

04.05.26

proof (of Proposition 2.12)

w.l.o.g. we can assume $g=1$. Indeed, $a \in \text{Sd}(g) \Leftrightarrow \frac{a}{g} \in \text{Sd}(1)$.

let's assume $a_j \in \text{Sd}(1) \quad \forall j \in \mathbb{N}$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp}(\chi) \in [-2, 2]$ and $\chi = 1$ on $[-1, 1]$.

let

$$a(x; h) := \sum_{j \geq 0} h^j (1 - \chi(\frac{\varepsilon_j}{h})) a_j(x; h).$$

Then a is a smooth function of $x \in \mathbb{R}^d$. Moreover $\forall \alpha \in \mathbb{N}^d$

$$|\partial^\alpha a(x; h)| \leq \sum_{j \leq |\alpha|} h^j |\partial^\alpha a_j(x; h)| + \sum_{j > |\alpha|} h^j \left| (1 - \chi(\frac{\varepsilon_j}{h})) \partial^\alpha a_j(x; h) \right|$$

By Lemma 2.13, $|\partial^\alpha a(x; h)| \leq C_\alpha + \sum_{j > |\alpha|} h^{j-1} \leq C'_\alpha$, and therefore $a \in \text{Sd}(1)$.

Moreover, $\forall \alpha \in \mathbb{N}^d$ and $N > |\alpha|$

$$\left| \partial^\alpha \left(a - \sum_{j=0}^N h^j a_j \right) \right| \leq \sum_{j=0}^N h^j \left| \chi(\frac{\varepsilon_j}{h}) \partial^\alpha a_j \right| + \sum_{j \geq N+1} h^j \left| (1 - \chi(\frac{\varepsilon_j}{h})) \partial^\alpha a_j \right|$$

Again by Lemma 2.13 Notice that

$$\begin{aligned} \sum_{j \leq N} h^j \left| \chi(\frac{\varepsilon_j}{h}) \partial^\alpha a_j \right| &\leq \sum_{j \leq N} h^j \frac{h^N}{\varepsilon_j^N} \left| \left(\frac{\varepsilon_j}{h} \right)^N \chi(\frac{\varepsilon_j}{h}) \right| C_{j, \alpha} \\ &= \sum_{j \leq N} h^{N+j} \varepsilon_j^{-N} \left| \left(\frac{\varepsilon_j}{h} \right)^N \chi(\frac{\varepsilon_j}{h}) \right| C_{j, \alpha} \end{aligned}$$

and by Lemma 2.13

$$\text{and } \sum_{j \geq N+1} h^j \left| (1 - \chi(\frac{\varepsilon_j}{h})) \partial^\alpha a_j \right| \leq \sum_{j \geq N+1} h^{j-1}.$$

Using the fact that $t \rightarrow \chi(t) t^N$ is a bounded function, we obtain

$$\left| \partial^\alpha \left(a - \sum_{j=0}^N h^j a_j \right) \right| \leq C_N h^N$$

where $C_N > 0$.

By the definition of asymptotic equivalence (see Definition 2.11)

we conclude that

$$a \sim \sum_{j=0}^{\infty} h^j a_j \quad \text{in } \text{Sd}(\frac{1}{h}).$$

EXAMPLE 2.14 (WKB solutions of 1D Schrödinger operator)

We consider the Schrödinger operator in $d=1$:

$$H = -\hbar^2 \frac{d^2}{dx^2} + V$$

where $V \in C^\infty(\mathbb{R}; \mathbb{R})$. Let $E \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ s.t. $V(x_0) < E$.

We aim at constructing wave functions Ψ s.t.

$$H\Psi = E\Psi,$$

that is Ψ is an eigenfunction of H with eigenvalue E .

To this end, we will construct approximate solutions of the eigenvalue problem.

To start with, observe that for $x \in U(x_0)$, where $U(x_0)$ is a neighborhood of x_0 , we can define

$$\varphi_\pm(x) := \pm \int_{x_0}^x \sqrt{E - V(y)} dy$$

such that $(\varphi'_\pm(x))^2 = E - V(x)$. This equation is usually referred to as the EIKONAL EQUATION.

Notice that, for $x \in U(x_0)$, $\varphi'_\pm(x) \neq 0$ and therefore $\sqrt{|\varphi'_\pm(x)|}$ is smooth for every $x \in U(x_0)$.

Now, for every $a = a(x)$ smooth in $U(x_0)$, we have

$$\begin{aligned} & \left(-\hbar^2 \frac{d^2}{dx^2} + V - E \right) (a e^{i\varphi_\pm/\hbar})(x) \\ &= \underbrace{\left[-2i\hbar a'(x) \varphi'_\pm(x) - i\hbar a \varphi''_\pm(x) - \hbar^2 a''(x) + (\varphi'_\pm(x))^2 + V(x) - E \right]}_{=0} a(x) \\ & \quad \cdot e^{i\varphi_\pm(x)/\hbar} \\ &= i\hbar \left[2a'(x) \varphi'_\pm(x) + a \varphi''_\pm(x) - i\hbar a''(x) \right] e^{i\varphi_\pm(x)/\hbar} \\ &= -2i\hbar \sqrt{|\varphi'_\pm(x)|} \left[(a(x) \sqrt{|\varphi'_\pm(x)|})' - i\hbar \frac{a''(x)}{2\sqrt{|\varphi'_\pm(x)|}} \right] e^{i\varphi_\pm(x)/\hbar} \end{aligned}$$

Let us now define $\{a_j^\pm(x)\}_{j \geq 0}$ by the recursive formula

$$(a_0^\pm(x) \sqrt{|\varphi'_\pm(x)|})' = 0; \quad a_j^\pm(x) \sqrt{|\varphi'_\pm(x)|} - i \frac{(a_{j-1}^\pm(x))'}{2\sqrt{|\varphi'_\pm(x)|}} = 0, \quad j \geq 1.$$

Let $a_{\pm}(x; \hbar)$ be a resummation of the formal series $\sum_{j \geq 0} \hbar^j a_j^{\pm}$.

By Proposition 2.12 such a resummation exists.

Let us define

$$u_{\pm}(x) := a_{\pm}(x) e^{i\varphi_{\pm}(x)/\hbar}.$$

We observe that $u_{\pm}$ satisfies

$$\left(-\hbar^2 \frac{d^2}{dx^2} + V(x) - E\right) u_{\pm}(x; \hbar) = \mathcal{O}(\hbar^{\infty})$$

uniformly in $x \in U(x_0)$ and for $\hbar > 0$ small, because

$$\sum_{j \geq 0} \hbar^j \left[\left(a_j^{\pm}(x) \sqrt{\varphi'_{\pm}(x)} \right)' - i \hbar \frac{(a_j^{\pm}(x))'}{2\sqrt{\varphi'_{\pm}(x)}} \right] \equiv 0.$$

Such approximate solutions are called WKB (Wentzel, Kramers, Brillouin) solutions. In particular, take

$$a_0^{\pm}(x) = \frac{\sqrt{\varphi'_{\pm}(x_0)}}{\sqrt{\varphi'_{\pm}(x)}} = \frac{(E - V(x_0))^{1/4}}{(E - V(x))^{1/4}}$$

and $\forall j \geq 1$

$$\begin{aligned} a_j^{\pm}(x) &= \frac{i}{\sqrt{\varphi'_{\pm}(x)}} \int_{x_0}^x \frac{(a_{j-1}^{\pm})'(y)}{\sqrt{\varphi'_{\pm}(y)}} dy \\ &= \frac{\pm i}{(E - V(x))^{1/4}} \int_{x_0}^x \frac{(a_{j-1}^{\pm})'(y)}{(E - V(y))^{1/4}} dy. \end{aligned}$$

Then the approximation $u_{\pm}$ satisfies $u_{\pm}(x_0; \hbar) = 1$.

2.2 Oscillatory Integrals

We now recall the basic properties of the Fourier transform.

Let $S(\mathbb{R}^d)$ be the space of Schwartz functions, i.e. functions which are smooth on $\mathbb{R}^d$ and rapidly decreasing at infinity, together with all their derivatives.

Let $u \in S(\mathbb{R}^d)$. We define the (semiclassical) Fourier transform of u as

$$(F_h u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi/h} u(x) dx$$

where $\xi \in \mathbb{R}^d$.

PROPOSITION 2.15

F_h is an isomorphism on $S(\mathbb{R}^d)$ with inverse

$$(F_h^{-1} w)(x) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi/h} w(\xi) d\xi$$

Moreover, F_h can be extended by duality to an isomorphism of $S'(\mathbb{R}^d)$, the space of Schwartz space of tempered distributions on $\mathbb{R}^d$.

Then F_h maps $L^2(\mathbb{R}^d)$ into itself and defines an isometry on $L^2(\mathbb{R}^d)$, i.e.

$$\forall u \in L^2(\mathbb{R}^d) \quad \|\hat{u}\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)}.$$

Furthermore,

$$F_h(h D_x u) = \xi F_h u$$

$$F_h(x u) = -h D_\xi F_h u.$$

proof. (exercise).

Let us now use the aforementioned properties to get some heuristics on the types of divergent integrals we aim at dealing with.