

Def. 2.30

For $a \in S_{3d}(\langle \xi \rangle^m)$, we define

$$a^*(x, y, \xi) = \overline{a(y, x, \xi)} \in S_{3d}(\langle \xi \rangle^m)$$

Then $(Op_h(a))^* := Op_h(a^*)$ is called the formal adjoint of $Op_h(a)$, and $\forall u, v \in S(\mathbb{R}^d)$ satisfies

$$\langle (Op_h(a))^* u, v \rangle_{L^2} = \langle u, Op_h(a) v \rangle_{L^2}$$

2.4 Composition of Pseudodifferential Operator

We are now in the position of defining the composition of two pseudo-differential operators: Theorem 2.29 guarantees that, under conditions on a , $Op_h(a)$ is a linear continuous operator from $S'(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$.

But what about the composition of $Op_h(a)$ and $Op_h(b)$? Is it a linear and continuous operator from $S'(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$, hence a pseudodifferential operator?

To answer this question, let us consider

$$a \in S_{3d}(\langle \xi \rangle^d), \quad b \in S_{3d}(\langle \xi \rangle^{d'}), \quad A = Op_h(a), \quad B = Op_h(b)$$

Then, at least formally, we get

$$(A \circ B)u(x) = \frac{1}{(2\pi h)^d} \int e^{i(x-y) \cdot \xi/h} a(x, y, \xi) B u(y) dy d\xi$$

$$= \frac{1}{(2\pi h)^d} \int e^{i(x-z) \cdot \eta/h} c(x, z, \eta) u(z) dz d\eta$$

where $c(x, z, \eta)$ depends on h and is given by

$$c(x, z, \eta) := \frac{1}{(2\pi h)^d} \int e^{i(x-y) \cdot (\xi - \eta)/h} a(x, y, \xi) b(y, z, \eta) dy d\xi$$

To answer the question, it is enough to show that

$$c \in S_{2d}(\langle \xi \rangle^m)$$

for some m .

□

To show that, we will make use of the following Lemma

Lemma 2.31 (Stationary Phase)

Let M be a $d \times d$ nondegenerate real symmetric matrix.

For all $u \in C_0^\infty(\mathbb{R}^d)$ and for all $n \geq 1$, it holds

$$\int e^{ix \cdot Mx/2h} u(x) dx = \sum_{j=0}^{n-1} \frac{(2\pi)^{d/2} h^{j+d/2} e^{i\frac{\pi}{4} \text{sgn}(M)}}{(2i)^j |\det(M)|^{1/2} j!} ((D_x M^{-1} D_x)^j u)(0) + R_n(u, h)$$

where

$$|R_n(u, h)| \leq \frac{C h^{n+\frac{d}{2}}}{2^n n! |\det M|^{1/2}} \sum_{|\alpha| \leq d+1} \|\partial_x^\alpha (D_x M^{-1} D_x)^n u\|_{L^1(\mathbb{R}^d)}$$

with $C = C(d)$, $\text{sgn}(M) = \text{signature of } M = \# \text{ of positive e.v.} - \# \text{ of neg. e.v.}$

Lemma 2.32

Let $F^{-1}(u) = F_h^{-1}|_{h=1}(u)$ be the inverse Fourier transform with $h=1$.

Then

$$F^{-1}(e^{ix \cdot Mx/2h}) = h^{d/2} e^{i\pi/4 \text{sgn}(M)} |\det(M)|^{-1/2} e^{-ih(K \cdot M^{-1} K)/2}$$

proof.

Let $z \in \mathbb{C}$ such that $\text{Re}(z) > 0$. Then

$$F^{-1}(e^{-zx^2/2}) = F(e^{-\bar{z}x^2/2}) = \frac{1}{\sqrt{z}} e^{-k^2/2\bar{z}}$$

Let us choose $z = \varepsilon \mp ib$, $b > 0$. Then

$$\lim_{\varepsilon \rightarrow 0} F^{-1}(e^{-zx^2/2}) = F^{-1}(e^{\pm ibx^2/2}) = \frac{1}{\sqrt{b}} e^{\pm i\pi/4} e^{\pm k^2/2ib}$$

The assumptions of M (see lemma 2.31) then guarantee that we can diagonalize M and obtain the statement. $\square$

proof of Lemma 2.31

By def of $\mathcal{F}$ we have

$$\int e^{ix \cdot Mx/2h} u(x) dx$$

$$= \langle e^{ix \cdot Mx/2h}, u \rangle_{S', S}$$

$$\xrightarrow{\text{(Parseval)}} = \langle \mathcal{F}^{-1}(e^{ix \cdot Mx/2h}), \mathcal{F}u \rangle_{S', S}$$

$$\xrightarrow{\text{(Lemma 2.32)}} = h^{d/2} e^{i\pi/4 \operatorname{sgn}(M)} |\det(M)|^{-1/2} \int e^{-ih \kappa \cdot M \kappa / 2} \hat{u}(\kappa) d\kappa$$

Observe that $|\partial_t^N e^{it}| = 1 \quad \forall N \in \mathbb{N}$, Taylor's expansion gives

$$\left| e^{it} - \sum_{j=0}^{n-1} \frac{(it)^j}{j!} \right| \leq \frac{|t|^n}{n!} \quad \forall t \in \mathbb{R}$$

Hence,

$$e^{-ih \kappa \cdot M^{-1} \kappa / 2} = \sum_{j=0}^{n-1} \frac{1}{j!} \left(\frac{h}{2i} (\kappa \cdot M^{-1} \kappa) \right)^j + \tilde{R}_n(\kappa, h)$$

$$\text{with } |\tilde{R}_n(\kappa, h)| \leq \frac{h^n}{2^n n!} |\kappa \cdot M^{-1} \kappa|^n$$

$$\text{Moreover, } \int (\kappa \cdot M^{-1} \kappa)^j \hat{u}(\kappa) d\kappa = (2\pi)^{d/2} ((D_x \cdot M^{-1} D_x)^j u)(0)$$

and therefore

$$|\tilde{R}_n| \leq \frac{h^{n+d/2}}{2^n n!} |\det(M)|^{-1/2} \int |(\kappa \cdot M^{-1} \kappa)^n \hat{u}(\kappa)| d\kappa$$

$$\text{Using } \|\mathcal{F}u\|_{L^1(\mathbb{R}^d)} \stackrel{(*)}{\leq} \sum_{|\alpha| \leq d+1} \|\partial^\alpha u\|_{L^1(\mathbb{R}^d)} \quad \text{the proof is}$$

achieved. $\square$

Exercise.

Show that inequality $(*)$ above holds true.

We now consider a particular case of Lemma 2.31, that we state as Corollary.

Corollary 2.33

For all $u \in C^\infty(\mathbb{R}^{2d})$ and $n \geq 1$, it holds

$$\frac{1}{(2\pi\hbar)^d} \int e^{-ixy/\hbar} u(x,y) dx dy = \sum_{j=0}^{n-1} \frac{\hbar^j}{i^j j!} \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} \partial_x^\alpha \partial_y^\alpha \right)^{\frac{1}{2}} u(0,0) + R_n$$

where

$$|R_n| \leq \frac{C \hbar^n}{n!} \sum_{|\alpha|+|\beta| \leq 2d+1} \|\partial_x^\alpha \partial_y^\beta (\partial_x \partial_y)^n u\|_{L^1(\mathbb{R}^{2d})}.$$

proof.

let $d \leftrightarrow 2d$, $M \leftrightarrow \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$ that implies $M = M^{-1}$, and satisfies $\text{sgn}(M) = 0$ and $|\det(M)| = 1$.

Then, applying lemma 2.31 gives the result. $\square$

Exercise.

Show that Corollary 2.33 is equivalent to

$$\text{LHS} = \sum_{|\alpha| \leq n-1} \frac{\hbar^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_x^\alpha \partial_y^\alpha u(0,0) + R_n$$

Theorem 2.34 (composition theorem)

$\forall a \in S(\langle \xi \rangle^{m+m'})$, $b \in S(\langle \xi \rangle^{m'})$, there exists $c \in S(\langle \xi \rangle^{m+m'})$

such that

$$O_{ph}(a) \circ O_{ph}(b) = O_{ph}(c).$$

A possible choice of c is given by the oscillatory integral

$$c(x,y,\xi) = \frac{1}{(2\pi\hbar)^d} \int e^{i(x-z)(\eta-\xi)/\hbar} a(x,z,\eta) b(z,y,\xi) dz d\eta$$

and we define it ~~with~~ ^{by} the notation $a \overset{\sim}{\#} b(x,y,\xi)$.

proof of Theorem 2.34. (exercise)

let $u \in C_0^\infty(\mathbb{R}^d)$ and consider $Op_h(b)$.

Step 1) show that, in the topology of $S(\mathbb{R}^d)$,

$$Op_h(b)u(z) = \lim_{\substack{\varepsilon \rightarrow 0^+ \\ \delta \rightarrow 0^+}} \frac{1}{(2\pi h)^d} \int e^{i(z-y)\xi/h - \varepsilon \langle \xi \rangle - \delta \langle z \rangle} b(z, y, \xi) u(y) dy d\xi$$

(hint: treat the RHS as an oscillatory integral using the operator L^* and integrating by parts)

Step 2) From Step 1, deduce that ^{the} continuity of $Op_h(a)$ on $S(\mathbb{R}^d)$ guarantees

$$\begin{aligned} & (2\pi h)^{2d} Op_h(a) \circ Op_h(b)u(x) \\ &= \lim_{\substack{\varepsilon \rightarrow 0^+ \\ \delta \rightarrow 0^+}} \int e^{i(x-z)\eta/h} a(x, z, \eta) \left(\int e^{i(z-y)\xi/h - \varepsilon \langle \xi \rangle - \delta \langle z \rangle} b(z, y, \xi) u(y) dy d\xi d\eta \right) \end{aligned}$$

Step 3) From Step 2) deduce that

$$Op_h(a) \circ Op_h(b)u(x) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \frac{1}{(2\pi h)^d} \int e^{i(x-y)\xi/h - \varepsilon \langle \xi \rangle} c_\delta(x, y, \xi) u(y) dy d\xi$$

with

$$c_\delta(x, y, \xi) := \frac{1}{(2\pi h)^d} \int e^{i(x-z)(\eta-\xi)/h - \delta \langle z \rangle - \delta \langle \eta \rangle} a(x, z, \eta) b(z, y, \xi) dz d\eta$$

Step 4) Show that $\lim_{\delta \rightarrow 0^+} c_\delta(x, y, \xi)$ exists and that

we can pass to the $\lim_{\delta \rightarrow 0^+}$ under the integral.

Step 5) Show that we can take $\lim_{\varepsilon \rightarrow 0^+}$ leading to an oscillatory integral.

(hint: show that $c_0 := \lim_{\delta \rightarrow 0^+} c_\delta \in S_{3d}(\langle \xi \rangle^{m+m'})$)

~~then divide into the following regions~~