

Usage of Schwartz kernel

The Schwartz kernel theorem gives us a distributional kernel $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ for every $A: \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ bounded operator. But this theorem is often used for other types of operators.

Example: Let $A: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be a bounded linear operator. People will now talk about its Schwartz kernel even though A does not satisfy the conditions of the theorem.

Solution: Consider

$$A' := A|_Y: Y(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \hookrightarrow Y'(\mathbb{R}^d)$$

Now A' has a Schwartz kernel $K_{A'}$ and A is the unique extension of A' and we write

$$\begin{aligned} \langle Au, v \rangle &= \lim_{n \rightarrow \infty} \langle Au_n, v \rangle = \lim_{n \rightarrow \infty} \langle K_{A'}, u_n \boxtimes v \rangle \\ &=: \langle K_A, u \boxtimes v \rangle \end{aligned}$$

which is well-defined as $\langle Au, v \rangle$ is so.

2.3 Pseudodifferential Operators

Definition 2.24

Let $a \in S_{3d}(\langle \xi \rangle^m)$, $u \in C_0^\infty(\mathbb{R}^d)$. Let

$$Op_h(a) u(x; h) = \frac{1}{(2\pi h)^d} \int e^{i(x-y) \cdot \xi / h} a(x, y, \xi) u(y) dy d\xi$$

and $\forall \mu \in \mathbb{R}$ we call $h^{-\mu} Op_h(a)$ the semiclassical pseudodifferential operator of symbol $h^{-\mu} a$ of degree m and order μ .

Remark 2.25

It follows from Definition 2.25 that

$$h^{-\mu} Op_h(a): C_0^\infty(\mathbb{R}^d) \rightarrow C_0^\infty(\mathbb{R}^d)$$

(Exercise)

Example 2.26

consider the case in which a has the particular form

$$a(x, y, \xi) = \sum_{|\alpha| \leq m} b_\alpha(x) \xi^\alpha$$

with $b_\alpha \in S_d(1)$. Then

$$Op_h(a) = Op_h\left(\sum_{|\alpha| \leq m} b_\alpha(x) \xi^\alpha\right) = \sum_{|\alpha| \leq m} b_\alpha(x) Op_h(\xi^\alpha)$$

(linearity)

and $Op_h(\xi^\alpha) = (hD_x)^\alpha$, hence

$$Op_h(a) = \sum_{|\alpha| \leq m} b_\alpha(x) (hD_x)^\alpha$$

If instead we consider

$$\tilde{a}(x, y, \xi) = \sum_{|\alpha| \leq m} \tilde{b}_\alpha(y) \xi^\alpha$$

then

$$Op_h(\tilde{a}) = \sum_{|\alpha| \leq m} (hD_x)^\alpha \tilde{b}_\alpha(x).$$

More generally, for $a(x, y, \xi) = \sum_{|\alpha| \leq m} b_\alpha(x, y) \xi^\alpha$, $b_\alpha \in S_{2d}(1)$, and $u \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} \mathcal{O}_{P_h} \left(\sum_{|\alpha| \leq m} b_\alpha(x, y) \xi^\alpha \right) u(x) \\ = \sum_{|\alpha| \leq m} \left(h D_x \right)^\alpha \left(b(x', x) u(x) \right) \Big|_{x'=x} \end{aligned}$$

Theorem 2.27

Let $a \in S_{3d}(\langle \xi \rangle^m)$. Then $\mathcal{O}_{P_h}(a)$ has a unique extension to a linear continuous operator from $S(\mathbb{R}^d)$ to $S(\mathbb{R}^d)$.

proof.

If $k > m + d$, $\mathcal{O}_{P_h}(a) I_k u$ is well defined for $u \in S(\mathbb{R}^d)$ and $I_k u \in C^\infty(\mathbb{R}^d)$.

By Leibniz formula

$$x^\beta \partial_x^\alpha I_k u(x) = \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} C_{\alpha/\beta'} \int e^{i(x-y) \cdot \xi / h} (x-y)^{\beta'} y^{\beta-\beta'} \xi^{\alpha'} (L^t)^k (\partial_x^{\alpha-\alpha'} u) dy d\xi$$

where $C_{\alpha/\beta'}$ is a combinatorial const.

By using that $e^{i(x-y) \cdot \xi / h} (x-y)^{\beta'} = (h D_\xi)^{\beta'} (e^{i(x-y) \cdot \xi / h})$

and by integrating by parts β' times wrt ξ :

$$x^\beta \partial_x^\alpha I_k u(x) = \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \mathcal{O} \left(\int \langle y \rangle^{|\beta-\beta'|} \langle \xi \rangle^{|\alpha'| - k + m} \sum_{|\gamma| \leq k} |\partial^\gamma u(y)| dy d\xi \right)$$

If we chose $k > m + |\alpha| + d$

$$(*) \quad x^\beta \partial_x^\alpha I_k u(x) = \mathcal{O} \left(\sup_{x \in \mathbb{R}^d} \sum_{|\gamma| \leq k} \langle y \rangle^{|\beta| + d + 1} |\partial^\gamma u(y)| \right)$$

as we where we used

$$\langle y \rangle^{|\beta-\beta'|} = \langle y \rangle^{d-1} \langle y \rangle^{|\beta-\beta'| + d + 1}$$

Thus $(*) \Rightarrow \mathcal{O}_{P_h}(a)u \in S(\mathbb{R}^d)$

and $|x^\beta \partial_x^\alpha \mathcal{O}_{P_h}(a)u(x)|$ can be bounded by a finite number of seminorms of $u \in S(\mathbb{R}^d)$.

To prove the uniqueness of the extension, we recall that $C_0^\infty(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$ (to be proven) $\square$

Prop. 2.28 $C_0^\infty(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$

By duality,

Thm 2.29

For all $a \in S_{3d}(\langle \xi \rangle^m)$, $\mathcal{O}_{ph}(a)$ has a unique extension to a linear and continuous operator from $S'(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$.

proof.

We let $A := \mathcal{O}_{ph}(a)$.

$\forall v \in S(\mathbb{R}^d)$ we define

$${}^tA v(y) = \frac{1}{(2\pi\hbar)^d} \int e^{i(x-y) \cdot \xi / \hbar} a(x, y, \xi) v(x) dx d\xi$$

Thm. 2.27 implies that ${}^tA v$ is in $S(\mathbb{R}^d)$ (observe that it follows from the fact that $a \in S_{3d}(\langle \xi \rangle^m)$).

If we denote by $\langle \cdot, \cdot \rangle_{S', S}$ the duality brackets between $S'(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, we can write $\forall u, v \in S(\mathbb{R}^d)$

$$\langle v, {}^tA u \rangle_{S', S} = \langle Au, v \rangle_{S', S}$$

and, if $u \in S'(\mathbb{R}^d)$ and $v \in S(\mathbb{R}^d)$, let

$$\langle Au, v \rangle_{S', S} := \langle u, {}^tA v \rangle_{S', S}$$

Since tA is continuous on $S(\mathbb{R}^d)$, we conclude that $Au \in S'(\mathbb{R}^d)$. Moreover if $\{u_n\}_{n \in \mathbb{N}} \subset S'(\mathbb{R}^d)$ s.t. u_n converges to $u \in S'(\mathbb{R}^d)$, then

$$Au_n \rightarrow Au.$$

The density of $C_0^\infty(\mathbb{R}^d)$ in $S(\mathbb{R}^d)$ then implies that there is a unique continuous extension of A on $S'(\mathbb{R}^d)$ □

Def. 2.30

For $a \in S_{3d}(\langle \xi \rangle^m)$, we define

$$a^*(x, y, \xi) = \overline{a(y, x, \xi)} \in S_{3d}(\langle \xi \rangle^m).$$

Then $(Op_h(a))^* := Op_h(a^*)$ and $\forall u, v \in S(\mathbb{R}^d)$ satisfies

$$\langle (Op_h(a))^* u, v \rangle_{L^2} = \langle u, Op_h(a) v \rangle_{L^2}$$

feng

$$(Op_h(a))^* = Op_h(a^*)$$

$$v \in S(\mathbb{R}^d) \text{ we define}$$

$$A^* v(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \overline{a(y, x, \xi)} \hat{v}(\xi) d\xi$$

Then A^* is adjoint of A in the sense that $\langle A^* v, u \rangle = \langle v, Au \rangle$ for all $u, v \in S(\mathbb{R}^d)$.

follows from the fact that $\langle A^* v, u \rangle = \langle v, Au \rangle$ for all $u, v \in S(\mathbb{R}^d)$.

It is clear that A^* is the adjoint of A in the sense that $\langle A^* v, u \rangle = \langle v, Au \rangle$ for all $u, v \in S(\mathbb{R}^d)$.

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Since A^* is continuous on $S(\mathbb{R}^d)$, we conclude that $A^* \in \mathcal{L}(S(\mathbb{R}^d))$.

Moreover, if $A \in \mathcal{L}(S(\mathbb{R}^d))$, then $A^* \in \mathcal{L}(S(\mathbb{R}^d))$ and $(A^*)^* = A$.

then

$$A^* \in \mathcal{L}(S(\mathbb{R}^d))$$

The density of $S(\mathbb{R}^d)$ in $S'(\mathbb{R}^d)$ is $(S(\mathbb{R}^d))^*$.

there is a unique extension of A to $S'(\mathbb{R}^d)$.

□

continuous