

We have seen that $p \geq 0$ does not imply $f_p \geq 0$.

This motivates us to define the Husimi transform

Definition. The Husimi transform of an operator p is defined as the ~~commutator~~ convolution of the Wigner function with a Gaussian:

$$\tilde{f}_p := g_h * f_p \geq 0$$

$$\text{where } g_h(x) = \left(\frac{2}{h}\right)^d e^{-|x|^2/h}$$

Remark.

$$\|\tilde{f}_p\|_{L^1} = \int_{\mathbb{R}^{2d}} g_h * f_p = \int \text{diag}(p) = h^d \text{tr}(p)$$

$$\begin{aligned} \int (g_h * f_p)(x, \xi) dx d\xi &= \int \left(\frac{2}{h}\right)^d e^{-|x-\xi|^2/h} f_p(y, \xi) dy d\xi dx \\ &= h^d \text{tr}(p). \end{aligned}$$

Let us now consider a sequence of operators $\{p_h\}_{h>0}$ such that

$$h^d \text{tr}(p_h) = \int \text{diag}(p_h) \text{ ~~is~~ uniformly bounded in } h.$$

Study $h \rightarrow 0$.

$$\text{Notice that } h^d \text{tr}(p_h) \text{ uniformly bounded} + \|\tilde{f}_{p_h}\|_{L^1} = h^d \text{tr}(p_h)$$

$$\Rightarrow \|\tilde{f}_{p_h}\|_{L^1} \text{ is uniformly bounded}$$

Hence, $\exists h_n \rightarrow 0$ and $\exists \mu \in \mathcal{M}(\mathbb{R}^{2d})^{*}$ (called Wigner measure) s.t.

$$\tilde{f}_{p_{h_n}} \xrightarrow{h \rightarrow 0} \mu \geq 0 \text{ weakly in the sense of measures.}$$

* This is a consequence of Banach-Alaoglu theorem + Riesz repr. Thm.

Indeed, $\|f_n\|_{L^1} < C \Rightarrow f_n = f_n(x) dx$ bounded in $\mathcal{H}(\Omega) = C_0(\Omega)^*$, the space

of finite measures, and then B.-A. (the closed ball of the dual is compact in the weak-* topology) yields

$\exists$ a subseq. $\mu_{n_k} \xrightarrow{*} \mu$ with μ finite measure.

The goal here is to obtain a bound on the Wigner measure. To this end let us consider the Banach space of test functions

$$X := \{\varphi \in C_0^\infty(\mathbb{R}^{2d}) : \|\varphi\|_X := \|\mathcal{F}_\xi \varphi\|_{L_x^1 L_\xi^\infty} < +\infty\}$$

Thm. (Lions-Paul '93)

let $M_0 > 0$ and $\{P_h\}_{h>0}$ be a family of nonneg operators st:
 $h^d \text{Tr}(P_h) = M_0$. Then, $\forall h > 0$, $\|f_{P_h}\|_X \leq M_0$ and $\exists h_n \rightarrow 0$ st

$$f_{P_h} \xrightarrow[h \rightarrow 0]{*} \mu \geq 0$$

in the weak-* topology of X' and

$$\int \mu \leq \liminf_{h \rightarrow 0} \int \text{diag}(P_h) dx$$

proof.

let $\varphi \in X$ test function with $\hat{\varphi} := \mathcal{F}_\xi \varphi$.

we have

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f_P \varphi dx d\xi &= \int_{\mathbb{R}^{2d}} P(x + \frac{y}{2}, x - \frac{y}{2}) \hat{\varphi}(x, \frac{y}{h}) dx dy \\ &\leq \sup_y \left(\int \sum_{j \in J} \lambda_j \psi_j(x + \frac{y}{2}) \overline{\psi_j(x - \frac{y}{2})} dx \right) \int_x \sup |\hat{\varphi}(x, \frac{y}{h})| dy \end{aligned}$$

where we used $P(x, y) = \sum_{j \in J} \lambda_j \psi_j(x) \overline{\psi_j(y)}$.

Then $\|f_P\|_{X'} \leq M_0$ because Cauchy-Schwarz ineq. leads to

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f_P \varphi dx d\xi &\leq \sup_y \left(\sum_{j \in J} \lambda_j \left(\int |\psi_j(x + \frac{y}{2})|^2 dx \right)^{1/2} \left(\int |\psi_j(x - \frac{y}{2})|^2 dx \right)^{1/2} \right) \int_x \sup |\hat{\varphi}(x, \frac{y}{h})| dy \\ &= \sup_y \sum_{j \in J} \lambda_j \|\psi_j\|_{L^2}^2 \int_x \sup |\hat{\varphi}(x, \frac{y}{h})| dy \leq \text{Tr}(P) h \|\varphi\|_X \end{aligned}$$

Since $\text{Tr}(\rho) < +\infty$,

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$$\int_{\mathbb{R}^{2d}} f_p \varphi dx d\xi \leq M_0 \|\varphi\|_X$$

$$\Rightarrow \|f_p\|_{X'} \leq M_0.$$

It follows that f_p converges up to subsequences to a measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$.

Up to subseq. we can assume that also the convergence of the Husimi measure holds, and therefore, by definition of $\tilde{f}$,

$$\begin{aligned} \langle f_p - \tilde{f}, \varphi \rangle_{X', X} &= \langle f_p, \varphi - g_h * \varphi \rangle_{X', X} \\ &\leq M_0 \|\varphi - g_h * \varphi\|_X \end{aligned}$$

$$\|\varphi - g_h * \varphi\|_X \leq I_1 + I_2$$

$$\text{where } I_1 = \int (1 - e^{-\frac{\pi}{2} h |\xi|^2}) \sup_x |\hat{\varphi}| d\xi$$

$$I_2 = \int \sup_x |\hat{\varphi} - g_h(x) * \hat{\varphi}| d\xi = \int \sup_x \left| \int \hat{\varphi}(x, \xi) - \hat{\varphi}(x-y, \xi) g_h(y) dy \right| d\xi$$

$$\text{using that } \mathcal{F}_\xi(g_h * \varphi) = (g_h(x) * \hat{\varphi}) e^{-\pi h |\xi|^2/2}.$$

I_1 and $I_2 \rightarrow 0$ by dom. conv. thm. + $\forall \phi \in C_0^\infty$ $g_h * \phi \rightarrow \phi$ a.e.

Then weak conv. of the Husimi transf. to μ implies

$$\int_{\mathbb{R}^{2d}} \mu \leq \liminf_{h \rightarrow 0} \int_{\mathbb{R}^d} h^d \rho(x, x) dx \quad \square$$

Similarly, one can also prove that $h^d \rho(x, x) =: \text{diag}(\rho)(x)$ is s.t.

$\|\text{diag}(\rho)\|_{L^1} = 1$ and therefore weakly converges to a measure ρ

(up to subseq.)