Reducts of $(\mathbb{Q}, <)$

Markus Junker, Martin Ziegler

April 2004

1 The result

This arcticle presents a new short proof of the following theorem:

Theorem 1.1 *Up to interdefinability, there are exactly five reducts of $(\mathbb{Q}, <)$, namely:*

1. *the dense linear order $(\mathbb{Q}, <)$ itself;*
2. *the unorientated dense linear order or "betweenness relation" coming from $<$;*
3. *the dense cyclic order coming from $<$;*
4. *the unorientated dense cyclic order or "separation relation" coming from $<$;*
5. *and the infinite set $\mathbb{Q}$ without structure.*

This follows immediately from a theorem of Cameron classifying the highly homogeneous permutation groups on a countable set (see [C2] theorem 3.10; originally with group theoretic methods in [C1]; a model theoretic proof is based on [HLS]). The first proof of the theorem was given by Frasnay in [F]. After having finished it, we learned that Higman in [Hi] works with the same basic idea.

2 The method

Let $\mathfrak{R}$ be a reduct of $(\mathbb{Q}, <)$. It is determined by the groups

$$G_n^{\mathfrak{R}} := \big\{\sigma \in S_n \;\big|\; (x_1, \ldots, x_n) \equiv (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ in } \mathfrak{R} \text{ for some (equivalently: each) } x_1 < x_2 < \cdots < x_n\big\},$$

because they determine the orbits of strict n-tuples. We call $(G_n^{\mathfrak{R}})_{n\in\mathbb{N}}$ the *structure sequence* of $\mathfrak{R}$. When is a sequence of groups a structure sequence? We need some terminology: For $i \in \{1, \ldots, n+1\}$, let

- $\beta_i : \{1, \ldots, n\} \to \{1, \ldots, i-1, i+1, \ldots, n+1\}$ be the order preserving bijection,
- $\pi_i : S_{n+1} \to S_n$ be the map $\sigma \mapsto \beta_{\sigma(i)}^{-1} \circ \sigma \circ \beta_i$.

We say that G_n *extends* to G_{n+1}, and write $G_n \sqsubset G_{n+1}$, if $\pi_i[G_{n+1}] = G_n$ for all i.

Lemma 2.1 *A sequence $(G_n)_{n\in\mathbb{N}}$ with $G_n \leqslant S_n$ is a structure sequence iff G_n extends to G_{n+1} for all n.*

PROOF: "$\Rightarrow$" is clear: choose a tuple $x_1 < \cdots < x_{n+1}$ and consider all extracted n-tuples.
"$\Leftarrow$": the automorphism group of the ($\aleph_0$-categorical) structure is given by all bijections acting like an element of G_n on ordered n-tuples, and the condition ensures that this automorphism group acts exactly like G_n. $\square$

Remark 2.2 *(a) Since $\pi_{\sigma(i)}(\tau) \circ \pi_i(\sigma) = \pi_i(\tau\sigma)$, we have $\bigcap_{i=1}^{n+1} \pi_i[G_{n+1}] \leqslant S_n$.*
It follows that G_{n+1} extends some group $G_n \leqslant S_n$ iff all $\pi_i[G_{n+1}]$ are equal.

(b) If $G_n \sqsubset G_{n+1}$, then $|G_{n+1}| \geqslant |G_n|$.

(c) If $G_n \sqsubset G_{n+1}$ and $|G_{n+1}| = |G_n|$, then all maps π_i must be bijective.

3 The proof

We start to determine all possible structure sequences. We need the following subgroups of S_n:

- the identity group I_n;
- the cyclic "swap group" of order 2 $Z_n^2 := \left\langle \big(1\ n\big)\big(2\ (n-1)\big) \cdots \big([\frac{n}{2}]\ [\frac{n+3}{2}]\big) \right\rangle$
- the cyclic "cycle group" of order n $Z_n^n := \big\langle (1\,2\,\ldots\,n) \big\rangle$;
- and the dihedral "square group" $D_n^{2n} := Z_n^2 \ltimes Z_n^n$.

3.1 The different orders

Lemma 3.1 *(a) (I_n) is the structure sequence of the dense order, and obviously I_{n+1} is the only possible extension of I_n if $n \geqslant 2$.*

(b) (Z_n^2) is the structure sequence of the betweenness relation, and it is the only possible structure sequence containing Z_3^2.

(c) (Z_n^n) is the structure sequence of the cyclic order, and it is the only possible structure sequence containing Z_3^3.

(d) (D_n^{2n}) is the structure sequence of the separation relation, and it is the only possible structure sequence containing D_4^8.

PROOF: Each time it is rather obvious that the sequences are the structure sequences of the given relations. Exemplarily, we treat the last case: It is clear that D_n^{2n} conserves the separation relation, hence that it is a subgroup of the n^{th} structure group. D_n^{2n} acts transitively, thus it is sufficient to show that the one-point stabilizers of the structure group equals those of D_n^{2n}, which are the "swap group around that point". But after fixing one point, the betweenness relation, whose structure group is the swap group, is definable from the separation relation.

(a) is clear anyway. For (b), if a structure sequence (G_n) contains Z_3^2, it has to preserve the betweenness relation, hence $G_n \leqslant Z_n^2$. By induction, G_n has to have at least 2 elements, whence $G_n = Z_n^2$.

Analogously for (c) and (d): if (G_n) contains Z_3^3 or D_4^8, we get $(G_n) \leqslant Z_n^n$ and $(G_n) \leqslant D_n^{2n}$ and by induction $|G_n| \geqslant n-1$ and $|G_n| \geqslant 2n-2$ respectively, whence equality. $\square$

3.2 The full symmetric group

Clearly, (S_n) is the structure sequence of the infinite set.

Lemma 3.2 *$\pi_i : G_{n+1} \to S_n$ is not injective for some i iff G_{n+1} contains a cycle ζ of consecutive elements.*

PROOF: If $\pi_i(\sigma) = \pi_i(\tau)$ and $\sigma \neq \tau$, then $\sigma(i) \neq \tau(i)$. Suppose $\sigma(i) < \tau(i)$. Then $\sigma \circ \tau^{-1} = \big(\sigma(i)\ (\sigma(i)+1) \ldots \tau(i)\big)$ is the desired cycle. Conversely, suppose $\zeta = \big(c\ (c+1) \ldots d\big) \in G_{n+1}$ and let $\sigma = \zeta \circ \tau$ for arbitrary τ. Then $\pi_i(\sigma) = \pi_i(\tau)$ for $i = \sigma^{-1}(c)$. □

Lemma 3.3 *(a) S_2 only extends to Z_3^2, Z_3^3 or S_3.*

(b) S_3 only extends to A_4, S_4 or one of the three dihedral Sylow-2-subgroups of S_4.

(c) If $n \geqslant 4$, then S_n only extends to A_{n+1} or S_{n+1}.

PROOF: (a) S_3 has five subgroups with at least two elements. By Remark 2.2 (c) and Lemma 3.2, S_2 does not extend to two of the one-point stabilizers.

(b),(c) If $S_n \sqsubset G_{n+1}$, then G_{n+1} has at least $n!$ elements, hence index $m \leqslant n+1$ in S_{n+1}. The action of S_{n+1} on the cosets of G_{n+1} provides a homomorphism $S_{n+1} \to S_m$ whose image has at least m elements. If $n \geqslant 4$, then A_{n+1} is simple. Hence either $m = 1$ and $G_{n+1} = S_{n+1}$, or $m = 2$ and $G_{n+1} = A_{n+1}$, or $m = n+1$, and then the homomorphism above is an isomorphism and G_{n+1} the one-point stabilizer of itself, hence $G_{n+1} \cong S_n$. If $n = 3$, the only further subgroups of S_4 of index at most 4 are the three Sylow-2-subgroups of index 3. This proves (b).

If $G_{n+1} \cong S_n$, G_{n+1} is a one-point stabilizer, except for $n = 5$, since S_6 has non-trivial outer automorphisms and therefore six "exotic" subgroups $S_6^{\mathrm{ex}1}, \ldots, S_6^{\mathrm{ex}6}$ isomorphic to S_5 (given by the action of S_5 on its six Sylow-5-subgroups). By Remark 2.2 (c) and Lemma 3.2, the one-point stabilizers are excluded. Inspection of the exotic groups shows, in some numbering, $(123456) \in S_6^{\mathrm{ex}1}$; $(2345) \in S_6^{\mathrm{ex}2}$; $(1234), (3456) \in S_6^{\mathrm{ex}3}$; $(1234), (23456) \in S_6^{\mathrm{ex}4}$; $(12345), (3456) \in S_6^{\mathrm{ex}5}$; $(2345) \in S_6^{\mathrm{ex}6}$. (This can also be deduced somewhat lengthily from the way the non-trivial outer automorphisms act on the conjugation classes of S_6). Hence $S_6^{\mathrm{ex}i}$ does not extend S_5 by Lemma 3.2.

This proves the "only"-parts of the lemma. One can check without difficulties that the groups really extend to the given ones, but this is not necessary for the proof of the theorem. □

3.3 The remaining cases

Finally we show that the remaining cases — two dihedral groups and the alternating groups — can't occur.

Lemma 3.4 *$D_4' = \langle (12), (1324) \rangle$ does not extend S_3.*

PROOF: It is straightforward to check that $\pi_4[D_4'] = \{\mathrm{id}_3, (12), (13), (123)\} \neq S_3$. □

Lemma 3.5 *$D_4'' = \langle (14), (1243) \rangle$ does not extend to any group.*

Proof: The 4-ary relation $R(a,b,c,d)$ induced by D''_4 is the following: "the unorientated intervals (a,d) and (b,c) lie one in the other". Given five elements $a < b < c < d < e$, R allows to identify a and e up to "swap" as the two elements such that $\models R(a,x,y,e)$ for each choice $x,y \in \{b,c,d\}$. Then d is determined from a by $R(a,x,y,d)$ with $x,y \neq e$, analogously b from e. Thus all the elements are identifiable up to swap, the betweenness relation is definable from R and the structure sequence of $(\mathbb{Q},R)$ must be (Z_n^2). □

Lemma 3.6 *A_n has no extension if $n \geqslant 4$.*

Proof: Consider all $\sigma \in S_{n+1}$ such that $\pi_5(\sigma) = (123)$. We show that some $\pi_i(\sigma)$ is odd. Note that σ is determined by $\sigma(5)$. 1st case: $\sigma(5) \geqslant 3$. Then $\pi_3(\sigma)$ has the same parity as σ and $\pi_1(\sigma)$ has a different parity, whence one is odd. 2nd case: $\sigma(5) = 2$. Then $\sigma = (1452)$ and $\pi_3(\sigma) = (1342)$ is odd. 3rd case: $\sigma(5) = 1$. Then $\sigma = (145)$ and $\pi_2(\sigma) = (1234)$ is odd. □

This proves the theorem. The information might be put together in a final picture as follows:

$$
\begin{array}{ccc|cc|cc|cc|cc|ccl}
I_1 & \sqsubset & I_2 & \sqsubset & I_3 & \sqsubset & I_4 & \sqsubset & I_5 & \sqsubset & I_6 & \sqsubset & \dots & \text{linear order} \\
 & & S_2 & \sqsubset & Z_3^2 & \sqsubset & Z_4^2 & \sqsubset & Z_5^2 & \sqsubset & Z_6^2 & \sqsubset & \dots & \text{betweenness} \\
 & & & & Z_3^3 & \sqsubset & Z_4^4 & \sqsubset & Z_5^5 & \sqsubset & Z_6^6 & \sqsubset & \dots & \text{cyclic order} \\
 & & & & S_3 & \not\sqsubset & D'_4 & & & & & & & \\
 & & & & & \sqsubset & D_4^8 & \sqsubset & D_5^{10} & \sqsubset & D_6^{12} & \sqsubset & \dots & \text{separation} \\
 & & & & & & D''_4 & \not\sqsubset & & & & & & \\
 & & & & & & A_4 & \not\sqsubset & & & & & & \\
 & & & & & & S_4 & \sqsubset & A_5 & \not\sqsubset & & & & \\
 & & & & & & & & S_5 & \not\sqsubset & \text{exotic } S_5 & & & \\
 & & & & & & & & & \sqsubset & A_6 & \not\sqsubset & & \\
 & & & & & & & & & & S_6 & \sqsubset & \dots & \text{infinite set}
\end{array}
$$

The proof of Theorem 1.1 also shows that the dense linear order, cyclic order, betweenness and separation relations eliminate quantifiers in their natural languages, since in each case, if n is the arity of the relation, the n^{th} group in the structure sequence already determines the whole sequence.

References

[C1] Peter Cameron, Transitivity of permutation groups on unordered sets, *Math. Z.* **148** (1976), pp. 127–139.

[C2] Peter Cameron *Oligomorphic permutation groups*, LMSLN 152, Cambridge University Press, Cambridge 1990.

[F] Claude Frasnay, quelques problèmes conbinatoires concernant les ordres totaux et les relations monomorphes, *Annales de l'INstitut Fourier (Grenoble)* **15** (1965), pp. 415–524.

[Hi] Graham Higman, Homogeneous relations. *Quart. J. Math.* **28** no. 109 (1977), pp. 31–39.

[HLS] Wilfrid Hodges, Alistair Lachlan, Saharon Shelah, Possible orderings of an indiscernible sequence, *Bull. London Math. Soc.* **9** (1977), pp. 212–125.