

We are now left with the proofs of the three Lemmas 2.48, 2.49, 2.50.

proof of Lemma 2.48 (partition of $\mathbb{R}^d$)

Let K be the compact set defined as follows

$$K := \left\{ z \in \mathbb{R}^d : |z_i| \leq \frac{1}{2}, i=1, \dots, d \right\}$$

Then there exists (by construction) $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi \geq 0$ and

$\phi|_K = 1$. Let

$$\Phi(z) = \sum_{k \in \mathbb{Z}^d} \phi(z-k)$$

and observe that the sum is finite due to the compact support of ϕ .

Moreover, $\forall k \in \mathbb{Z}^d$ we have $\Phi(z+k) = \Phi(z)$ and $\Phi(z) \geq 1$.

If we now choose

$$\chi_0(z) = \frac{\phi(z)}{\Phi(z)}$$

we are done.

proof of Lemma 2.49 (Cotlar-Stein Lemma)

Fix $M \geq 0$ and let

$$S := \sum_{|k| \leq M} A_k$$

Notice that

S^*S is a non-negative self-adjoint operator

and denote by $\sigma(S^*S)$ the spectrum of S^*S .

Then

$$\|S\|^2 = \sup_{\substack{u \in \mathcal{H} \\ \|u\|=1}} \|Su\|^2 = \sup_{\substack{u \in \mathcal{H} \\ \|u\|=1}} \langle S^*S u, u \rangle = \sup_{\lambda \in \sigma(S^*S)} |\lambda| = \|S^*S\|.$$

Thus, for $m \geq 1$,

$$\|S\|^{2m} = \|S^*S\|^m = \left(\sup_{\lambda \in \sigma(S^*S)} |\lambda| \right)^m = \sup_{\lambda \in \sigma(S^*S)} |\lambda|^m$$

$$= \sup_{\lambda' \in \sigma((S^*S)^m)} |\lambda'| = \|(S^*S)^m\|.$$

Moreover, by definition of S , we have

$$\begin{aligned}(S^*S)^m &= \left(\left(\sum_{|k| \leq M} A_k^* \right) \left(\sum_{|k'| \leq M} A_{k'} \right) \right)^m \\ &= \left(\sum_{|k|, |k'| \leq M} A_k^* A_{k'} \right)^m \\ &= \sum_{|k_1|, |k'_1| \leq M} A_{k_1}^* A_{k'_1} A_{k_2}^* A_{k'_2} \dots A_{k_m}^* A_{k'_m}\end{aligned}$$

Let

$$A := \sum_{|k|, |k'| \leq M} A_k^* A_{k'}$$

Then, by assumption⁽ⁱ⁾, it holds

$$\begin{aligned}\|A\| &\leq \sum_{|k_1|, |k'_1| \leq M} \|A_{k_1}^* A_{k'_1}\| \dots \|A_{k_m}^* A_{k'_m}\| \\ &\stackrel{(*)}{\leq} \omega(k_1 - k'_1) \dots \omega(k_m - k'_m) \quad (*)\end{aligned}$$

Moreover, observe that

$$\|A_k\|^2 = \|A_k^* A_k\| \leq \omega(k - k) = \omega(0) \Rightarrow \|A_k\| \leq \sqrt{\omega(0)}.$$

Then, from

$$\|A\| \leq \sum_{|k_1|, |k'_1| \leq M} \sqrt{\omega(0)} \omega(k'_1 - k_2) \dots \omega(k'_{m-1} - k_m) \sqrt{\omega(0)} \quad (**)$$

↑

where we used: $\|A\| \leq \|A_{k_1}^*\| \|A_{k'_1}\| \dots \|A_{k'_{m-1}}\| \|A_{k_m}\|$

The geometric mean of (*) and (**) yields

$$\|A\| \leq \sqrt{\omega(0)} \sqrt{\omega(k'_1 - k_2)} \sqrt{\omega(k'_2 - k_3)} \dots \sqrt{\omega(k'_{m-1} - k_m)} \sqrt{\omega(0)}$$

Hence,

$$\begin{aligned}\|(S^*S)^m\| &\leq \sum_{|k_1| \leq M} \sum_{|k'_1| \leq M} \dots \sum_{|k_m| \leq M} \sum_{|k'_m| \leq M} \sqrt{\omega(0)} \sqrt{\omega(k_1 - k'_1)} \sqrt{\omega(k'_1 - k_2)} \dots \sqrt{\omega(k'_{m-1} - k_m)} \\ &\leq \sum_{|k_1| \leq M} \sum_{|k_2| \leq M} \dots \sum_{|k_m| \leq M} \sqrt{\omega(0)} \sqrt{\omega(k_1 - k'_1)} \dots \sqrt{\omega(k'_{m-1} - k_m)} \cdot C_0\end{aligned}$$

$$\text{where } C_0 := \sum_{k \in \mathbb{Z}^d} \sqrt{\omega(k)}.$$

Iterating this argument, we get

$$\|(S^*S)^m\| \leq \sum_{|k_i| \leq M} \sqrt{w(0)} C_0^{2m-1} \leq (2M+1)^m \sqrt{w(0)} C_0^{2m-1}$$

This implies

$$\|S\| \leq \left[(2M+1)^m \sqrt{w(0)} C_0^{2m-1} \right]^{1/2m} \quad \forall m \geq 1$$

$$\uparrow$$

$$\left(\text{because } \|S\|^{2m} = \|(S^*S)^m\| \right)$$

By taking the limit $m \rightarrow +\infty$, we get

$$\lim_{m \rightarrow +\infty} \|S\| \leq \lim_{m \rightarrow +\infty} \left[(2M+1)^m \sqrt{w(0)} C_0^{2m-1} \right]^{1/2m} = C_0 \sqrt{2M+1}$$

$$\Rightarrow \|S\| \leq C_0 \quad \blacksquare$$

proof of lemma 2.50 (Schur lemma)

By Cauchy-Schwarz

$$|Au(x)|^2 \leq \left(\int |k(x,y)|^{1/2} |k(x,y)|^{1/2} |u(y)| dy \right)^2$$

$$\leq \left(\int |k(x,y)| dy \right) \left(\int |k(x,y)| |u(y)|^2 dy \right)$$

Thus

$$\|Au\|^2 \leq \int \left(\int |k(x,y)| dy \right) \left(\int |k(x,y)| |u(y)|^2 dy \right) dx$$

$$\leq \left(\sup_x \int |k(x,y)| dy \right) \left(\iint |k(x,y)| |u(y)|^2 dy dx \right)$$

$$\leq \left(\sup_x \int |k(x,y)| dy \right) \left(\sup_y \int |k(x,y)| dx \right) \|u\|_2^2$$

$$\Rightarrow \|A\|_{\mathcal{L}(L^2)} = \sup_{\|u\|_2 \neq 0} \frac{\|Au\|_{L^2}}{\|u\|_2} \leq \left(\sup_x \int |k(x,y)| dy \right) \left(\sup_y \int |k(x,y)| dx \right) \quad \blacksquare$$

We are now interested in some applications of the Calderón-Vaillancourt theorem.