

Let us now consider the family of seminorms

$$N_\alpha(a) := \sup |\partial^\alpha a|.$$

They induce a topology on $S_d(g)$, and moreover

PROPOSITION 2.6

$S_d(g)$ endowed with the topology induced by the seminorms $N_\alpha(\cdot)$ is a Fréchet space.

Definition 2.7

Fréchet space X is a topological vector space such that:

- (i) X is an Hausdorff space; (ii) its topology is induced by a family of countable seminorms (i.e. $U \subset X$ is an open set $\Leftrightarrow \forall u \in U, \exists \beta \geq 0$ and $\varepsilon > 0$ s.t. the set $\{v : \|v - u\|_\alpha < \varepsilon \ \forall \alpha \leq \beta\} \subset U$); (iii) X is a complete space with respect to the family of seminorms.

PROPOSITION 2.8

Let g_1, g_2 be two ~~function~~ order functions on $\mathbb{R}^d$.

Let $a \in S_d(g_1)$, $b \in S_d(g_2)$. Then $g_1 \cdot g_2$ is an order function and $ab \in S_d(g_1 g_2)$.

proof. (exercise)

Definition 2.9

A symbol $a \in S_d(g)$ is **elliptic** if $\exists C_0 > 0$ s.t.

$$|a| \geq \frac{1}{C_0} g \quad \text{uniformly on } \mathbb{R}^d \times (0, h_0].$$

PROPOSITION 2.10

If $a \in S_d(g)$ is elliptic, then $\frac{1}{a} \in S_d(\frac{1}{g})$.

proof

Let $b := \frac{1}{a}$. Then the ellipticity assumption reads $|\frac{1}{a}| \leq \frac{C_0}{g}$. Consider $ab = 1$.

By induction on α :

$$\alpha = 1 : \partial(ab) = (\partial a)b + a(\partial b) = 0 \Rightarrow (\partial b) = -\frac{1}{a}(\partial a)b. \text{ By assumption}$$

$$|\partial b| \leq \left(\frac{C_0}{g}\right)^2 O(g) = O\left(\frac{1}{g}\right).$$

Assume it for α and prove it for $\alpha+1$:

$$0 = \partial^{\alpha+1} (ab) = \sum_{\beta \leq \alpha+1} \binom{\alpha+1}{\beta} (\partial^\beta a) (\partial^{\alpha+1-\beta} b)$$

Since by induction ^{step} we know that $\partial^\beta b \in S_d(\frac{1}{g}) \quad \forall |\beta| \leq |\alpha|$, we conclude that $\partial^{\alpha+1} b \in S_d(\frac{1}{g})$ from Leibniz formula. $\square$

2.1 $\hbar$ -DEPENDENCE

We are now interested in the dependence of the symbols from the semiclassical parameter $\hbar$. Our focus will be on asymptotic behavior of symbols wrt $\hbar$ small. For this reason, we introduce the following notation of asymptotic equivalence:

Definition 2.11

Let g be an order function on $\mathbb{R}^d$, and let $a \in S_d(g)$ and $\{a_j\}_{j \in \mathbb{N}}$ a sequence of symbols s.t. $a_j \in S_d(g) \quad \forall j \in \mathbb{N}$.

We say that a is asymptotically equivalent to the formal sum $\sum_{j=0}^{\infty} \hbar^j a_j \in S_d(g)$ and we denote it by

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j$$

if $\forall N \in \mathbb{N}, \forall \alpha \in \mathbb{N}^d, \exists h_{N,\alpha} > 0$ and $C_{N,\alpha} > 0$ such that

$$\left| \partial^\alpha \left(a - \sum_{j=0}^N \hbar^j a_j \right) \right| \leq C_{N,\alpha} h^N g$$

uniformly on $\mathbb{R}^d \times (0, h_{N,\alpha}]$.

Asymptotic equivalence means that, for $N > 0$ the symbol a can be approximated by $\sum_{j=0}^N \hbar^j a_j$, up to a symbol vanishing as h^N for $h \rightarrow 0$.

In particular, if a_j 's are zero, we will use the notation $a = O(h^\infty)$

in $S_d(g)$ if $a \sim 0$ in $S_d(g)$.

Furthermore, we observe that the series $\sum_{j \geq 0} h^j a_j$ is not necessarily convergent. However, one can always find a symbol a that is asymptotically equivalent to it. This is proven in the following

PROPOSITION 2.12 Let $\{a_j\}_{j \in \mathbb{N}}$ be a sequence of symbols $a_j \in S_d(g)$ $\forall j \in \mathbb{N}$. Then there exists $a \in S_d(g)$ such that a is asymptotically equivalent to $\sum_{j=0}^{\infty} h^j a_j$ in $S_d(g)$. Moreover, a is unique up to $O(h^\infty)$.

The symbol a satisfying this property is called a resummation of the formal symbol $\sum_{j \geq 0} h^j a_j$.

The proof of Proposition 2.12 is based on the following

LEMMA 2.13

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp}(\chi) \subset [-2, 2]$ and $\chi = 1$ on $[-1, 1]$.

Then there exists a decreasing sequence of numbers $\{\epsilon_j\}_{j \in \mathbb{N}}$, $\epsilon_j \geq 0$, s.t. $\epsilon_j \rightarrow 0$, such that $\forall j \in \mathbb{N}$, $\forall \alpha \in \mathbb{N}^d$ with $|\alpha| \leq j$

$$\sup_{x \in \mathbb{R}^d} \left| \left(1 - \chi\left(\frac{\epsilon_j}{h}\right) \right) \partial^\alpha a_j(x; h) \right| \leq h^{-1}$$

for h small enough.

proof (of Lemma 2.13)

let $C_j := \sup_{|\alpha| \leq j} \sup_{x \in \mathbb{R}^d} |\partial^\alpha a_j(x; h)|$. Then

$$h \sup_{x \in \mathbb{R}^d} \left| \left(1 - \chi\left(\frac{\epsilon_j}{h}\right) \right) \partial^\alpha a_j(x; h) \right| \leq \epsilon_j C_j \leq 1$$

where we used that $1 - \chi\left(\frac{\epsilon_j}{h}\right) \neq 0$ only for $h \leq \epsilon_j$, and that $\{\epsilon_j\}$ is

$$\text{s.t. } \epsilon_j \leq \frac{1}{C_j} \quad \blacksquare$$

We are now ready to prove the proposition: