_1/E_1$ and $V_2 = W_2/E_2$ are varieties, then the product $V_1 \times V_2 = (W_1 \times W_2)/(E_1 \times E_2)$ is a variety.

It is clear that a morphism should be a definable and continuous map, but in general this is not sufficient to prove the fundamental properties of proposition 2.3 (b) below. A slightly stronger property has to be required:

Definition 2.2 *Let V_1, V_2 be two varieties. A* ***morphism*** *is a definable map $f : V_1 \to V_2$ such that for each $n \in \omega$ the map $f \times \mathrm{id}_{Z^n} : V_1 \times Z^n \to V_2 \times Z^n$ is continuous.*

The identity map, constant maps, projections and diagonal maps $\delta_k : V \to V^k$, $v \mapsto (v, \ldots, v)$ are obvious examples of morphisms. If E is a closed equivalence relation on V, then the canonical surjection $p : V \to V/E$ is a morphism.

Now it is straightforward to prove to the following proposition:

Proposition 2.3 *(a) If $f : V_1 \to V_2$ is a morphism, then for every variety V, the map $f \times \mathrm{id}_V : V_1 \times V \to V_2 \times V$ is continuous.*

(b) Products, pairs and compositions of morphisms are morphisms. If $f : V_1 \times V_2 \to V$ is a morphism, then for all $c \in V_1$, the map $f_c : V_2 \to V$, $v \mapsto f(c, v)$ is a morphism.

(c) The graph of a morphism is closed.

A Noetherian space T satisfies the **dimension formula** if for all closed irreducible subsets F_1, F_2 and each irreducible component X of $F_1 \cap F_2$ the following holds:

$$\dim X \geq \dim F_1 + \dim F_2 - \dim T$$

Because irreducible components are non empty by definition, this inequality holds in particular, if $F_1 \cap F_2 = \emptyset$.

Definition 2.4 *A variety V is* ***smooth*** *iff the dimension formula holds in $V \times Z^n$ for each $n \in \omega$.*

If an $\aleph_0$-saturated Zariski geometry is smooth, then any $\aleph_0$-saturated elementary equivalent Zariski geometry is smooth, too.

Let $F \subseteq Z^n$ be a closed subset and $\pi : Z^n \to Z^l$ a projection. Define $\boldsymbol{\pi[F, \geq k]}$ to be the set of all $\bar{a}$ such that the π-fibre over $\bar{a}$ is of dimension at least k. The dimension is **definable** iff the sets $\pi[F, \geq k]$ are definable for all F, π and k, and **semi-continuous** iff these sets are closed in $\pi[F]$.

If the dimension is definable and $\pi[F]$ is irreducible, then there is exactly one k such that $\pi[F, \geq k] \setminus \pi[F, \geq k+1]$ is dense in $\pi[F]$. It is called the π-generic fibre dimension of F, denoted by **π-gdim F**. Semi-continuity (for all F) is equivalent to the fact that π-gdim F is the minimal dimension of non void π-fibres of F.

Finally, a Zariski geometry is called **additive** iff for all π and irreducible F, the equation $\pi\text{-gdim}\, F = \dim F - \dim \pi[F]$ holds.

Definition 2.5 *A Zariski geometry Z is **rich** if for any n and any $\bar{a} \in Z^n$, the intersection of all hypersurfaces of Z^n containing $\bar{a}$ is finite.*

Proposition 2.6 *Let Z be a rich, smooth and additive Zariski geometry. Let $F \subseteq Z^n$ be closed irreducible, $\pi : Z^n \to Z^l$ a projection and $a \in \pi[F]$. Then $\dim G \geq \dim F - \dim \pi[F]$ for each irreducible component G of $\pi^{-1}(a) \cap F$. In particular, dimension is semi-continuous.*

■ Let $F \subseteq Z^n$ be closed irreducible and $a \in \pi[F]$. Construct by induction on $i \leq \dim \pi[F]$ closed irreducible sets X_i containing a such that $\dim X_i = \dim \pi[F] - i$ and $\dim G \geq \dim F - i$ for every irreducible component G of $\pi^{-1}[X_i] \cap F$:

Let $X_0 := \overline{\pi[F]}$. Suppose X_i is constructed for $i < \dim \pi[F]$. Choose a hypersurface H_i of Z^l such that $a \in H_i$ but $X_i \not\subseteq H_i$. This is possible by richness and because X_i is infinite ($\dim X_i > 0$). Let X_{i+1} be an irreducible component of $X_i \cap H_i$ containing a. Then $X_{\dim \pi[F]} = \{a\}$, which gives the result. ■

Definition 2.7 *Let Z be a Zariski geometry. A variety C is **complete** iff for every $n \geq 1$, the projection $C \times Z^n \to Z^n$ maps closed sets onto closed sets.*

Many properties of complete algebraic varieties are satisfied in the general context, namely:

Proposition 2.8 *Let C be a complete variety.*

(a) The projection $\pi : C \times V \to V$ is a closed map for every variety V.

(b) If $f : C \to V$ is a morphism, then $f[C]$ is complete and closed in V.

(c) Any closed subset of C is itself complete.

(d) A finite Cartesian product of complete varieties is complete.

In fact, if either $V_1 \times V_2$ is a smooth variety or V_2 is complete, then a map $g : V_1 \to V_2$ is a morphism iff the graph of g is closed in $V_1 \times V_2$.

3 Zariski groups

Definition 3.1 *A **Zariski group** is an elementary Zariski geometry G in a countable language together with to morphisms $\mu : G^2 \to G$ and $\iota : G \to G$ being the multiplication and the inverse of a group law on G and such that dimension equals Morley rank in $\aleph_0$-saturated models.*

Recall that Morley rank is definable and additive in $\aleph_0$-stable groups of finite rank. So a Zariski group is an additive Zariski geometry with definable dimension. It is not very difficult to see that Morley rank equals dimension in any model. But to avoid any trouble, one may assume that all Zariski groups considered are $\aleph_0$-saturated.

Examples:

- Any algebraic group over an algebraically closed field is a Zariski group, the topology being the Zariski topology.
- Any $\aleph_0$-stable one-based group of finite rank is a Zariski group, closed sets being the cosets of definable subgroups.

Many proofs for algebraic groups go through for Zariski groups, e.g. (cf. [Hu] 7.3 and 7.4):

Proposition 3.2 *A definable subgroup H is closed. Its irreducible components are the cosets of the connected component H°.*

In particular, a Zariski group is a connected group by definition (because it is irreducible).

A definable subgroup H gives rise to two isomorphic varieties: left and right coset space. These are well behaved varieties.

Theorem 3.3 *(a) The natural surjection $p : G^{n+1} \to G/H \times G^n$ maps constructible sets on constructible sets.*
(b) If $Q \subseteq G/H \times G^n$ is constructible, then $\dim Q = \dim p^{-1}[Q] - \dim H$.
In particular, $\dim G = \dim G/H + \dim H$.
(c) Dimension in $G/H \times G^n$ equals Morley rank.
(d) Definability and additivity of dimension hold in $G/H \times G^n$.

■ First note that G/H is really a variety: the corresponding equivalence relation E_H is the inverse image of H under the morphism $(g, h) \mapsto g^{-1}h$, hence closed. It is sufficient to consider the case $n = 0$.

(a) is as in algebraic geometry, where one shows that p is even an open map which is a stronger property. (c) and (d) follow easily from (b).

(b) Let $Q \subseteq G/H$ be constructible and irreducible. A chain $F_0 \subset \cdots \subset F_k$ of relatively closed irreducible subsets of Q lifts to a chain $p^{-1}[F_0] \subset \cdots \subset p^{-1}[F_k]$. These sets are not necessarily irreducible, but it is easily seen that dimension increases at each step, whence $\dim p^{-1}[Q] \geq \dim Q + \dim p^{-1}[F_0] = \dim Q + \dim H$.

If S is an irreducible subset of G, there is an open subset U in $p[S]$ such that the dimension of the fibres $p^{-1}(u) \cap S$ is constant. Call this dimension the H-gdim S. Then the following equality holds: $\dim S + \dim H = \dim SH + H\text{-gdim}\, S$. Choosing subsets generically, it is possible to find a chain $S_0 \subset \cdots \subset S_{\dim Q}$ of relatively closed irreducible subsets of Q such that H-gdim $S_i \leq H$-gdim S_{i+1} for all i. Then $p[S_0] \subset \cdots \subset p[S_{\dim Q}]$ will be of length at least $\dim Q - \dim H$. ■

Corollary 3.4 *Let G be a smooth Zariski group. If H is a definable subgroup, then G/H is a smooth variety.*

Proposition 3.5 *If G is a Zariski group and N a definable normal subgroup, then G/N is a Zariski group. If G is smooth, then G/N also.*

■ G/N is a variety, hence there is a family of finite dimensional Noetherian topologies on the powers of G/N and the diagonal is closed. G/N is irreducible as continuous image of G. Quantifier elimination is immediate from theorem 3.3 (a).
The compatibility maps, multiplication and inverse are morphisms: straightforward. Morley rank equals dimension by theorem 3.3 (c).
If G is smooth, G^n is smooth by definition and $(G/N)^n \cong G^n/N^n$ is a smooth variety by corollary 3.4, in particular the dimension formula holds in $(G/N)^n$. ■

Complete and parabolic subgroups

Proposition 3.6 *(a) Let C be an irreducible complete subset in a Zariski group G such that $e \in C$. Then the normal subgroup generated by C is definable and complete.*
(b) G contains a maximal complete connected subgroup G^{c}. This subgroup is unique and normal in G.

■ Use Zil'ber's indecomposability theorem and proposition 2.8. ■

The following theorem generalizes the corollary of theorem 14 of [R]. In fact, it is possible to prove it without the semi-continuity property (i.e. in a not necessarily smooth Zariski group).

Theorem 3.7 *Assume dimension is semi-continuous. Then G^{c} is central in G.*

■ The proof is essentially the same as of the rigidity lemma in algebraic geometry. The set $T = \{(g,h) \mid \exists c \in G^{\mathsf{c}}, h = [c,g]\} \subseteq G \times G^{\mathsf{c}}$ is irreducible and closed by completeness of G^{c}. The projection π onto G gives rise to a finite fibre over e, hence the π-fibres are generically finite, i.e. the centralizer of generic elements are finite. Then it is not difficult to conclude that all centralizers are trivial. ■

Corollary 3.8 *A simple smooth Zariski group does not contain any infinite complete subset.*

■ If C is infinite complete and $c \in C$, then $c^{-1}C$ in still complete and contains e. By simplicity, G equals the normal subgroup generated by $c^{-1}C$, which is complete by proposition 3.6. It suffices to show that a simple smooth Zariski group is rich which will be proved in the following lemma. Then dimension is semi-continuous by proposition 2.6 and the preceding theorem applies, G is Abelian: contradiction. ■

Lemma 3.9 *A simple smooth Zariski group is rich.*

■ It suffices to verify richness for $(e, \ldots, e)$, because any point in G^k can be translated on $(e, \ldots, e)$ by an isomorphism.

Define $G^{\mathsf{h}} := \bigcap \{H \mid H \text{ a hypersurface of } G \text{ and } e \in H\}$. By Noetherianity, G^{h} is a definable normal proper subgroup of G, thus $G^{\mathsf{h}} = \{e\}$. If H is a hypersurface in G, a set $G^m \times H \times G^{k-m-1}$ is a hypersurface in G^k by additivity. Hence the intersection of all hypersurfaces in G^k containing $(e, \ldots, e)$ equals $G^{\mathsf{h}} \times \cdots \times G^{\mathsf{h}} = \{(e, \ldots, e)\}$. ■

Definition 3.10 *A definable subgroup H of a Zariski group G is* ***parabolic****, if G/H is a complete variety.*

Because left and right coset space are isomorphic, this notion is unambiguous: H is left parabolic iff it is right parabolic.

Theorem 3.11 *Let G be a smooth Zariski group with semi-continuous dimension. Suppose H is a connected definable subgroup such that $H \cap H^g = \{e\}$ for any $g \in G/H$. Then H is a parabolic subgroup of G.*

■ The hypothesis implies that $H = \mathbb{N}_G(H)$ and that conjugacy defines an equivalence relation on $\bigcup_{g \in G} H^g \backslash \{e\}$. The quotient $\mathbb{H}$ can be identified with the set $\{H^g \mid g \in G/\mathbb{N}_G(H)\}$, hence with $G/\mathbb{N}_G(H)$. In fact, this is an embedding of varieties, and $\mathbb{H}$ is easily shown to be complete. But it is more direct to show the result without constructing $\mathbb{H}$ explicitly:

Let $\Gamma := \bigcup_{g \in G} (Hg \times H^g) \times \Delta(G)^n$. This is a closed irreducible subset of G^{2n+2}. Let π_1 (resp. π_2) :$G^{2n+2} \to G^{n+1}$ be the projection on the coordinates with odd (even) index and $p : G^{n+1} \to G/H \times G^n$ the natural surjection. Consider $G/H = \{Hg \mid g \in G\}$ as a right coset space. To every set $X \subseteq G/H \times G^n$ associate the set $\gamma(X) := \pi_2[\Gamma \cap (p \circ \pi_1)^{-1}[X]] = \{(h^g, \bar{g}) \mid h \in H, (Hg, \bar{g}) \in X\} \subseteq G \times G^n$. In some sense, γ is a map whose graph is Γ.

Suppose X is closed irreducible. Then it is possible to show that $\gamma(X)$ is still closed irreducible. The proof works by calculating the dimensions of irreducible components of $\overline{\gamma(X)} \setminus \gamma(X)$. It makes essential use of the smoothness condition, additivity and semi-continuity of dimension applied to $\Gamma \cap (p \circ \pi_1)^{-1}[X]$.

Now it is easy to conclude: let $\pi : G \times G^n \to G^n$ and $\pi' : G \times G^n \to G$ be the projections. Obviously, $(p \circ \pi)[X] = \pi[\gamma(X)] = \pi'^{-1}(e) \cap \gamma(X)$ which is a closed set. So H is a parabolic subgroup. ■

Recall that an $\aleph_0$-stable group of finite rank is called **bad** if it is connected, non solvable, and all connected solvable subgroups are nilpotent. A **Borel** subgroup is a maximal connected solvable subgroup.

Proposition 3.12 *If G is a simple bad smooth Zariski group, then its Borel subgroups are parabolic.*

■ In a simple bad group, Borel subgroups are auto-normalizing and two distinct Borels intersect in $\{e\}$ — see [P] 3.31. A simple group is rich: see lemma 3.9. As dimension is semi-continuous (proposition 2.6), theorem 3.11 applies. ■

Proposition 3.13 *There are no bad smooth Zariski groups.* (confer [Hu] 21.4)

■ If G is a such a group, it has a simple bad factor group ([P] 3.31), which is still a smooth Zariski group (proposition 3.5). By corollary 3.8, G has no complete infinite subset.
Let B be a Borel, B is nilpotent, hence its center is non trivial. Take $c \in \mathbb{Z}(B)$ and let $\phi : G \to G$ be the morphism $g \mapsto g^c g^{-1}$. Then ϕ is constant on B-cosets, in particular $\phi[B] = \{e\}$. Thus ϕ factors through G/B providing a morphism $\bar{\phi} : G/B \to G$. Now $\bar{\phi}[G/B]$ is irreducible and complete, because G/B is complete, therefore $\bar{\phi}[G/B] = \{e\}$. It follows that $\phi = e$ and $c \in \mathbb{Z}(G)$, that is $\mathbb{Z}(B) \subseteq \mathbb{Z}(G)$ contradicting the simplicity of G. ■

By Zil'ber's theorem ([Z1], see [P] 3.20), any connected $\aleph_0$-stable group of finite rank that is neither bad nor nilpotent interprets an algebraically closed field. Hence the preceding proposition yields as an immediate corollary the main theorem:

Theorem 3.14 *Any smooth non nilpotent Zariski group interprets an algebraically closed field.*

In fact, the proof of theorem 3.11 (and hence the main theorem 3.14) goes through for Zariski groups interpretable in Zariski geometries that are additive and smooth with semi-continuous dimension.

Open problems

(A) A non singular algebraic variety (over an algebraically closed field) satisfies the dimension formula. Because algebraic groups can't have singularities, they are smooth Zariski groups. Question: Are all Zariski groups smooth?
The dimension formula holds at least for cosets of definable subgroups.

(B) As pointed out by Poizat ([P] p. 144), a group interpretable in an algebraically closed field is a Zariski group in a canonical way. Is this true for any $\aleph_0$-stable group of finite rank?

(C) By results of Hrushovski, a simple group of finite Morley rank is interpretable in any infinite field that is interpretable in the group, provided that the field is endowed with the full structure coming from the group. To solve completely Cherlin's conjecture for Zariski groups, it remains to show that the group in interpretable in the pure field structure. Hence the problems reduces to the following question: Is the field interpretable in a simple smooth Zariski group a pure field?

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