

Theorem 2.51 (Egorov's Thm.)

Let  $A_t = \text{Op}(a_t)$ ,  $a_t \in S_{2d}(\langle \xi \rangle^m)$  for each  $t \in \mathbb{R}$ , such that

$$A_t = A_1 + A_2$$

with  $\sigma(A_1) \in S_{2d}(\langle \xi \rangle^m)$  and  $a_2 \in S_{2d}(1)$ .

Consider the equation

$$\partial_t u = i A_t(x, D_x) u \quad (1)$$

Let  $S(t, s)$  be the propagator associated with (1) mapping  $u(s)$  to  $u(t)$ .

Let  $B_t = S(t, 0) B S(0, t)$ , for  $B = \text{Op}(b)$ ,  $b \in S_{2d}(\langle \xi \rangle^m)$ .

Then, for each  $t$ ,  $\sigma(B_t) \in S_{2d}(\langle \xi \rangle^m)$  and  $\sigma(B_t)(x_0, \xi_0) = b_t(y_0, \eta_0)$

where  $(y_0, \eta_0)$  are obtained from  $(x_0, \xi_0)$  by the flow  $\Phi_{0,t}$  generated by the Hamiltonian vector field

$$H_{A_1}(t, x, \xi) = \sum_{j=1}^d \left( \frac{\partial \sigma(A_1)}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial \sigma(A_1)}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

proof.  $S(t, s)$  is continuous (bounded) operator from  $H^s$  to  $H^s$ .

From  $B_t = S(t, 0) B S(0, t)$  we get

$$\begin{cases} B'_t = i [A_t(x, D_x), B_t] \\ B_0 = B \end{cases} \quad (2)$$

Our goal is to construct  $Q_t$  approx. solution to (2) and show that  $Q_t - B_t$  is a smoothing operator.

To this end, we look for  $Q_t$  such that  $\sigma(Q_t) \in S_{2d}(\langle \xi \rangle^m)$  and

$$\begin{cases} Q'_t = i [A_t(x, D_x), Q_t] + R_t \\ Q_0 = B \end{cases}$$

with  $R_t = \text{Op}(r_t)$  with  $r_t = \mathcal{O}(t^\infty)$ .

Let us construct the symbol  $q(t, x, \xi)$  as asymp. expansion:

$$q(t, x, \xi) \sim q_0(t, x, \xi) + q_1(t, x, \xi) + \dots$$

The symbol of  $i [A_t(x, D_x), Q_t]$  is of the form

$$H_{A_1} q + \{A_2, q\} + \sum_{|\alpha| \geq 2} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha A_t D_x^\alpha q - D_\xi^\alpha q D_x^\alpha A_t)$$

We then define  $q_0$  s.t.

$$(\partial_t - H_{A_t}) q_0 = 0 \quad \text{and} \quad q_0(0, x, \xi) = \psi(B_t)(x, \xi) \in S_{2d}(\langle \xi \rangle^m)$$

where  $(x, \xi)$  are chosen according to the theorem statement, i.e.

$$q_0(t, x_0, \xi_0) = \psi(B_t)(y_0, \eta_0).$$

Recursively,  $\forall j \geq 1$ ,

$$\begin{cases} (\partial_t - H_{A_t}) q_j^{(j)}(t, x, \xi) = b_t^{(j)}(x, \xi) \\ q_0^{(j)}(x, \xi) = 0 \end{cases}$$

with solutions  $q_j^{(j)} \in S_{2d}(\langle \xi \rangle^{m-j})$ , leading to

$$\begin{cases} Q_t' = i[A_t(x, D_x), Q_t] + R_t \\ Q_0 = B \end{cases}$$

We now show that  $B_t - Q_t$  is a smoothing operator, in other words

we show that for any  $f \in H^s(\mathbb{R}^d)$

$$v(t) - w(t) := S(t, 0)Bf - Q_t S(t, 0)f \in H^\infty(\mathbb{R}^d)$$

$$\text{where } H^\infty(\mathbb{R}^d) := \bigcap_{s \geq 0} H^s(\mathbb{R}^d).$$

Observe that

$$\begin{cases} \partial_t v = i A_t(x, D_x) v \\ v(0) = Bf \end{cases} \quad \begin{cases} \partial_t w = i A_t(x, D_x) w + g \\ w(0) = Bf \end{cases}$$

where  $g = R_t S(t, 0)w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^d))$ .

Then  $(v-w)$  satisfies

$$\begin{cases} \partial_t (v-w) = i A_t(x, D_x) (v-w) - g \\ v(0) - w(0) = 0 \end{cases}$$

Then using the hyperbolic structure of the eq. we get  $v-w \in H^\infty$ , for every  $f \in H^s(\mathbb{R}^d)$ .

Indeed

$$\partial_t u = i A u - g \Rightarrow u(t) = S(t, 0) u_0 - \int_0^t S(t, s) g(s) ds$$

Notice that  $S(t, s)$  is continuous on  $H^s(\mathbb{R}^d) \forall s \in \mathbb{R}$ , thus

$$u_0 \in H^s \Rightarrow S(t, 0) u_0 \in H^s.$$

Recall the def. of  $H^\infty$ : it suffice to show that  $u(t) \in H^s \forall s$ .

But, indeed, if  $g \in H^\infty$  in particular  $g \in H^s \forall s \geq 0$ , and therefore

$$\int_0^t S(t,s)g(s)ds \in H^s(\mathbb{R}^d) \quad \forall s$$

hence  $u(t) \in H^\infty(\mathbb{R}^d)$ .

### Remark 2.52

Egorov's Thm says that quantum observables evolve, to leading order, by the classical Hamiltonian flow. In other words, conjugation by the quantum propagator correspond to transporting the symbol along the classical trajectories.

### Example 2.53

Let  $A$  be a s.a. <sup>pseudodiff.</sup> operator with principal symbol  $a(x,\xi)$  and let  $S(t) = e^{-itA/\hbar}$ .

If  $B$  is a pseudodifferential operator,  $B_t = S(t)^* B S(t)$  is again a pseudodifferential operator with principal symbol  $b_t = b \circ \Phi_t$  where  $\Phi_t$  is the Hamiltonian flow generated by  $A$ :

$$\begin{cases} \dot{x} = D_\xi A \\ \dot{\xi} = -D_x A \end{cases}$$

Take the semiclassical Schrödinger operator

$$A = -\frac{\hbar^2}{2} \Delta + V(x)$$

with symbol  $a(x,\xi) = \frac{|\xi|^2}{2} + V(x)$ .

The Hamiltonian flow is

$$\begin{cases} \dot{x} = \xi \\ \dot{\xi} = -\nabla V(x) \end{cases}$$

Egorov's theorem gives that if the observable  $B$  initially measures a phase space quantity  $b(x,\xi)$ , then at time  $t$  it measures approximately  $b(\Phi_t(x,\xi))$ , i.e. quantum observables evolve according to classical mechanics in the semiclassical limit.