

20.04.26

MATH. PHYSICS II : SEMICLASSICAL ANALYSIS

SS26 Mo. 14:15 - 16:00

Exercise session: Wed.

Exam: 50% points + oral examination

1. Introduction

Semiclassical analysis is a branch of analysis and mathematical physics that studies the behaviour of quantum systems in the approximation $\hbar$, where $\hbar$ is the reduced Planck constant, $\hbar = \frac{h}{2\pi}$ and $h = 6.6 \times 10^{-34}$ Joules/second.

It shows that quantum mechanics is a generalization of classical mechanics.

Typically one studies (pseudo-) differential operators dependent on $\hbar$ and describes their behaviour as $\hbar \rightarrow 0$.

EXAMPLE.

Schrödinger operator $H_{\hbar} = -\hbar^2 \Delta + V(x)$

- study spectral properties of $H_{\hbar}$ as $\hbar \rightarrow 0$
- study dynamical properties of $H_{\hbar}$ as $\hbar \rightarrow 0$

Why is semiclassical analysis interesting?

- ① Bridge between quantum and classical mechanics
- ② Spectral properties $\rightarrow$ Weyl law
- ③ PDEs: propagation of singularities
scattering phenomena
- ④ Connection to classical dynamics $\rightarrow$ chaotic systems (quantum ergodicity)

$\rightarrow$ integrable systems & inv. manifolds

$\rightarrow$ coherence / decoherence

1.1 Short review of classical mechanics

Let us consider a particle of mass m . Its position at time $t \in \mathbb{R}$, is a function $t \rightarrow x(t) \in \mathbb{R}^d$, while its momentum at time t is given by $t \rightarrow p(t) \in \mathbb{R}^d$.

Given F , a conservative force field (i.e. $F = -\nabla V$, $V: \mathbb{R}^d \rightarrow \mathbb{R}$ pot.)

Newton's laws give the evolution of the point particle:

$$m\ddot{x}(t) = F$$

that is $\ddot{x}(t) = -\frac{1}{m} \nabla V(x(t))$.

Noticing that $p(t) := m\dot{x}(t)$, we have

$$\begin{cases} \dot{x}(t) = \frac{1}{m} p(t) \\ \dot{p}(t) = -\nabla V(x(t)) \end{cases}$$

The curve $t \mapsto (x(t), p(t)) \in \mathbb{R}^{2d}$ represents a phase-space trajectory.

We say that a function of x and p is an observable if it is a smooth function of x and p , i.e. $a := a(x, p)$ is smooth.

Within classical mechanics, all measurements should give be associated with an observable $a(x, p)$, but empirically we know that this is not the case. One example is the Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2\pi}$$

which gave rise to a new interpretation of the elementary particles having a double nature, wavelike and corpuscular.

The theory that aims at describing such double nature is quantum mechanics.

1.2 Short review of Quantum Mechanics

In order to describe the wavelike and corpuscular nature of particles, in quantum mechanics a particle is represented by a function $\psi_t: x \rightarrow \psi_t(x) \in L^2(\mathbb{R}^d)$, for each $t \in \mathbb{R}$. Moreover, $\|\psi_t\|_{L^2(\mathbb{R}^d)} = 1$. ψ_t is said to be the STATE of the particle at time t .

The position of the particle in the state ψ_t is given by the expectation of the position operator X :

$$\langle X \rangle_{\psi_t} := \langle X \psi_t, \psi_t \rangle_{L^2(\mathbb{R}^d)}$$

while the momentum

$$\langle P \rangle_{\psi_t} := \langle -i\hbar \nabla \psi_t, \psi_t \rangle_{L^2(\mathbb{R}^d)}$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ denotes the inner product on $L^2(\mathbb{R}^d)$.

We define the $\hbar$ -Fourier transform $F_\hbar$ as

$$F_\hbar \psi_t(p) := \hat{\psi}_t(p) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{-i x \cdot p / \hbar} \psi_t(x) dx$$

PROPOSITION 1.1

- $F_\hbar$ is an isometry of $L^2(\mathbb{R}^d)$
- If $\psi_t \in H^1(\mathbb{R}^d)$, $F_\hbar(-i\hbar \nabla \psi_t)(p) = p F_\hbar \psi_t(p)$
- $\langle P \rangle_{\psi_t} = \langle p \hat{\psi}_t, \hat{\psi}_t \rangle_{L^2(\mathbb{R}^d)}$

Recall that OBSERVABLES are self-adjoint operators (see the definition of self-adjoint operators for unbounded operators).

Mirroring classical mechanics, we see that given $(x, p) \in \mathbb{R}^{2d}$, we can associate to them the position and momentum operators X and P .

A natural question is whether this correspondence between classical and quantum observables can be generalized. More precisely, given a classical observable $a(x, p)$, is there a natural way to associate with it a quantum observable $a(x, -i\hbar \nabla)$?

Pseudodifferential calculus provides the algebraic correspondence

between the space of classical observables, endowed with the multiplication, and the space of quantum observables, endowed with the composition of operators.

EXAMPLE 1.2

- The quantum analogue of the kinetic energy $\frac{p^2}{2m}$ is $-\frac{\hbar^2}{2m} \Delta$.
- The quantum analogue of the total energy $\frac{p^2}{2m} + V(x)$ is $-\frac{\hbar^2}{2m} \Delta + V(x)$, where $V(x)$ has to be understood as a multiplication operator.

Remark 1.3

- (1) If $a = a(x)$, then $\psi \rightarrow a\psi$ the associated quantum observable is a multiplication operator.
- (2) If $a = a(p)$, then $a(-i\hbar\nabla) = \mathcal{F}_\hbar^{-1} a(p) \mathcal{F}_\hbar$.

2. PSEUDO-DIFFERENTIAL CALCULUS

Recall that a classical observable a is a smooth function of x and p , i.e. $a(x,p)$ smooth.

We will refer to classical observables as SYMBOLS.

Definition 2.1

Let $g \in C^\infty(\mathbb{R}^d; \mathbb{R}_{+} \setminus \{0\})$ be s.t. $\partial^\alpha g = O(g)$ for any $\alpha \in \mathbb{N}^d$ and uniformly on $\mathbb{R}^d$. Then g is called ORDER FUNCTION on $\mathbb{R}^d$.

EXAMPLES 2.2

- (1) $\langle z' \rangle^m := (1 + |z'|^2)^{m/2}$, $m \in \mathbb{R}$, $z' = (z_1, \dots, z_k)$ with $k \leq d$, is an order function.
- (2) $e^{\alpha \langle z \rangle}$, $\alpha \in \mathbb{R}$, is an order function
- (3) $e^{f(z)}$, for f smooth and bounded with bounded derivatives, is an order function.
- (4) e^{z^2} is not an order function.

PROPOSITION 2.3

If g is an order function on $\mathbb{R}^d$, then $1/g$ is an order function on $\mathbb{R}^d$.

proof. the goal is to show that, for any $\alpha \in \mathbb{N}^d$, $\partial^\alpha (1/g) = O(1/g)$.

Let $\tilde{g} = 1/g$ so that $g\tilde{g} = 1$.

Using Leibniz formula α times, we obtain

$$0 = \partial^\alpha (1) = \partial^\alpha (g\tilde{g}) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\partial^\beta \tilde{g})(\partial^{\alpha-\beta} g)$$

By induction: for $\alpha=1$, $0 = \partial(\tilde{g}g) = \tilde{g}\partial g + \partial\tilde{g} \cdot g = \tilde{g} O(g) + \partial\tilde{g} O(g)$

$$\Rightarrow d\tilde{g} = O(\tilde{g}), \text{ that is } \partial(1/g) = O(1/g).$$

For α assume it is true. Prove it for $\alpha+1$:

$$0 = \sum_{\beta \leq \alpha+1} \frac{(\alpha+1)!}{\beta!(\alpha+1-\beta)!} (\partial^\beta \tilde{g}) (\partial^{\alpha-\beta} g)$$

$$\text{when } \beta = \alpha+1: 0 = \frac{(\alpha+1)!}{(\alpha+1)! \cdot 0!} \partial^{\alpha+1} \tilde{g} O(g) + \underbrace{\sum_{\beta \leq \alpha} \dots}_{O(g)O(g)}$$

$$\text{Then } O(g) \partial^{\alpha+1} \tilde{g} = O(\tilde{g}) O(g)$$

$$\Rightarrow \partial^{\alpha+1} \tilde{g} = O(\tilde{g}) \quad \square$$

Definition 2.4 (SPACE of SYMBOLS)

A function $a = a(x; h)$ defined on $\mathbb{R}^d \times (0, h_0]$, for some $h_0 > 0$, is in $S_d(g)$ if a depends smoothly on x and $\forall \alpha \in \mathbb{N}^d$

$$\partial_x^\alpha a(x; h) = O(g(x))$$

uniformly with respect to $(x, h) \in \mathbb{R}^d \times (0, h_0]$, where g is an order function.

EXAMPLES. 2.5

- (1) $S_d(1)$ is the set of families of C^∞ functions on $\mathbb{R}^d$ parametrized by $h \in (0, h_0]$, that are uniformly bounded together with all their derivatives.
- (2) $\eta \in C_0^\infty(\mathbb{R}^d)$ is in $S_d(1)$.
- (3) If $V(x) \in S_d(1)$, then $p^2 + V(x) \in S_d(\langle p \rangle^2)$.
- (4) For $m \in \mathbb{R}$, $\langle x \rangle^m \in S_d(\langle x \rangle^m)$
- (5) $(x, p) \rightarrow e^{ix \cdot p} \notin S_d(\langle x \rangle^m)$, $m \in \mathbb{R}$
but $e^{ix \cdot p} \in S_d(e^{\varepsilon \langle x \rangle})$, for $\varepsilon > 0$.