

2.6 Pseudodifferential Operators acting on L^2

Since one of the main applications of pseudodifferential calculus is quantum mechanics, we are interested in studying the action of pseudodifferential operators on L^2 functions.

The main result in this direction is the Calderón - Vaillancourt theorem, that states that pseudodifferential operators are continuous on L^2 :

Theorem 2.47

Let $a \in S_{3d}(1)$. Then $Op_h(a)$ is continuous on $L^2(\mathbb{R}^d)$ and there exist constants C and M (only dimension dependent) such that

$$\|Op_h(a)\|_{\mathcal{L}(L^2)} \leq C \left(\sum_{|\alpha| \leq M} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^{3d})} \right)$$

where $\mathcal{L}(L^2)$ is the space of bounded operators on $L^2(\mathbb{R}^d)$.

The proof makes use of important lemmas, that we write here below and postpone their proof.

Lemma 2.48 (partition of $\mathbb{R}^{3d}$)

There exists $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ such that

$$\sum_{k \in \mathbb{Z}^d} \chi_k = 1 \text{ on } \mathbb{R}^d$$

with $\chi_k(z) = \chi_0(z - k)$ for $z \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$.

Lemma 2.49 (Cotlar - Stein Lemma)

Let H be an Hilbert space and $\{A_k\}_{k \in \mathbb{Z}^d}$ a family of operators in $\mathcal{L}(H)$. Let $\omega: \mathbb{Z}^d \rightarrow \mathbb{R}_+$ be a map s.t.

$$\|A_k A_{k'}^*\| + \|A_k^* A_{k'}\| \leq \omega(k - k') \quad \forall k, k' \in \mathbb{Z}^d,$$

and let $C_0 = \sum_{k \in \mathbb{Z}^d} \sqrt{w(k)} < +\infty$

Then $\forall M \geq 0$,

$$\left\| \sum_{|k| \leq M} A_k \right\| \leq C_0$$

Lemma 2.50 (Schur lemma)

Let

$$Au(x) = \int K(x,y) u(y) dy$$

with $K \in C^0(\mathbb{R}^{2d})$. Then

$$\|A\|_{L^2(\mathbb{R}^d)} \leq \left(\sup_x \int_y |K(x,y)| dy \right)^{1/2} \left(\sup_y \int_x |K(x,y)| dx \right)^{1/2}$$

We are now ready to prove Theorem 2.47:

proof of Thm 2.47

w.l.o.g. we can restrict to prove the Theorem for $A := Op_s^w(a)$

(check by exercise)

Lemma 2.48 guarantees the existence of $\chi_0 \in C_0^\infty(\mathbb{R}^{2d})$ such that

$$\sum_{k \in \mathbb{Z}^{2d}} \chi_k = 1. \quad \text{Set}$$

$$a_k = a \chi_k$$

and notice that $\forall \alpha \in \mathbb{N}^{2d}$

$$|\partial^\alpha \chi_k(z)| = |(\partial^\alpha \chi_0)(z-k)| \leq \sup_{z \in \mathbb{R}^d} |\partial^\alpha \chi_0|$$

Leibniz formula then gives

$$\partial^\alpha a_k = O\left(\sup_{\beta \leq \alpha} \|\partial^\beta a\|_{L^\infty}\right) \quad \text{uniformly in } k \in \mathbb{Z}^{2d}$$

Set $A_k = Op_s^w(a_k)$ such that $\forall u \in C_0^\infty(\mathbb{R}^d)$

$$Au = \sum_{k \in \mathbb{Z}^d} A_k u$$

where the limit of the series can be taken in $L^2(\mathbb{R}^d)$ (exercise)

We want to apply Lemma 2.50. To this end, we need and Lemma 2.49

for proving the theorem. To this end, we need to verify the assumptions

of Lemma 2.49:

observe that for $k, k' \in \mathbb{Z}^{2d}$,

$$A_k A_{k'}^* u(x) = \int K_{k,k'}(x,y) u(y) dy$$

where

$$K_{k,k'}(x,y) = \frac{1}{(2\pi)^{2d}} \int e^{i(x\xi - y\eta - z\xi + z\eta)} a_k\left(\frac{x+z}{2}, \xi\right) \overline{a_{k'}}\left(\frac{y+z}{2}, \eta\right) dz d\eta d\xi \quad (*)$$

By assumptions $a_k, a_{k'}$ are C^∞ , thus $K_{k,k'} \in C^\infty(\mathbb{R}^{2d})$.

To bound $A_k A_{k'}^*$ we need to bound $\int |K_{k,k'}(x,y)| dy$. Thus we focus on (*) and define the operator

$$L := \frac{(1 + (x-z) \cdot D_\xi - (y-z) \cdot D_\eta - (\xi-\eta) \cdot D_z)}{1 + |x-z|^2 + |y-z|^2 + |\xi-\eta|^2}$$

such that $L(e^{i(x\xi - y\eta - z\xi + z\eta)}) = e^{i(x\xi - y\eta - z\xi + z\eta)}$.

Then, by integration by parts through L , we have $\forall N \geq 0$

$$K_{k,k'}(x,y) = \frac{1}{(2\pi)^{2d}} \int e^{i(x\xi - y\eta - z\xi + z\eta)} (L^N) \left(a_k\left(\frac{x+z}{2}, \xi\right) \overline{a_{k'}}\left(\frac{y+z}{2}, \eta\right) \right) dy dz d\xi d\eta$$

Notice that for $|k-k'|$ large, there exists $C > 0$ such that

$$\frac{1}{C} |k-k'| \leq |x-y| + |\xi-\eta| \leq C |k-k'| \quad \text{on } \text{supp}(a_k(x, \xi) \overline{a_{k'}}(y, \eta))$$

let $k = (k_1, k_2)$ and $k' = (k'_1, k'_2)$ in $\mathbb{Z}^{2d}$. Then

$$\int |K_{k,k'}(x,y)| dy = \int_{\mathcal{D}_{k,k'}} \mathcal{O} \left(\frac{\sup_{|\alpha| \leq N} \| \partial^\alpha a \|_{L^\infty}^2}{(1 + |x-z| + |\xi-\eta|)^d (|\xi-\eta|)^N} \right) dz d\eta d\xi$$

with $\mathcal{D}_{k,k'}$ the domain defined by

$$\mathcal{D}_{k,k'} := \left\{ \frac{1}{C} |k-k'| \leq |x-y| + |\xi-\eta| \leq C |k-k'|, |\xi-k_2| \leq C', |\xi-k'_2| \leq C' \right\}$$

Since $|x-z| + |y-z| \geq |x-y|$, we obtain

$$\int |K_{k,k'}(x,y)| dy = \int \frac{\mathcal{O}((1 + |k-k'|)^{2d+2N-N} \sup_{|\alpha| \leq N} \| \partial^\alpha a \|_{L^\infty}^2)}{(1 + |x-z|)^{d+1} (1 + |x-y|)^{d+1}} dy dz$$

In the same way we obtain $\forall N \geq 0$

$$\sup_{x \in \mathbb{R}^d} \int |K_{k,k'}(x,y)| dy = O\left((1+|k-k'|)^{2d+2-N} \sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}^2\right)$$

Analogously, for $N \geq 0$

$$\sup_{y \in \mathbb{R}^d} \int |K_{k,k'}(x,y)| dx = O\left((1+|k-k'|)^{2d+2-N} \sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}^2\right)$$

By Schur lemma we conclude that

$$\|A_k A_{k'}^*\|_{\mathcal{L}(L^2)} = O\left((1+|k-k'|)^{2d+2-N} \sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}^2\right)$$

$$\|A_k^* A_{k'}\|_{\mathcal{L}(L^2)} = O\left((1+|k-k'|)^{2d+2-N} \sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}^2\right)$$

Let us choose $N = 4d+3$ and apply Cotlar-Stein Lemma:

$$\left\| \sum_{|k| \leq M} A_k u \right\|_{L^2} \leq C_0 \|u\|_{L^2}$$

$\forall M \geq 0, \forall u \in C_0^\infty$ and with

$$C_0 = \sqrt{C} \sum_{k \in \mathbb{Z}^{2d}} (1+|k|)^{-d-\frac{1}{2}} \sup_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^\infty}$$

Let $M \rightarrow +\infty$, then $\|Au\|_{L^2} \leq C_0 \|u\|_{L^2}$, that yields

$$\|A\|_{\mathcal{L}(L^2)} \leq C_0$$

□