

Proposition 2.35

Let $a \in S_{3d}(\langle \xi \rangle^m)$ be an elliptic symbol (see Definition 2.9).

Then there exists $b \in S_{3d}(\langle \xi \rangle^{-m})$ such that

$$\mathcal{O}_h(a) \circ \mathcal{O}_h(b) = 1 + \mathcal{O}_h(r)$$

$$\mathcal{O}_h(b) \circ \mathcal{O}_h(a) = 1 + \mathcal{O}_h(r')$$

where $r, r' = \mathcal{O}(h^\infty)$ in $S_{3d}(1)$.

proof.

$a \in S_{3d}(\langle \xi \rangle^m)$ elliptic $\Rightarrow \frac{1}{a} \in S_{3d}(\langle \xi \rangle^{-m})$.

Let $b_0 := \frac{1}{a}$ and consider the expression for c given by $a \# b_0$

as in Theorem 2.34:

$$\sum_{\alpha \geq 0} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_x^\alpha \partial_\eta^\alpha (a(x, z, \eta) b_0(z, y, \xi)) \Big|_{\substack{z=x \\ \eta=\xi}} \text{ in } S_{3d}(\langle \xi \rangle^{m+m'})$$

where $m' = -m$.

Hence, we can define $b_j \in S_{3d}(\langle \xi \rangle^{-m})$ such that if

$$b_j \sim \sum_{j \geq 0} h^j b_j$$

then

$$a \# \tilde{b} = 1 + \mathcal{O}(h^\infty) \text{ in } S_{3d}(1).$$

Analogously, there exists $b' \in S_{3d}(\langle \xi \rangle^{-m})$ such that

$$b' \# a = 1 + \mathcal{O}(h^\infty) \text{ in } S_{3d}(1).$$

The composition theorem 2.34 then implies

$$\mathcal{O}_h(a) \circ \mathcal{O}_h(b) = 1 + \mathcal{O}_h(r)$$

$$\mathcal{O}_h(b') \circ \mathcal{O}_h(a) = 1 + \mathcal{O}_h(r')$$

with $r, r' = \mathcal{O}(h^\infty)$ in $S_{3d}(1)$.

Observe moreover that $\mathcal{O}_h(b) = \mathcal{O}_h(b') + \mathcal{O}_h(r_1)$ with $r_1 = \mathcal{O}(h^\infty)$

in $S_{3d}(1)$. Indeed, by the associative property of composition we have

$$\mathcal{O}_h(b') \circ \mathcal{O}_h(a) \circ \mathcal{O}_h(b) = (1 + \mathcal{O}_h(r')) \circ \mathcal{O}_h(b)$$

$$= \mathcal{O}_h(b') (1 + \mathcal{O}_h(r))$$

thus

$$\begin{aligned} \text{Op}_h(b) &= \text{Op}_h(b') + \text{Op}_h(b) \text{Op}_h(r) - \text{Op}_h(r') \text{Op}_h(b) \\ &= \text{Op}_h(b') + \text{Op}_h(r_1) \end{aligned}$$

with $r_1 = \mathcal{O}(h^\infty)$ in $S_{3d}(1)$ $\square$

2.5 Pseudodifferential

2.5 Symbolic Calculus

The goal of this section is to associate a symbol of $2d$ variables to a pseudodifferential operator on $\mathbb{R}^d$ and viceversa, i.e. to establish a one-to-one correspondence.

Observe however, that the quantization of a symbol is not unique.

Indeed, consider the symbol $x_j \xi_j = \xi_j x_j$ for $j=1, \dots, d$. We have two possible quantization of the same symbol, namely $x_j h D_{x_j}$ and $h D_{x_j} x_j$. Because of the non-commutativity of $x_j h D_{x_j}$ and $h D_{x_j} x_j$, $x_j h D_{x_j} \neq h D_{x_j} x_j$. More precisely, $h D_{x_j} x_j = x_j h D_{x_j} - i$. This explains why there are different ways of quantizing a symbol.

Definition 2.36

Let $a \in S_{3d}(\langle \xi \rangle^m)$. We define the quantization of parameter $t \in [0, 1]$ of the symbol a as

$$\text{Op}_h^t(a) := \text{Op}_h(a((1-t)x + ty, \xi)).$$

Remark 2.37

Notice that for $t \in [0, 1]$, $a((1-t)x + ty, \xi) \in S_{3d}(\langle \xi \rangle^m)$ if $a \in S_{3d}(\langle \xi \rangle^m)$.

Definition 2.38

For $t=0$, we call $\text{Op}_h^0(a) = \text{Op}_h(a)$ standard quantization or

Left quantization.

For $t = \frac{1}{2}$, we call $O_{\hbar}^{1/2}(a) = O_{\hbar}^W(a)$ Weyl quantization.

For $t = 1$, we call $O_{\hbar}^1(a)$ right quantization.

Theorem 2.39

Fix $t \in [0, 1]$. Let $b \in S_{2d}(\langle \xi \rangle^m)$. There exists a unique

$b_t(x, \xi) \in S_{2d}(\langle \xi \rangle^m)$ such that $O_{\hbar}(b) = O_{\hbar}^t(b_t)$.

Moreover, b_t is given by the oscillatory integral

$$b_t(x, \xi) = \frac{1}{(2\pi\hbar)^d} \int e^{i(\xi - \xi') \cdot \theta / \hbar} b(x + \theta t, x - (1-t)\theta, \xi') d\xi' d\theta$$

and satisfies

$$b_t(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|} \hbar^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_{\xi}^{\alpha} \partial_{\theta}^{\alpha} b(x + t\theta, x - (1-t)\theta, \xi) \Big|_{\theta=0}$$

in $S_{2d}(\langle \xi \rangle^m)$.

Remark 2.40

b_t is the symbol of index t of $B := O_{\hbar}(b)$ and it is denoted by $b_t = \sigma_t(B)$.

proof of theorem 2.39

We look for $b_t \in S_{2d}(\langle \xi \rangle^m)$ such that

$$\int e^{i(x-y) \cdot \xi / \hbar} b(x, y, \xi) d\xi = \int e^{i(x-y) \cdot \xi / \hbar} b_t((1-t)x + ty, \xi) d\xi$$

where the identity holds in $\mathcal{D}'(\mathbb{R}_x^d \times \mathbb{R}_y^d)$.

The change of variable

$$\begin{cases} \theta \rightarrow x - y \\ z \rightarrow (1-t)x + ty = x - t\theta \end{cases}$$

leads to the following rewriting of the left hand side:

$$\int e^{i\theta\xi/h} b(z+t\theta, z-(1-t)\theta, \xi) d\xi = \underbrace{\int e^{i\theta\xi/h} b_t(z, \xi) d\xi}_{\text{in the sense of oscillatory integrals}}$$

The right hand side is the inverse

$\hbar$ -Fourier transform of $\xi \rightarrow b_t(z, \xi)$:

$$b_t(z, \zeta) = \frac{1}{(2\pi\hbar)^d} \int e^{i(\xi-\zeta)\theta/h} b(z+t\theta, z-(1-t)\theta, \xi) d\xi d\theta \quad (*)$$

that is the $\hbar$ -Fourier transform in θ .

With these manipulations the identity holds in $S'(\mathbb{R}^{2d})$.

Notice that if $b_t \in S_{2d}(\langle \xi \rangle^m)$, then the stationary phase theorem 2.31 gives

$$b_t(z, \zeta) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|} \hbar^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_{\xi}^{\alpha} \partial_{\theta}^{\alpha} (b(z+t\theta, z-(1-t)\theta, \xi)) \Big|_{\substack{\theta=0 \\ \xi=\zeta}}$$

Moreover, by uniqueness of (*), b_t is unique.

We are left with the proof that $b_t \in S_{2d}(\langle \zeta \rangle^m)$:

Let $a_t(z, \theta, \xi) := b(z+t\theta, z-(1-t)\theta, \xi)$.

Since the change of variable is linear ~~into~~ and $t \in [0, 1]$, we conclude that $a_t \in S_{2d}(\langle \xi \rangle^m)$ uniformly in t .

We want to show that

$$b_t(z, \zeta) = \frac{1}{(2\pi\hbar)^d} \int e^{i(\xi-\zeta)\cdot\theta/h} a_t(z, \theta, \xi) d\theta d\xi \in S_{2d}(\langle \zeta \rangle^m)$$

where the r.h.s. has to be understood in the sense of oscillatory integrals. To this end, we "regularize" ~~the~~ b_t in the following way: let $\chi \in C_c^{\infty}(\mathbb{R}^{2d})$, $\chi(0) = 1$ and define

$$b_t^{\varepsilon}(z, \zeta) := \frac{1}{(2\pi\hbar)^d} \int e^{i(\xi-\zeta)\cdot\theta/h} \chi(\varepsilon\theta) \chi(\varepsilon(\xi-\zeta)) a_t(z, \theta, \xi) d\theta d\xi.$$

For any $\varepsilon > 0$ the integral on the r.h.s. is convergent.

We need estimates uniform in ε to conclude.

Let then $\alpha, \beta \in \mathbb{N}^d$ and consider

$$\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} b_{\epsilon}^{\epsilon} = \int e^{i(\xi-\eta)\theta/h} \Lambda_{\alpha\beta}^{\epsilon}(z, \eta, \theta, \xi) d\theta d\xi$$

Notice that ∂_{η}^{β} can land on the phase or on the cut-off function $\chi(\epsilon(\xi-\eta))$. In the latter case, we gain factors of ϵ . When ∂_{η}^{β} lands on the phase instead we get

$$\partial_{\eta}^{\beta} e^{i(\xi-\eta)\theta/h} = \left(-\frac{i}{h}\theta\right)^{\beta} e^{i(\xi-\eta)\theta/h}$$

We therefore deduce that it is enough to bound (unif. in ϵ) integrals of the form

$$I_{\epsilon} = \int e^{i(\xi-\eta)\theta/h} \theta^{\beta} a_{\epsilon}^{(\alpha)}(z, \theta, \xi) \underbrace{\chi_{\epsilon}(\theta, \xi-\eta)}_{=\chi(\epsilon\theta)\chi(\epsilon(\xi-\eta))} d\theta d\xi$$

By integrating by parts using the operator $L := \frac{1 - h^2 \Delta_{\theta}}{1 + |\xi - \eta|^2}$

and noticing that $(1 - h^2 \Delta_{\theta}) e^{i(\xi-\eta)\theta/h} = (1 + |\xi - \eta|^2) e^{i(\xi-\eta)\theta/h}$, we get

$$I_{\epsilon} = \int e^{i(\xi-\eta)\theta/h} (L^k)^* [\theta^{\beta} a_{\epsilon}^{(\alpha)} \chi_{\epsilon}] d\theta d\xi$$

Each L produces a factor $(1 + |\xi - \eta|^2)^{-1}$ and at most ∂_{θ}^2 , but
 $\uparrow$
 $(L^k)^* = L^k$ formally autoadjoint

we know that

$$|\partial_{\theta}^{\beta} a_{\epsilon}| \leq C_{\beta} \langle \xi \rangle^m$$

Moreover, we gain ϵ from $\partial^k \chi_{\epsilon}$, *since* $\epsilon^k (\partial^k \chi)(\epsilon \cdot)$ which is uniformly bounded. Then

$$|L^k [\theta^{\beta} a_{\epsilon}^{(\alpha)} \chi_{\epsilon}]| \leq C_k (1 + |\xi - \eta|^2)^{-k} (1 + |\theta|)^{|\beta|} \langle \xi \rangle^m$$

uniformly in ϵ .

For $k > d + |\beta|$

$$|I_{\epsilon}| \leq C \int (1 + |\xi - \eta|^2)^{-k} (1 + |\theta|)^{|\beta|} \chi_{\epsilon} \langle \xi \rangle^m d\theta d\xi$$

and integrating by parts in ξ with $\tilde{L} := \frac{1 - \hbar^2 \Delta_\xi}{1 + |\theta|^2}$ we get

$$|I_\varepsilon| \leq C \int (1 + |\xi - \eta|^2)^{-\tilde{\kappa}} (1 + |\theta|^2)^{-\tilde{\kappa}} \langle \xi \rangle^m d\theta d\xi.$$

For $\tilde{\kappa} > d/2$ we have convergence of the integral in θ :

$$\int (1 + |\theta|^2)^{-\tilde{\kappa}} d\theta < +\infty$$

and therefore

$$|I_\varepsilon| \leq C \int (1 + |\xi - \eta|^2)^{-\tilde{\kappa}} \langle \xi \rangle^m d\xi.$$

For $\kappa > d/2$, $(1 + |\xi - \eta|^2)^{-\kappa} \in L^1(\mathbb{R}^d)$, but it is better than that because

$$|I_\varepsilon| \leq C \left((1 + |\cdot|^2)^{-\kappa} * \langle \cdot \rangle^m \right) (\eta)$$

↑
convolution

Using

$$\langle \xi \rangle^m \leq C \langle \eta \rangle^m \langle \xi - \eta \rangle^m$$

we obtain

$$\begin{aligned} \int (1 + |\xi - \eta|^2)^{-\kappa} \langle \xi \rangle^m d\xi &\leq C \langle \eta \rangle^m \int (1 + |\xi - \eta|^2)^{-\kappa} \langle \xi - \eta \rangle^m d\xi \\ &= C \langle \eta \rangle^m \int (1 + |\eta|^2)^{-\kappa} \langle \eta \rangle^m d\eta \end{aligned}$$

Choosing $\kappa > \frac{d}{2} + \frac{m}{2}$ the integral converges:

$$|I_\varepsilon| \leq C \langle \eta \rangle^m.$$

Same estimate holds for $\partial_z^\alpha \partial_\eta^\beta$.

Hence, we can pass to the limit $\varepsilon \rightarrow 0$ and, by def. of oscillatory integrals we get

$$b_c = \lim_{\varepsilon \rightarrow 0} b_c^\varepsilon$$

The dominated convergence theorem then implies that the limit exists and satisfies

$$|\partial_z^\alpha \partial_\eta^\beta b_c(z, \eta)| \leq C_{\alpha, \beta} \langle \eta \rangle^m$$

which implies $b_c \in S_{2d}(\langle \eta \rangle^m)$ q.e.d. $\square$