

for $u \in \mathcal{S}(\mathbb{R}^d)$, we have (by Proposition 2.15)

$$u = F_h^{-1} F_h u.$$

Then, in the sense of distributions,

$$\langle \delta_{y=x}, u \rangle = u(x) = \frac{1}{(2\pi h)^d} \int \left(\int e^{i(x-y) \cdot \xi / h} u(y) dy \right) d\xi$$

If we could apply Fubini's theorem, we would have

$$\delta_{y=x} = \frac{1}{(2\pi h)^d} \int e^{i(x-y) \cdot \xi / h} d\xi.$$

We need to give a meaning to the above formula as a well-defined object. To this end, let us consider $a = a(x, y, \xi) \in S_{3d}(\langle \xi \rangle^m)$, for $m \in \mathbb{R}$, as per Definition 2.4 where the order function is chosen to be $g(x, y, \xi) = \langle \xi \rangle^m$.

The goal is to give a meaning to the possibly divergent integral

$$I(a) := \int e^{i(x-y) \cdot \xi / h} a(x, y, \xi) d\xi.$$

We first observe that for $m < -d$ the integral is absolutely convergent.

We are left with $m \geq -d$. To give a meaning to $I(a)$ for $m \geq -d$, we need some notions about distribution kernels of operators.

Theorem 2.16 (Schwartz kernel theorem)

Let $A : C_0^\infty(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ be a linear and continuous from the space of C^∞ and compactly supported functions on $\mathbb{R}^d$ to the space of tempered distributions on $\mathbb{R}^d$.

Then there exists a unique kernel $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that for every $u, w \in C_0^\infty(\mathbb{R}^d)$

$$\langle Au, w \rangle_{\mathcal{D}', \mathcal{D}} = \langle K, w \otimes u \rangle_{\mathcal{D}', \mathcal{D}}$$

with $\mathcal{D} = C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

K is called the distribution kernel (or the Schwartz kernel) of A .

To clarify the notations :

$\mathcal{D}'(\mathbb{R}^d)$ = space of distributions on $\mathbb{R}^d$.

$w \otimes u \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ is the tensor product between w and u .

$\langle \cdot, \cdot \rangle_{\mathcal{D}', \mathcal{D}}$ is the duality bracket between $\mathcal{D}'$ and $\mathcal{D} = C_0^\infty$.

For $u \in \mathcal{D} = C_0^\infty(\mathbb{R}^d)$ and $a \in S_{3d}(\langle \xi \rangle^m)$ with $m < -d$, let

$$(A_a u)(x, h) := \int e^{i(x-y) \cdot \xi / h} a(x, y, \xi) u(y) dy d\xi$$

Observe that

$$(1 - h\xi \cdot D_y)(e^{i(x-y) \cdot \xi / h}) = (1 + \xi^2) e^{i(x-y) \cdot \xi / h}$$

Let

$$L := \frac{1}{1 + \xi^2} (1 - h\xi \cdot D_y) = L(\xi, h D_y)$$

$$\text{then } L(e^{i(x-y) \cdot \xi / h}) = e^{i(x-y) \cdot \xi / h}$$

$$\text{Moreover } \forall k \in \mathbb{N} \setminus \{0\}, \quad L^k(e^{i(x-y) \cdot \xi / h}) = e^{i(x-y) \cdot \xi / h}$$

Thus, by k integrations by parts in $(A_a u)(x, h)$, we get

$$(A_a u)(x, h) = \int e^{i(x-y) \cdot \xi / h} (L(\xi, h D_y)^k)(a u) dy d\xi =: I_k u(x)$$

where

$$(L^k)^k(a u) = \left(\frac{1 + h\xi D_y}{1 + \xi^2} \right)^k(a u) = \mathcal{O}(\langle \xi \rangle^{m-k})$$

uniformly as $|\xi| \rightarrow \infty$. Therefore $I_k u(x)$ is absolutely convergent for $m - k < -d$, i.e. $m < k - d$.

We now notice that $I_{k+l} u = I_k u \quad \forall l \geq 0$. Therefore the following definition makes sense:

Definition 2.17

For all $m \in \mathbb{R}$, $a \in S_{3d}(\langle \xi \rangle^m)$ and $u \in \mathcal{D}_0^\infty(\mathbb{R}^d)$

$$(A_a u)(x) := \int e^{i(x-y) \cdot \xi / h} (L^t(\xi, h D_y))^k(a u) dy d\xi$$

$\forall k \geq m + d$.

Theorem 2.18

A_a defines a continuous linear operator from $C_0^\infty(\mathbb{R}^d)$ to $C^\infty(\mathbb{R}^d)$.

proof.

$\forall \ell \in \mathbb{N} \setminus \{0\}$, let $k := [m] + d + \ell + 1$.

For $u \in C_0^\infty(\mathbb{R}^d)$, $(A_a u)(x) = I_k u(x)$ and by Lebesgue's dominated convergence theorem we know that this is a C^ℓ function on $\mathbb{R}^d$.

Thus, $A_a u \in C^\infty(\mathbb{R}^d)$ and it is linear in u .

Let $K \subset \mathbb{R}^d$ be a compact set of $\mathbb{R}^d$ with $\text{supp}(u) \subset K$.

Since $(L^t)^k$ is a differential operator of order k with coefficients $O(\langle \xi \rangle^{-k})$ and since $\partial_x^\alpha e^{i(x-y) \cdot \xi / h} = O(\langle \xi \rangle^{|\alpha|}) \quad \forall \alpha \in \mathbb{N}^d$,

we conclude that $\forall K'$ compact set of $\mathbb{R}^d$

$$\begin{aligned} \sup_{x \in K'} |\partial^\alpha A_a u(x)| &\leq \sup_{x \in K'} |\partial^\alpha I_{[m]+d+|\alpha|+1} u(x)| \\ &\leq C_{k,k',\alpha} \sum_{|\beta| \leq [m]+d+|\alpha|+1} \sup_{y \in K} |\partial^\beta u(y)| \end{aligned}$$

where $C_{k,k',\alpha}$ constant depending only on the choice of K, K', α $\square$

Definition 2.19

We define the distribution kernel of A_a by

$$I(a) := \int e^{i(x-y) \cdot \xi / h} a(x, y, \xi) d\xi \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$$

Remark 2.20

Notice that $a \in S_{3d}(\langle \xi \rangle^m) \mapsto I(a) \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$

is a continuous map and $I(a)$ defined as in Def. 2.19 coincides with the absolutely convergent integral for $m \leq -d$.

(exercise).

Example 2.21

Let $a=1$, then for any $k \geq d+1$

$$\begin{aligned} \tilde{A}_a u(x) &= \iint e^{i(x-y) \cdot \xi / h} (L^t(\xi, h D_y))^k u(y) dy d\xi \\ &= \int e^{ix \cdot \xi / h} \langle (L^t(\xi, h D_y))^k u, e^{-iy \cdot \xi / h} \rangle_{\mathcal{D}', \mathcal{D}} d\xi \\ &= \int e^{ix \cdot \xi / h} \langle u, L(\xi, h D_y)^k e^{-iy \cdot \xi / h} \rangle_{\mathcal{D}', \mathcal{D}} d\xi \\ &= \int e^{ix \cdot \xi / h} \langle u, e^{-iy \cdot \xi / h} \rangle_{\mathcal{D}', \mathcal{D}} d\xi \\ &= (2\pi h)^d F_h^{-1} F_h u(x) = (2\pi h)^d u(x) \end{aligned}$$

$$\Rightarrow \int e^{i(x-y) \cdot \xi / h} d\xi = (2\pi h)^d \delta_{y=x} \quad \text{in the sense of oscillatory integrals.}$$

The definition of oscillatory integral can be generalized by means of the so-called phase functions:

Definition 2.22

A function $\varphi = \varphi(x, \theta) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d'})$ is a phase function if

- $\text{Im } \varphi \geq 0$
- $\partial^\alpha \varphi = O(1)$ on $\mathbb{R}^{d+d'}$ when $|\alpha| \geq 2$
- $\exists C > 0$ and $p > 0$ s.t. $|\nabla_{x, \theta} \varphi| \geq \frac{C \langle \theta \rangle^p}{C}$ for $|\theta|$ large enough.

Definition 2.23

Given a phase function $\varphi = \varphi(x, \theta)$ and a symbol $a = a(x, \theta) \in S_{d+d'}(\langle \theta \rangle^m)$ we define the oscillatory integral $I_\varphi(a) = \int e^{i\varphi(x, \theta)/h} a(x, \theta) d\theta \in \mathcal{D}'(\mathbb{R}^d)$ by the formula

$$\langle I_\varphi(a), u \rangle = \int e^{i\varphi(x, \theta)/h} (L^t(x, \theta, h D_x, h D_\theta))^k [a(x, \theta) u(x)] dx d\theta$$

with $k \geq \frac{m+d'}{p}$ and

$$L = \frac{1}{1 + |\nabla \varphi|^2} \left(1 + h \frac{\partial \bar{\varphi}}{\partial x} D_x + h \frac{\partial \bar{\varphi}}{\partial \theta} D_\theta \right)$$

such that

$$L(e^{i\varphi/h}) = e^{i\varphi/h}$$

$$(L^t)^k (au) = O \left(\frac{(1 + |\nabla \varphi|)^k \langle \theta \rangle^m}{(1 + |\nabla \varphi|^2)^k} \right) = O(\langle \theta \rangle^{m-kp})$$