Inscribed radius bounds for metric measure spaces with mean convex boundary

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The Heintze-Karcher inequality for Riemannian manifolds

M^n compact Riemannian manifold, $\mathrm{ric}_M \geq K$, assume $K > 0$, $n \geq 2$.

$S^{n-1} \subset M$ submanifold, compact, embedded, two-sided.

Theorem (Heintze-Karcher, 1978)

$$\mathrm{vol}_M(S_\epsilon^+) \leq \int_S \int_0^\epsilon J_{H(p),K,n}(t)dtd\,\mathrm{vol}_S(p)$$

where $S_\epsilon^+ := \left\{\exp_x(tN^+(x)) : t \in (0,\epsilon), x \in S\right\}$, N^+ *is one of two unit normal fields on* S, H *is the mean curvature of* S *and the Jacobian*

$$J_{H,K,n}(t) = \left(\cos(t\sqrt{K/(n-1)}) + \tfrac{H}{n-1}\sin(t\sqrt{K/(n-1)})\right)_+^n.$$

Also

$$\mathrm{vol}_M(M) \leq \int \int J_{H(p),K,n}(t)dtd\,\mathrm{vol}_S(p)$$

with "=" iff $M = \mathbb{S}^n$ *and* S *has constant mean curvature.*

InRadius bounds for Riemannian manifolds with boundary

M^n compact Riemannian manifold with boundary. $\mathrm{ric}_M \geq 0$, $n \geq 2$.

Theorem (Kasue, 1984)

Assume the mean curvature $H_{\partial M}$ of ∂M is bounded from below by $n-1$. Then

$$\sup_{x \in M} \inf_{y \in \partial M} d(x,y) \leq 1$$

and "$=$" iff $M = B_1(0) \subset \mathbb{R}^n$.

Theorem (Kasue, 1984)

Assume ∂M is disconnected and satisfies $H_{\partial M} \geq 0$. Then

$$M = N \times [0,1]$$

for a closed Riemannian manifold N with $\mathrm{ric}_N \geq 0$.

Curvature-dimension condition for metric measure spaces

(X, d, m) a metric measure space (compact, geodesic space, $\mathrm{m}(X) < \infty$).

For $N > 1$ the N-Renyi entropy is

$$\mu \in \underbrace{\mathcal{P}(X)}_{\text{prob. meas. on } X} \mapsto S_N(\mu) = \begin{cases} -\int_X \rho^{1-\frac{1}{N}} \, d\,\mathrm{m} & \text{if } \mu = \rho\,\mathrm{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition (Lott-Sturm-Villani)

(X, d, m) satisfies the curvature-dimension condition $CD(0, N)$ if $\forall \mu_0, \mu_1 \in \mathcal{P}(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that

$$S_N(\mu_t) \leq (1-t)S_N(\mu_0) + tS_N(\mu_1).$$

The curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ is defined similarly using the notion of "(K, N)-convexity."

Properties of *CD* spaces

- M^n a Riemannian manifold s.t. $M\backslash\partial M$ is geodesically convex and $e^{-f}\,\mathrm{vol}_M =: \mathrm{m}$, $f \in C^\infty(M)$. $K \in \mathbb{R}$, $N \geq n$. Then

$$(M^n, d_M, \mathrm{m}) \text{ satisfies } CD(K,N)$$
$$\Leftrightarrow \quad \mathrm{ric}_M^{f,N} := \mathrm{ric}_M + \nabla^2 f - \frac{1}{N-n} df \otimes df \geq K.$$

- $[a,b] \subset \mathbb{R}$. $K \in \mathbb{R}$ and $N > 1$.

$$([a,b], |\cdot|_2, m) \text{ satisfies } CD(K,N)$$
$$\Longleftrightarrow$$
$$m = h d\mathcal{L}^1 \text{ with } h \text{ continuous } \;\&\; \frac{d^2}{dt^t} h^{\frac{1}{N-1}} + \frac{K}{N-1} h^{\frac{1}{N-1}} \leq 0.$$

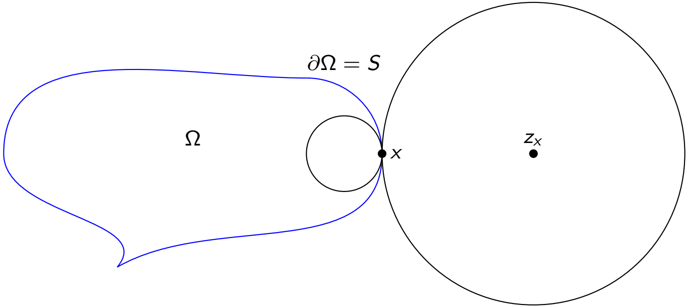
Hypersurfaces in metric measure spaces, interior ball condition

$\Omega \subset X$ open, and $S = \partial\Omega$. Assume $\mathrm{m}(S) = 0$

Ω *satisfies an interior ball condition* if $\forall x \in S$ there exists $z_x \in \Omega$ and $\eta_x > 0$ such that

$$B_{\eta_x}(z_x) \subset \Omega \quad \text{and} \quad x \in \partial B_{\eta_x}(z_x).$$

S *satisfies an exterior/interior ball condition* if Ω and $X \backslash \overline{\Omega}$ satisfy an interior ball condition.



$1D$ localisation method (Cavalletti-Mondino)

Let u be 1-Lipschitz. Define

$$\Gamma_u = \left\{(x, y) \in X^2 : u(y) - u(x) = d(x, y)\right\}$$

If $\gamma : [a, b] \to X$ is a (minimal) geodesic and $(\gamma(a), \gamma(b)) \in \Gamma_u$, then

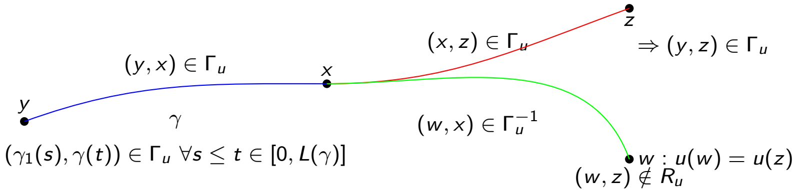
$$(\gamma(s), \gamma(t)) \in \Gamma_u \ \forall s \leq t \in [a, b].$$

Γ_u is transitive but not symmetric.

$\Gamma_u^{-1} = \{(x, y) : (y, x) \in \Gamma_u\}$. Define *transport relation*

$$R_u := \Gamma_u \cup \Gamma_u^{-1}, \quad P_1(R_u \backslash \{(x, y) : x = y\}) = \mathcal{T}_u.$$

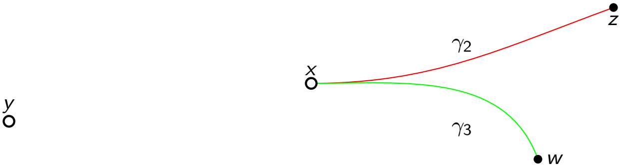
R_u is symmetric but not transitive.



Forward and backward branching points:

$$A_+ = \{x \in \mathcal{T}_u : \exists y, z \in \mathcal{T}_u \text{ s.t. } (x,y),(x,z) \in \Gamma_u, (y,z) \notin R_u\}$$
$$A_- = \left\{x \in \mathcal{T}_u : \exists y, z \in \mathcal{T}_u \text{ s.t. } (x,y),(x,z) \in \Gamma_u^{-1}, (y,z) \notin R_u\right\}$$

Define the non-branched transport set $\mathcal{T}_u^b = \mathcal{T}_u \backslash (A_+ \cup A_-)$



R_u restricted to $\mathcal{T}_u^b$ is an equivalence relation with quotient space Q,

$\mathfrak{Q} : \mathcal{T}_u^b \to Q$ quotient map.

Each equivalence class is given by the image of a distance preserving map $\gamma : I_\gamma \subset \mathbb{R} \to X$.
$\left(\mathcal{T}_u^b = \dot{\bigcup}_{\gamma \in Q} \mathrm{Im}(\gamma)\right)$

Disintegration formula:

$$m|_{\mathcal{T}_u^b} = \int \mathrm{m}_\gamma \, d\mathfrak{q}(\gamma)$$

where $\mathfrak{q} = \mathfrak{Q}_\# m$ and the measures m_γ are concentrated on $\mathrm{Im}(\gamma)$.

Theorem (Cavalletti-Mondino)

Let (X, d, m) be an essentially non-branching $CD(K, N)$-space. Then

- $m(A_+ \cup A_-) = 0$,
- *For $\mathfrak{q}$-a.e. γ the metric measure space $(\overline{Im(\gamma)}, d, \mathrm{m}_\gamma)$ is $CD(K, N)$.*

Remark: $\mathrm{m}_\gamma = \gamma_\# \left(h_\gamma d\mathcal{L}^1|_{I_\gamma} \right)$ for $h_\gamma : \overline{I_\gamma} \to [0, \infty)$ continuous such that

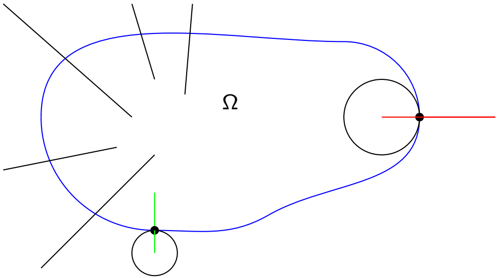
$$\frac{d^2}{dt} h_\gamma^{\frac{1}{N-1}} + \frac{K}{N-1} h_\gamma^{\frac{1}{N-1}} \leq 0 \text{ in distrib. sense.}$$

$\Omega \subset X$ open, $S = \partial\Omega$ satisfying an ext/int ball condition.

Signed distance function: $d_S = d_\Omega - d_{X\setminus\overline{\Omega}}$ where $d_\Omega(\cdot) = \inf_{x\in\Omega} d(x,\cdot)$

d_S is 1-Lipschitz since X is a length space.

Hence, apply $1D$ localisation method to $u = d_S$:



Surface measure, Mean curvature

$\mathfrak{q}$-a.e. needle $\gamma : I_\gamma \to X$ does NOT intersect with S at its endpoits.

Choose (arclength) parametrisation s.t. $0 \in \mathrm{Int}(I_\gamma)$ and $S \cap \mathrm{Im}(\gamma) = \gamma(0)$ for $\mathfrak{q}$-a.e. γ.

Identify Q with $\{p \in S : p = \gamma(0), \gamma \in Q\} \subset S$ via $\gamma \mapsto \gamma(0)$.

Define surface measure m_S on S via

$$d\,\mathrm{m}_S := h_\gamma(0) d\mathfrak{q}(\gamma).$$

Recall h_γ is semi-concave: Left and right derivatives $\frac{d^\pm}{dt}$ exist $\forall t \in \mathrm{Int}(I_\gamma)$.

Define the mean curvature of S as

$$H(p) := \max\left\{ \frac{d^+}{dt} \log h_\gamma(0), \frac{d^-}{dt} \log h_\gamma(0) \right\},\ p = \gamma(0)$$

Heintze-Karcher inequality for metric measure spaces

Theorem (K. 2019)

Let (X, d, m) be an essentially nonbranching $CD(K, N)$ space, and let S be as before. $S_\epsilon^+ = B_\epsilon(\Omega)\backslash\overline{\Omega}$. Then

$$\mathrm{m}(S_\epsilon^+) \le \int_S \int_0^\epsilon J_{H(p),K,N}(t)dtd\,\mathrm{m}_S(p)$$

where

$$J_{H,K,N}(t) = \left(\cos(t\sqrt{K/(N-1)}) + \tfrac{H}{N-1}\sin(t\sqrt{K/(N-1)})\right)_+^N.$$

Also

$$\mathrm{m}(M) \le \int\int J_{H(p),K,N}(t)dtd\,\mathrm{m}_S(p).$$

For X satisfying $RCD(K, N)$ "=" if and only if there exists a $RCD(K, N-1)$ space Y such that X is an $N-1$-suspension over Y.

Proof of the first inequality

$$
\begin{aligned}
\mathrm{m}(S_\epsilon^+) &= \int \mathrm{m}_\gamma(B_\epsilon(\Omega)\backslash\Omega)d\mathfrak{q}(\gamma) \\
&= \int \left(\int_{I_\gamma \cap (0,\epsilon)} h_\gamma(t)dt \right) d\mathfrak{q}(\gamma) \\
&= \int \left(\int_{I_\gamma \cap (0,\epsilon)} \frac{h_\gamma(t)}{h_\gamma(0)} dt \right) h_\gamma(0) d\mathfrak{q}(\gamma) \\
&\leq \int \left(\int_0^\epsilon J_{H(\gamma(0)),K,N}(t)dt \right) h_\gamma(0) d\mathfrak{q}(\gamma) \\
&= \int \int_0^\epsilon J_{H(p),K,N}(t)dtd\,\mathrm{m}_S(p)
\end{aligned}
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Thank you!