

Remark 2.41

we can rewrite the formula for b_t as

$$b_t(x, \xi, \hbar) = e^{-i\hbar D_\theta D_\xi} b(x+t\theta, x-(1-t)\theta, \xi) \Big|_{\theta=0}$$

This implies that we can pass from the t -quantization to the t' -quantization with the formula

$$b_{t'}(x, \xi) = e^{-i\hbar(t'-t) D_x D_\xi} b_t(x, \xi).$$

The reduction to symbols depending only on $2d$ variables is important because symbols are uniquely determined by the pseudodifferential operator they quantize. In particular,

Theorem 2.42 (Symbolic Calculus)

Let $a \in S_{2d}(\langle \xi \rangle^m)$, $b \in S_{2d}(\langle \xi \rangle^{m'})$. Then for every $t \in [0, 1]$, there exists a unique $c_t \in S_{2d}(\langle \xi \rangle^{m+m'})$ such that

$$Op_{\hbar}^t(a) \circ Op_{\hbar}^t(b) = Op_{\hbar}^t(c_t)$$

where

$$c_t(x, \xi, \hbar) = e^{i\hbar [D_\eta D_\sigma - D_u D_\xi]} \left(a((1-t)x + t\eta, \eta) b(tx + (1-t)u, \xi) \right) \Big|_{\substack{u=v=x \\ \eta=\xi}} \\ =: a \#^t b$$

and

$$c_t(x, \xi, \hbar) \sim \sum_{\substack{h \geq 0 \\ i \in \mathbb{N}^d}} \frac{\hbar^i}{i!} (\partial_\eta \partial_\sigma - \partial_\xi \partial_u)^i \left(a((1-t)x + t\eta, \eta) b(tx + (1-t)u, \xi) \right) \Big|_{\substack{u=v=x \\ \eta=\xi}} \\ \text{in } S_{2d}(\langle \xi \rangle^{m+m'})$$

proof

Theorem 2.34 implies

$$Op_{\hbar}^t(a) \circ Op_{\hbar}^t(b) = Op_{\hbar}(c)$$

with

$$c(x, y, \xi) = \frac{1}{(2\pi\hbar)^d} \int e^{i(x-y)(\eta-\xi)/\hbar} a((1-t)x + t\eta, \eta) b((1-t)z + ty, \xi) d\eta d\eta$$

$$\text{and } c(x, y, \xi) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_x^\alpha \partial_\eta^\alpha (a((1-t)x + tz, \eta) b((1-t)z + ty, \xi)) \Big|_{\substack{z=x \\ \eta=\xi}}$$

We also have $O_{Ph}(c) = O_{Ph}^E(c_E)$ with

$$c_E(x, \xi) = \frac{1}{(2\pi h)^d} \int e^{i(\xi' - \xi)\theta/h} c(x + t\theta, x - (1-t)\theta, \xi') d\xi' d\theta$$

and

$$c_E(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|} h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_\xi^\alpha \partial_\theta^\alpha (c(x + t\theta, x - (1-t)\theta, \xi)) \Big|_{\theta=0}$$

Thus

$$c_E(x, \xi) = \frac{1}{(2\pi h)^d} \int e^{i(\xi' - \xi)\theta/h + (x + t\theta - z)(\eta - \xi')/h} a((1-t)(x + t\theta) + tz, \eta) \cdot b((1-t)z + t(x - (1-t)\theta), \xi') dz d\eta d\xi' d\theta$$

We perform the change of variables

$$(z, \theta) \rightarrow (u, v) = (z + (1-t)\theta, z - t\theta)$$

and obtain

$$c_E(x, \xi) = \frac{1}{(2\pi h)^d} \int e^{i(x(\eta - \xi') + u(\xi' - \xi) - v(\eta - \xi'))/h} a((1-t)x + tu, \eta) \cdot b(tx + (1-t)v, \xi') dz d\eta d\xi' d\theta$$

By inverse Fourier formula we can write

$$\begin{aligned} & e^{i h [D_\eta D_v - D_u D_\xi]} (a((1-t)x + tu, \eta) b(tx + (1-t)v, \xi')) \Big|_{\substack{u=v=x \\ \eta=\xi}} \\ &= \frac{1}{(2\pi h)^d} \int e^{i[(\xi - \eta)\eta' + (x - v)v' + (x - u)u' + (\xi - \xi')\xi' + \eta'v' - u'\xi']/h} \\ & \quad \cdot a((1-t)x + tu, \eta) b(tx + (1-t)v, \xi') du dv d\eta d\xi' du' dv' d\eta' d\xi' \end{aligned}$$

Integrals in u' and v' give $\delta(\xi' - (x - u))$ and $\delta(\eta' - (v - x))$. Hence

$$\begin{aligned} & e^{i h [D_\eta D_v - D_u D_\xi]} (a((1-t)x + tu, \eta) b(tx + (1-t)v, \xi')) \Big|_{\substack{u=v=x \\ \eta=\xi}} \\ &= \frac{1}{(2\pi h)^{2d}} \int e^{i((\xi - \eta)(v - x) + (\xi - \xi')(x - u))/h} a((1-t)x + tu, \eta) b(tx + (1-t)v, \xi') \\ & \quad du dv d\eta d\xi' \end{aligned}$$

Putting these expressions together we get

$$c_E(x, \xi) = e^{i h [D_\eta D_v - D_u D_\xi]} (a((1-t)x + tu, \eta) b(tx + (1-t)v, \xi')) \Big|_{\substack{u=v=x \\ \eta=\xi}}$$

Corollary 2.43

Let $a \in S_{2d}(\langle \xi \rangle^m)$, $b \in S_{2d}(\langle \xi \rangle^{m'})$. Denote $A := \text{Op}_h(a)$ and $B := \text{Op}_h(b)$.
For $t \in [0, 1]$

$$\sigma_t(A \circ B) = \sigma_t(A) \sigma_t(B) + \mathcal{O}(h) \quad \text{in } S_{2d}(\langle \xi \rangle^{m+m'})$$

proof.

By exercise.

Example 2.44

Let A, B be two pseudodifferential operators with symbols $a \in S_{2d}(\langle \xi \rangle^m)$ and $b \in S_{2d}(\langle \xi \rangle^{m'})$, respectively. For $t \in [0, 1]$

$$\sigma_t([A, B]) = -i\hbar \{a, b\} + \mathcal{O}(\hbar^2) \quad \text{in } S_{2d}(\langle \xi \rangle^{m+m'})$$

where

$$[A, B] := AB - BA \quad (\text{commutator of } A \text{ and } B)$$

and

$$\{a, b\} := \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial \xi} \quad (\text{Poisson brackets of } a \text{ and } b).$$

Definition 2.45

A symbol $a \in S_{2d}(\langle \xi \rangle^m)$ is called classical if it can be written in asymptotic expansion, i.e.

$$a(x, \xi; \hbar) \sim \sum_{j \geq 0} \hbar^j a_j(x, \xi).$$

$\hbar^\mu a_0(x, \xi)$ is called PRINCIPAL SYMBOL of the classical pseudodifferential operator $A = \sum_{\mu \in \mathbb{R}} \hbar^\mu \text{Op}_h^\mu(a)$, with $a_0 \neq 0$ and for every $\mu \in \mathbb{R}$.

Remark 2.46

Observe that $\text{Op}_h(b) = \text{Op}_h^\pm(b_\pm)$ implies that the principal symbol does not depend on the quantization.