# Difference normal forms in topology and propositional logic

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July 23, 2009

### Abstract

The article shows a normal form for constructible sets in arbitrary topological spaces. This yields a new normal form for propositional logic and some applications to the model theory of equational and ordered structures.

## 1 Topology

### Constructible sets and difference chains

Let $(U, \tau)$ be a fixed topological space. The closure of a subset $X \subseteq U$ is denoted by $\overline{X}$, the interior by $\overset{\circ}{X}$. A subset of the space is called *constructible* if it is a Boolean combination of open sets, i.e., an element of the smallest family containing all open sets and stable under finite intersections and complements. The following lemmas might be well-known:

**Lemma 1** *Disjoint constructible sets, not both empty, have distinct closures.*

This property is equivalent to the following: Two disjoint constructible sets $X, Y$ cannot be dense in the same non-empty superset $Z$. For if they were, $X \subseteq Z \subseteq \overline{Y}$ and $Y \subseteq Z \subseteq \overline{X}$, whence $\overline{X} = \overline{Y}$. Conversely, if $\overline{X} = \overline{Y} = Z$, both $X$ and $Y$ are dense in $Z$.

PROOF: Let $X$ and $Y$ be disjoint, constructible, and dense in $Z := X \cup Y$. Consider the coarsening $\tau'$ of $\tau$ generated by finitely many open sets which allow to express $X$ and $Y$ as Boolean combinations. Then $X$ and $Y$ are constructible in $\tau'$, which is a finite, hence Noetherian topology. We may assume that $Z$ is the whole space (pass to the induced topology) and that it is irreducible (pass to the finitely many irreducible components). Then $X$ and $Y$ are still constructible, disjoint and dense in $Z$. The constructible sets form a Boolean algebra; with the disjunctive normal form and the decomposition into irreducible components one sees that $X$ is a union of finitely many locally closed irreducible sets: $X = \bigcup_{j=1}^{k} C_j \setminus D_j$ with closed $D_j$, closed irreducible $C_j$ and $D_j \subset C_j$. Then $\overline{X} = \bigcup \overline{C_j \setminus D_j} = \bigcup C_j$. Since the space is irreducible, $Z = C_j$ for some $j$. But then $Y \subseteq D_j$ and hence $\overline{Y} \subseteq D_j \neq Z$: contradiction. □

**Lemma 2** *The interior of a constructible set is dense in that set.*

PROOF: Let $X$ be constructible. We may again suppose that $X$ is dense in $U$ (by passing to the induced topology on $\overline{X}$) and that $\tau$ is Noetherian, because we may replace it by a finite topology

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generated by the interior $\overset{\circ}{X}$ and finitely many open sets of which $X$ is as a Boolean combination. Then $X$ decomposes into its finitely many relatively closed irreducible components $X_1, \ldots, X_k$. From the irreducibility of the $X_i$ and Lemma 1 we get that $\overset{\circ}{X_i}$ is dense in $X$. Now the union of the $\overset{\circ}{X_i}$ is contained in $\overset{\circ}{X}$, which shows the lemma. $\qquad\square$

Remark: There are also direct proofs of Lemma 2, of which Lemma 1 is an immediate consequence.

**Definition 3** *An expression $C_0 \setminus C_1 \setminus C_2 \setminus \cdots \setminus C_n$ is called a* difference chain (of length $n$) *and meant to be parenthesised from the right, i.e., to equal $C_0 \setminus (C_1 \setminus (C_2 \setminus \cdots \setminus (C_{n-1} \setminus C_n)\cdot))$. One may assume, and will do so henceforth, that $C_0 \supset C_1 \supset \cdots \supset C_n$.*

The first apparition of difference chains is probably in Felix Hausdoffs book on set theory [H] (chapter 1, § 5 "Differenzketten"), where he shows that the sets represented by difference chains form a Boolean algebra. With other words:

**Proposition 4** *Every constructible set is given by a difference chain of closed sets.*

Here comes a proof, for the sake of completeness.

PROOF: Let $C$ be constructible sets and consider again a finite topology $\tau'$ such that $C$ is constructible in $\tau'$. Define inductively $C_0 := C$ and $C_{i+1} := \overline{C_i} \setminus C_i$. Then $C_i = \overline{C_i} \setminus C_{i+1}$ and hence inductively

$$C = \overline{C_0} \setminus \overline{C_1} \setminus \cdots \setminus \overline{C_{i-1}} \setminus C_i$$

As $C_{i+1}$ cannot be dense in $\overline{C_i}$ by Lemma 1, $\overline{C_0} \supset \overline{C_1} \supset \cdots \supset \overline{C_{i+1}}$ is strictly decreasing as long as $C_i \neq \emptyset$. Since the space is Noetherian, there is an $n$ such that $C_{n+1} = \emptyset$. Hence

$$C = \overline{C_0} \setminus \overline{C_1} \setminus \cdots \setminus \overline{C_{n-1}} \setminus \overline{C_n}. \qquad\square$$

Finally we note two useful remarks that are easy to prove:

$$C_0 \setminus C_1 \setminus \cdots \setminus C_n \;=\; (C_0 \setminus C_1) \cup (C_2 \setminus C_3) \cup \cdots \tag{1}$$

$$C_0 \setminus C_1 \setminus \cdots \setminus C_n = X \iff C_{n-1} \setminus \cdots \setminus C_0 \setminus X = C_n. \tag{2}$$

## The difference normal form

**Definition 5** *Let $\partial : \mathfrak{P}(U) \to \mathfrak{P}(U)$ be the map $X \mapsto \overline{X} \setminus X$. That is, $\partial X$ is the part of the border of $X$ not belonging to $X$. Define $\partial^0 X := X$ and inductively $\partial^{i+1} X := \partial(\partial^i X)$. Moreover, let $X_{[i]} := \overline{\partial^0 X} \setminus \cdots \setminus \overline{\partial^i X}$.*

**Lemma 6** *For all $i$, $X = \overline{\partial^0 X} \setminus \cdots \setminus \overline{\partial^i X} \setminus \partial^{i+1} X$, and*

$$\overline{X} = X_{[0]} \supseteq X_{[2]} \supseteq \cdots \supseteq X \supseteq \cdots \supseteq X_{[3]} \supseteq X_{[1]} \supseteq \overset{\circ}{X}.$$

PROOF: First note that $\partial^{i+1} X = \overline{\partial^i X} \setminus \partial^i X \subseteq \overline{\partial^i X}$. Inductively, this shows $\partial^{i+1} X = \overline{\partial^i X} \setminus \cdots \setminus \overline{\partial^0 X} \setminus \partial^0 X$, which gives the first part of the lemma by equality 2 above. The second part then follows easily with equality 1. $\qquad\square$

The following theorem is proved for Noetherian topologies in my doctoral thesis [J1], fact 1.11. Soon after I have noticed that the general case can be reduced to the Noetherian one as shown in this note. Some years later I discovered the paper [A] in which the same result is shown by a different approach. Similar results (from which the theorem might easily been derived) have been shown by other authors, e.g. [St], but the description as a difference chain in (c), which allows the applications in Sections 2 and 3, is usually missing.

For example [DM] proves an analogous result "$X$ is constructible $\Leftrightarrow X^{(n)} = \emptyset$" for the operator $X \mapsto X^{(1)}$ which they define to be $X \setminus X \setminus \overline{\overline{X} \setminus X} = X \cap \overline{\partial X}$. This is a subset of $X$. Therefore their result is different from the result above and does not allow a characterisation like (c).

**Theorem 7** *The following are equivalent:*
*(a) $X$ is constructible;*
*(b) there is an $n$ such that $\partial^{n+1} X = \emptyset$;*
*(c) for some $n$*

$$X = \overline{\partial^0 X} \setminus \overline{\partial^1 X} \setminus \cdots \setminus \overline{\partial^n X}.$$

*Moreover, the number $n$ in (b) is the smallest length of a difference chain of closed sets expressing $X$.[1]*

The expression under (c) will be called the *difference normal form* of $X$ (as a Boolean combination of closed sets), and the number under (b) will be called the *degree of constructibility* $\mathrm{dc}(X)$ of $X$. The degree of constructibility of a non-constructible set is defined to be $\infty$.

PROOF: (b)$\Rightarrow$(c) and (c)$\Rightarrow$(a) are clear. For (a)$\Rightarrow$(b), let $X$ be constructible, expressed as a difference chain $C_0 \setminus C_1 \setminus \cdots \setminus C_n$ of closed sets. Then $\overline{X} \subseteq C_0$, hence

$$X \;=\; (C_0 \setminus C_1 \setminus \cdots \setminus C_n) \cap \overline{X} \;=\; (C_0 \cap \overline{X}) \setminus (C_1 \cap \overline{X}) \setminus \cdots \setminus (C_n \cap \overline{X})$$

and we may assume $C_0 = \overline{X} = \overline{\partial^0 X}$. Assume inductively that we can write

$$X = \overline{\partial^0 X} \setminus \overline{\partial^1 X} \setminus \cdots \setminus \overline{\partial^i X} \setminus C'_{i+1} \setminus \cdots \setminus C'_n.$$

Then $\overline{\partial^i X} \setminus \cdots \setminus \overline{\partial^0 X} \setminus X = C'_{i+1} \setminus \cdots \setminus C'_n$ and in the same way as above me may assume that $C'_{i+1} = \overline{\overline{\partial^i X} \setminus \cdots \setminus \overline{\partial^0 X} \setminus X} = \overline{\partial^{i+1} X}$. It follows that if $X$ is constructible, then $\partial^{n+1} X$ for some $n$. This proof also shows the "moreover"–clause. $\qquad\square$

**Corollary 8 ([A])** *Any constructible set $X$ is a disjoint union of $n = \lceil \mathrm{dc}(X)/2 \rceil$ canonically defined locally closed sets, namely*

$$X = \bigcup_{i=0}^{n} \left( \overline{\partial^{2i} X} \setminus \overline{\partial^{2i+1} X} \right).$$

Allouche has shown that moreover $n = \lceil \mathrm{dc}(X)/2 \rceil$ is the minimal number such that a constructible set can be written as a union of $n$ locally closed sets. Of course it is clear from the disjunctive normal form that a constructible set can be written as union of locally closed sets. The additional information here is the canonicity.

We have $\mathrm{dc}(X) = -1$ if and only if $X = \emptyset$
$\phantom{We have }\mathrm{dc}(X) = 0$ if and only if $X$ is closed and non-empty
$\phantom{We have }\mathrm{dc}(X) = 1$ if and only if $X$ is locally closed, but not closed

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[1] Define the empty difference chain to have length $-1$ and to equal $\emptyset$.

For non constructible sets, there are several possible phenomena: for example for $X = \mathbb{Q}$ in $\mathbb{R}$, the sequence of the $\partial^i X$ is periodic without reaching $\emptyset$. If $X_i$ is a constructible set of degree $i$ in a topological space $U_i$ and $X$ the union of the $X_i$ in the disjoint union of the $U_i$, then the sequence of the $\partial^i X$ is neither eventually stationary nor periodic.

At this point, some generalisations are possible: Call a subset $X$ of a topological space *hyper-constructible* if $X = \bigcap_{i=0}^{\infty} X_{[2i]}$, and *hypo-constructible* if $X = \bigcup_{i=0}^{\infty} X_{[2i+1]}$. Are they the same? Is there a characterisation of hyper-/hypo-constructible sets?

## Bounds on the degree of constructibility

Let us call a sub-topology of $\tau$ a *decomposition topology of $X$* if all the $\overline{\partial^i X}$ are closed in it. These are exactly the sub-topologies in which $X$ has the same difference normal form as in $\tau$.

**Lemma 9** *The degree of constructibility of a set is the minimal dimension of this set in its Noetherian decomposition topologies, or, equivalently, the dimension of this set in its minimal decomposition topology.*

PROOF: From Lemma 1 it follows that in each Noetherian decomposition topology of $X$, $\overline{\partial^{i+1}(X)}$ does not contain an irreducible component of $\overline{\partial^i(X)}$ (immediate if irreducible, otherwise decompose into irreducible components). Hence $\mathrm{dc}(X) \leqslant \dim(X)$. On the other hand, the sub-topology generated by the $\overline{\partial^i(X)}$ as closed sets, i.e., the smallest decomposition topology for $X$, is Noetherian and the $\overline{\partial^i(X)}$ are irreducible in it, whence $\mathrm{dc}(X) = \dim(X)$ in that topology. □

It follows from Lemma 9 that the degree of constructibility is bounded in a Noetherian space of finite dimension. As a sort of a converse one gets that the length of strictly decreasing chains of closed irreducible sets is bounded by any bound on dc. However, a discrete infinite topological space is an example of a non Noetherian space with bounded dc.

**Examples:** (a) The degree of constructibility of a constructible set in the Zariski topology on $\mathbb{C}^n$ is bounded by the dimension.

(b) In the Euclidean topology on $\mathbb{R}^n$, the degree of constructibility of definable constructible sets is bounded by the dimension. For non-definable sets, there is no bound, even for $n = 1$:[2]

Denote by $C^{(k)}$ the $k^{\text{th}}$ Cantor–Bendixson derivative of a closed set $C$ and suppose $C \subseteq R$ is such that $C^{(2n+1)} = \emptyset \neq C^{(2n)}$. All $C^{(k)}$ are closed, therefore $C_k := C^{(k)} \setminus C^{(k+1)}$ is constructible. Let $X := C_0 \cup C_2 \cup \cdots \cup C_{2n}$. Then $\partial X = C_1 \cup C_3 \cup \cdots \cup C_{2n-1}$ and $\partial^2 X = C_0 \cup C_2 \cup \cdots \cup C_{2n-2}$, whence $\mathrm{dc}(X) = n$.

The degree of constructibility of a constructible subset $X$ is bounded by the Cantor–Bendixson rank of $\overline{X}$: Assume the rank is $n > 1$ (all other cases are clear). Then no point $x \in \overline{X}^{(n)}$ is an accumulation point of $\overline{X} \setminus \{x\}$. Thus $x$ must already be an element of $X$, whence $\partial X \subseteq \overline{X} \setminus \overline{X}^{(n)}$ has Cantor–Bendixson rank at most $n - 1$.

**Proposition 10** *If $Z$ is a Boolean combination of $X_1, \ldots, X_n$, then $\mathrm{dc}(Z_i)$ is bounded by $\mathrm{dc}(X_1) + \cdots + \mathrm{dc}(X_n) + n$. If in addition $Z \subseteq \overline{X_1 \cup \cdots \cup X_n}$, and either $Z \neq U$ or $n > 0$, then $\mathrm{dc}(Z_i)$ is bounded by $\mathrm{dc}(X_1) + \cdots + \mathrm{dc}(X_n) + n - 1$.*

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[2]Question of Martin Ziegler during a seminar presentation of this work, answered afterwards by Antongiulio Fornasiero.

PROOF: Take the smallest common decomposition topology for $X_1, \ldots, X_n$. Then $Z$ is constructible in it, and $\mathrm{dc}(Z)$ is bounded by its dimension, which in turn is bounded by the dimension of the free topology generated by $\big(\mathrm{dc}(X_1)+1\big) + \cdots + \big(\mathrm{dc}(X_n)+1\big)$ sets. Closed irreducible sets therein are intersections of complements of the generators, therefore this dimension is $\mathrm{dc}(X_1) + \cdots + \mathrm{dc}(X_n) + n$. In the second case, the bound can be improved by one as $U$ is irreducible in the free situation and does not appear as a factor $\overline{\partial^i Z}$ except for $Z = U$. $\qquad \square$

For $(X_1 \cup \cdots \cup X_n)^{\complement}$ and $(X_1 \cup \cdots \cup X_n)$ in case all $\overline{\partial^j X_i}$ are independent, the bounds are attained (except $n-1$ of the $X_i$ have degree 0). For example for two locally closed sets,

$$(\overline{X} \setminus \overline{\partial X}) \cup (\overline{Y} \setminus \overline{\partial Y}) \;=\; (\overline{X} \cup \overline{Y}) \setminus (\overline{\partial X} \cup \overline{\partial Y}) \setminus (\overline{X} \cap \overline{Y} \cap (\overline{\partial X} \cup \overline{\partial Y})) \setminus (\overline{\partial X} \cap \overline{\partial Y}),$$

and this is the normal form in a free situation.

Note that if $X \neq \emptyset$ is constructible, of degree of constructibility $n$, then $\mathrm{dc}(\partial X) = n-1$ and $\overline{\partial^1 X} \setminus \cdots \setminus \overline{\partial^n X}$ is the difference normal form of $\partial X$.

**Proposition 11** *(a) The following bounds hold:* [3]

$$
\begin{aligned}
\mathrm{dc}(X \cup Y) &\leqslant \mathrm{dc}(X) + \mathrm{dc}(Y) + 1 \\
\mathrm{dc}(X \cap Y) &\leqslant \mathrm{dc}(X) + \mathrm{dc}(Y) \\
\mathrm{dc}(Y^{\complement}) &\leqslant \mathrm{dc}(Y) + 1 \\
\mathrm{dc}(X \setminus Y) &\leqslant \mathrm{dc}(X) + \mathrm{dc}(Y) + 1 \\
\mathrm{dc}(X \bigtriangleup Y) &\leqslant \mathrm{dc}(X) + \mathrm{dc}(Y) + 1
\end{aligned}
$$

*(b) If $\overline{X}$ and $\overline{Y}$ are disjoint, then $\mathrm{dc}(X \cup Y) \leqslant \max\{\mathrm{dc}(X), \mathrm{dc}(Y)\}$.*

PROOF: (a) All inequalities except for intersection follow from Proposition 10. The intersection computes as $X \cap Y = (\overline{X} \cap \overline{Y}) \setminus (\partial X \cup \partial Y)$, then use the first inequality and the remark above.

The rules for complements and differences also follow from the equalities $Y^{\complement} = U \setminus Y$ and $\mathrm{dc}(X \setminus Y) = \mathrm{dc}(X \cap Y^{\complement})$.

(b) Assume $C_0 \setminus C_1 \setminus \cdots \setminus C_n = X$ and $D_0 \setminus D_1 \setminus \cdots \setminus D_n = Y$ (same length can be assumed by filling up with $\emptyset$'s), then $(C_0 \cup D_0) \setminus (C_1 \cup D_1) \setminus \cdots \setminus (C_n \cup D_n) = X \cup Y$. $\qquad \square$

## The dual version

Let, with a notation borrowed from logic, $A \Rightarrow B$ denote $A^{\complement} \cup B$.

**Definition 12** *Let $\varrho : \mathfrak{P}(U) \to \mathfrak{P}(U)$ be the map $X \mapsto (X \Rightarrow \overset{\circ}{X})$. That is, $\varrho X$ is the complement of the part of the border of $X$ that belongs to $X$. Define $\varrho^0 X := X$ and inductively $\varrho^{i+1} X := \varrho(\varrho^i X)$.*

Remark that $\varrho$ is the dual operator to $\partial$, that means $\varrho X = \partial(X^{\complement})^{\complement}$.

**Proposition 13** *The following are equivalent:*
*(a) $X$ is constructible*

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[3] $\bigtriangleup$ stands for the symmetric difference.

*(b) there is an $n$ such that $\varrho^{n+1}X = U$*

*(c) for some $n$*

$$X = \left( \cdot \left( \overset{\circ}{\overbrace{\varrho^n X}} \Rightarrow \overset{\circ}{\overbrace{\varrho^{n-1}X}} \right) \Rightarrow \cdots \right) \Rightarrow \overset{\circ}{\overbrace{\varrho^0 X}}.$$

Part (c) yields a normal form of $X$ as Boolean combination of open sets.

PROOF: Analogously to Theorem 7, or dualise as in the proof of Theorem 14. $\square$

## 2   An application to Propositional Logic

The following is a translation of Theorem 7 into propositional logic. It is hard to imagine that this is a new result, but in spite of discussions with many logicians, I haven't found any trace of it.

**Theorem 14** *If $\varphi$ is a formula in classical propositional calculus, then $\varphi$ is equivalent to a formula of the form*

$$(\cdot((\gamma_n \to \gamma_{n-1}) \to \gamma_{n-2}) \to \cdots \to \gamma_1) \to \gamma_0$$

*where the $\gamma_i$ are positive Boolean combinations in the propositional variables (possibly $\top$ or $\bot$).*

PROOF[4]:    We embed the Tarski–Lindenbaum algebra via the Stone representation into a power set algebra, and identify formulae with subsets. The positive Boolean combinations in the propositional variables occuring in $\varphi$ form (the open sets of) a finite topology in which $\neg\varphi$ is constructible. With Theorem 7, we can write $\neg\varphi$ up to logical equivalence in the form

$$\neg\gamma_0 \,\wedge\neg\, (\neg\gamma_1 \,\wedge\neg\, \cdots \,\wedge\neg\, (\neg\gamma_{n-1} \,\wedge\neg\, \neg\gamma_n)\cdot)$$

with $\gamma_i$ as above. Negating both sides and applying de Morgan's rule transforms the expression into the desired form. $\square$

**Definition 15** *If we use in the proof the smallest set of propositional variables such that $\varphi$ can be expressed as a formula in these variables, we get the* implication normal form *of $\varphi$. The number of arrows is the* implication degree *of $\varphi$.[5]*

The implication degree is an invariant attached to a propositional formula. What a possible meaning of this invariant could be, remains a mystery to me.

In the normal form, we have in addition $\vdash (\gamma_i \to \gamma_{i+1})$, and hence from Corollary 8:

**Corollary 16** *A propositional formula $\varphi$ is classically equivalent to $\bigwedge_{i\geqslant 0}(\gamma_{2i+1} \to \gamma_{2i})$ and to $\bigwedge_{i\geqslant 0}(\gamma_{2i+1} \leftrightarrow \gamma_{2i})$, with the $\gamma_i$ from the implication normal form.*

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[4]It might be worth to note that I have not succeeded in giving a direct proof of Theorem 14 by an induction on the construction of formulae.

[5]To be coherent with the rest, the empty expression is defined to equal $\top$, which therefore has implication degree $-1$.

**Proposition 17** *The following bounds for the implication degree* id *hold:*

$$
\begin{aligned}
\mathrm{id}(\neg\varphi) &\leqslant \mathrm{id}(\varphi) + 1 \\
\mathrm{id}(\varphi \wedge \psi) &\leqslant \mathrm{id}(\varphi) + \mathrm{id}(\psi) + 1 \\
\mathrm{id}(\varphi \vee \psi) &\leqslant \mathrm{id}(\varphi) + \mathrm{id}(\psi) \\
\mathrm{id}(\varphi \rightarrow \psi) &\leqslant \mathrm{id}(\varphi) + \mathrm{id}(\psi) + 1 \\
\mathrm{id}(\varphi \leftrightarrow \psi) &\leqslant \mathrm{id}(\varphi) + \mathrm{id}(\psi) + 1
\end{aligned}
$$

*The maximal implication degree of a formula with $n$ variables is $n$.*

PROOF: Dualise Proposition 11. The second part comes from Proposition 10. □

**Examples:** Consider the 16 formulae in two propositional variables $A$ and $B$: $\top$ has implication degree $-1$, the other five positive formulae have degree 0. There are seven formulae of degree 1, namely $A \rightarrow B$ with normal form $(A \vee B) \rightarrow B$, then $B \rightarrow A$, four negations of positive formulae $\varphi$ of with normal form $\varphi \rightarrow \bot$, and $A \leftrightarrow B$ with normal form $(A \vee B) \rightarrow (A \wedge B)$. Finally, three formulae have degree 2, namely $\neg(A \leftrightarrow B)$ with normal form $((A \vee B) \rightarrow (A \wedge B)) \rightarrow \bot$, then $\neg(A \rightarrow B)$ with normal form $((A \vee B) \rightarrow B) \rightarrow \bot$, and $\neg(B \rightarrow A)$.

**Remarks: (a)** There is at most one occurrence of $\bot$, namely at the right end of an implication normal form. $\top$ is only needed as the normal form of a tautology and could be replaced by $\bot \rightarrow \bot$, if one would like to minimise the use of connectives.

Mark Weyer has found a non-topological proof of the fact that every formula can be put in a form like in Theorem 14. Moreover, he gets positive formulae that are built with conjunctions and falsum only. The number of arrows in his case is however exponential in the number of propositional variables.

**(b)** There is no implication normal form in intuitionistic logic. In one propositional variable $A$, there are up to intuitionistic equivalence infinitely many formulae, but only three positive formulae ($\bot$, $\top$ and $A$) and only five implication normal forms (in addition $A \rightarrow \bot$ and $(A \rightarrow \bot) \rightarrow \bot$).

**(c)** The implication normal form is not yet a normal form in the strict sense as the positive formulae $\gamma_i$ are only determined up to logical equivalence. To get a real normal form, one could choose either the "positive DNF" or the "positive KNF", that is the result of applying the distributive law to a positive expression to get either a disjunction of conjunctions or vice versa (and then of course some ordering of the conjuncts and disjuncts, which always depends on some chosen ordering of the propositional variables).

Another point of view is to see the implication normal from as a normal form in a sort of "positive Tarski–Lindenbaum algebra", where the free algebra of all formulae has already been reduced by "positive equivalence". That should be an equivalence relation inducing logical equivalence on positive formulae. It seems to be most reasonably to take the equivalence relation generated by the "positive laws" of Boolean algebras, i.e., the absorption, associativity, commutativity, distributivity and idempotence laws for $\wedge$ and $\vee$, and the neutrality and absorption laws for $\top$ and $\bot$ with respect to $\wedge$ and $\vee$.

# 3 Applications to Model Theory

Theorem 7 can be an elegant tool to show properties of structures in which topologies are naturally involved, in particular ordered structures with the order topology and equational structures with the Srour topology.

**Application to elimination of imaginaries**

Let $\mathfrak{M}$ be a structure and $\tau$ a topology on $U = M^n$ that is $\mathrm{Aut}(\mathfrak{M})$-invariant. Assume for simplicity that $\mathfrak{M}$ is saturated (otherwise one needs that $\tau$ extends in a compatible way to elementary extensions). A canonical parameter $\mathrm{cp}(X)$ of a set $X \subseteq M^n$ is an object in some set on which $\mathrm{Aut}(\mathfrak{M})$ acts naturally such that $X$ is setwise invariant under exactly thoses automorphisms fixing $\mathrm{cp}(X)$. The theory of $\mathfrak{M}$ eliminates imaginaries if each definable set has a tuple of elements of $\mathfrak{M}$ as canonical parameter (see [CF] for an account on elimination of imaginaries). Theorem 7 shows that an $A$-invariant constructible set $X \subseteq M^n$ is canonically a Boolean combination of $A$-invariant closed sets. Therefore it follows:

**Proposition 18** *If closed sets have canonical parameters and $X$ is constructible of degree $n$, then the tuple $\big(\mathrm{cp}(\overline{\partial^0 X}), \ldots, \mathrm{cp}(\overline{\partial^n X})\big)$ is a canonical parameter of $X$.*

There are cases where all closed sets are definable (then the topology is necessarily Noetherian). Examples are the Zariski topology in fields or the d-topology in differential fields. In these cases, an $A$-definable constructible set is canonically a Boolean combination of $A$-definable closed sets. Moreover, if all closed sets have canonical bases in the home sort, then also all constructible sets do (all this follows of course also easily from the Noetherianity of the topology). If in addition all definable sets are constructible as in the two examples above, then the structure eliminates imaginaries.

This generalises to equational theories: A definable set $X$ is Srour-closed if —in every elementary extension— the conjugates of $X$ satisfy the descending intersection property, and a structure is called equational if all definable sets are Boolean combination of definable Srour-closed sets. In particular, all definable sets are constructible in the topology generated by the Srour-closed sets as a basis of closed sets (see [J2] for details and related definitions). Most "natural" stable structures are equational, e.g. algebraically closed fields, differentially closed fields, separably closed fields of finite degree of imperfection, all modules, free pseudo-spaces. (A stable non-equational theory has been constructed by Hrushovski and Srour as an expansion of a free pseudo-space. Contrary to the belief of some, the pseudo-space constructed by Baudisch and Pillay has nothing to do with non-equationality.)

If $\Phi$ is a set of formulae $\varphi(\bar{x}; \bar{y})$, let a $\Phi$-*set* be a set defined by an instance $\varphi(\bar{x}, \bar{a})$. An equation (in Srour's sense) is a formula $\varphi(\bar{x}; \bar{y})$ such that every $\{\varphi\}$-set is Srour-closed set. In fact, the Srour-closed set are exatly the $\Phi$-sets for all equations $\Phi$ (for details see [J2]).

**Proposition 19** *Assume $\Phi$ is a family of equations such that every definable set is a boolean combination of $\Phi$-sets. If all $\Phi$-sets have canonical parameters, then the theory eliminates imaginaries.*

*Special case: If all definable Srour-closed sets in an equational theory have canonical parameters (in the home sort), then the theory eliminates imaginaries.*

PROOF: Proposition 18 does not apply directly, because closures of definable sets need not to be definable. If $X$ is definable, it is a Boolean combination of Srour-closed sets defined by instances of equations $\varphi_1, \ldots, \varphi_n \in \Phi$. The topology $\tau_{\varphi_1, \ldots, \varphi_n}$ generated by the $\Phi$-sets as a subbasis of closed sets in Noetherian (see [J2]). Then Proposition 18 applies for this topology as the $\overline{\partial^n X}$ computed in $\tau_{\varphi_1, \ldots, \varphi_n}$ are invariant under all automorphisms fixing $X$. $\quad\square$

One might also recall from [JL] that arbitrary Srour-closed sets have canonical parameters in the form of tuples in $T^{\mathrm{eq}}$ of size at most $|T|$. Proposition 18 thus implies:

**Remark 20** *If $X$ is an arbitrary Srour-constructible set, then it has a canonical parameters in form of a possibly infinite, but bounded tuple in $T^{eq}$.*

**Definable closed sets**

Model theory is interested in definable sets, and only in special situations as above all constructible sets are definable or controlled by definable sets. Therefore a generalisation is useful. Let $\Lambda$ be a sub-lattice of $\mathfrak{P}(U)$ containing $\emptyset$ and $U$. Call a subset of $U$ $\Lambda$-*constructible* if it is an element of the Boolean sub-algebra of $\mathfrak{P}(U)$ generated by $\Lambda$.

**Lemma 21** *Every $\Lambda$-constructible set can be written as a difference chain of elements of $\Lambda$.*

PROOF: Only finitely many elements of $\Lambda$ are needed to express a $\Lambda$-constructible set. These elements generate, as closed sets, a finite topology. Then apply Proposition 4. $\quad\square$

Thus every $\Lambda$-constructible set has a *degree of $\Lambda$-constructibility*, namely the shortest length of a difference chain as in Lemma 21. However, there need not to be a normal form, since in general there is no smallest or otherwise canonical sub-lattice of $\Lambda$ over which a given set is a Boolean combination (in contrast to the situation in Theorem 14).

The following two types of problems occur in model theory: Let $\mathcal{B}$ be a Boolean sub-algebra of $\mathfrak{P}(U)$ —typically the definable sets of a structure on $U$— such that every element in $\mathcal{B}$ is $\Lambda$-constructible for some lattice $\Lambda$ as above (e.g. $\Lambda = $ the positively definable sets).

1. Given a sub-lattice $\Lambda'$ of $\Lambda$, find conditions for the elements of $\mathcal{B}$ to be $\Lambda'$-constructible.

2. Given a Boolean sub-algebra $\mathcal{B}'$ of $\mathcal{B}$, find conditions for the element of $\mathcal{B}'$ to be $(\mathcal{B}' \cap \Lambda)$-constructible.

Induction on the degree of constructibility helps easily to give a partial answer to problem 2 as in the following proposition, which was proved in a more special situation in [J2]. That article also offers three applications to equational theories: With the help of this technique I have proved for example that a theory is equational if some expansion by constants is, and that the uniform and the non-uniform versions of equationality are equivalent.

**Proposition 22** *Assume that in a situation as above, the following property holds:*
$\qquad$*For all $X \in \mathcal{B}'$ and $X \subseteq Y \in \Lambda$, there is a $Y' \in \mathcal{B}' \cap \Lambda$ with $X \subseteq Y' \subseteq Y$.*

*Then every element of $\mathcal{B}'$ is $(\mathcal{B}' \cap \Lambda)$-constructible. The $(\mathcal{B}' \cap \Lambda)$-degree of constructibility equals the $\Lambda$-degree of constructibility.*

PROOF: By induction on the degree $n$ of $\Lambda$-constructibility of $X \in \mathcal{B}'$. For $n \leqslant 0$, everything is clear. Otherwise let $X = Y_0 \backslash \cdots \backslash Y_n$ with $Y_i \in \Lambda$. By hypothesis, there is a $Y_0' \in \mathcal{B}' \cap \Lambda$ such that $X \subseteq Y_0' \subseteq Y_0$. Then $X = Y_0' \backslash (Y_1 \cap Y_0') \backslash \cdots \backslash (Y_n \cap Y_0')$, and $(Y_1 \cap Y_0') \backslash \cdots \backslash (Y_n \cap Y_0') = Y_0' \backslash X \in \mathcal{B}'$ is $(\mathcal{B}' \cap \Lambda)$-constructible by induction. $\qquad\square$

If $\Lambda$ is given by the closed sets of a topology, we get the following special case:

**Corollary 23** *Let $\tau$ be a topology on $U$ and $\mathcal{B}'$ a Boolean sub-algebra of the constructible sets. If for all $X \in \mathcal{B}'$ also $\overline{X} \in \mathcal{B}'$, then all sets in $\mathcal{B}'$ are $(\tau \cap \mathcal{B}')$-constructible.*

**Application to constructible sets in ordered structures**

If we consider the field of real numbers with the Euclidean topology, or more generally totally ordered structures with the order topology and its product topologies, then the closure of a definable set is again definable (and over the same parameters). With $\mathcal{B}' =$ the Boolean algebra of the definable constructible sets, corollary 23 shows:

**Proposition 24** *In an ordered structure, a definable constructible set is definably constructible, i.e., a Boolean combination of definable closed sets. Moreover, an A-definable constructible set is in a canonical way a Boolean combination of A-definable closed sets.*

In o-minimal structures, this follows from cell decomposition, which by the way shows all definable sets to be constructible. The first part of the proposition is the content of the [DM].

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