Heisenberg algebras and Hilbert schemes of points on surfaces

The aim of this seminar is to give a first introduction to the beautiful subject of Hilbert schemes of varieties. The aspect we will be focusing on is the construction of a representation of the Heisenberg superalgebra on the homology groups of the Hilbert scheme of points on a surface.

1 Preliminaries

The goal of the first two talks is to introduce the Hilbert polynomial of a variety. Not only is this an interesting object in its own right, but it is also the first concept we need to understand in order to be able to define Hilbert schemes.

Talk 1: The degree of a projective variety

Date: 20.04

- Define the degree of a projective variety using the notion of linear subvarietes and discuss the examples of plane conics, hypersurfaces in $\mathbb{P}^n$, and if time permits complete intersections.
- Introduce the notion of a scheme.
- Discuss the fact that the degree is a projective invariant but not an invariant of the isomorphism class of a projective variety.
- If time permits, give Bézout's theorem as application of the notion of degree.

References: For definition of linear subvariety see the beginning of 5.4 from [1]. For Bézout's theorem, look at Lecture 18 of [2] or any other classical reference. For the rest follow 5.5 from [1].

Talk 2: The Hilbert polynomial of a projective variety

Date: 27.04

- Define the Hilbert function h_X of a projective variety X and give its geometric interpretation.
- Compute h_X for a variety consisting of three or four points in the projective plane.
- Introduce the Hilbert polynomial and give some examples.
- Discuss the connection between the degree of a projective variety and its Hilbert polynomial.

References: Chapter 5.6 of [1], Lecture 13 of [2].

2 The Hilbert scheme of points on the affine plane

In the next few talks we get some practice with Hilbert schemes of points by looking at a toy example, namely the affine plane $\mathbb{A}^2$ over $\mathbb{C}$.

Talk 3: The moduli problem

Date: 04.05

- Introduce the problem of parametrising subvarieties of $\mathbb{P}^n$ having the same Hilbert polynomial P.
- Illustrate this by defining the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ of n points on $\mathbb{A}^2$ as a set.
- Compare with the symmetric product $S^n\mathbb{A}^2$.
- State our first goal: "We want to describe $\mathrm{Hilb}^n(\mathbb{A}^2)$ as a subvariety of a Grassmannian."

References: Chapter 5.6 of [1] for the parametrisation problem. For the rest, see [3]: Chapter 18.1 for the Hilbert scheme and beginning of Chapter 3.1 for explanation of the staircase diagram appearing in the discussion from 18.1.

Talk 4: Grassmannians and Gröbner bases - technical interlude

Date: 11.05

- Introduce Grassmannians and Plücker coordinates.
- Introduce (reduced) Gröbner bases and initial ideals.
- If time permits, discuss the Gröbner basis of the ideal of Plücker relations.

References: From [3], Chapters 2.2 for Gröbner bases, 14.1 for Grassmannians, and 14.2 for Plücker relations.

Talk 5: The quasi-projective variety structure of $\mathrm{Hilb}^n(\mathbb{A}^2)$

Date: 18.05

- Following 18.1 of [3] endow $\mathrm{Hilb}^n(\mathbb{A}^2)$ with the structure of a quasi-projective variety by constructing a suitable open covering.
- State Theorem 18.7 of loc.cit.
- If time permits, discuss briefly the proof of the connectivity part of Theorem 18.7.

References: Chapters 18.1, 18.2 of [3].

3 Hilbert schemes of points on surfaces

In this section we generalise what we know about Hilbert schemes by reformulating our concepts in the more sophisticated language of moduli spaces. We also see how our results about the Hilbert scheme of points of the affine plane helps us conclude that the Hilbert scheme of points of any smooth surface is itself smooth.

Talk 6: The moduli problem revisited

Date: 01.06

- Discuss families of geometric objects, fine moduli spaces, moduli functors, and representability.
- Introduce Hilbert schemes in this general moduli setting following 1.1 of [4]. Write out the corresponding moduli functor and state Grothendieck's theorem.
- Recover Hilbert schemes of points using the new language we introduced (as explained in the paragraph after Definition 1.2 in [4]).

References: For background on moduli spaces, use Sections 0.2.1-0.2.8 of [5]. For the rest, see 1.1 of [4].

Talk 7: Hilbert schemes of points on surfaces

Date: 08.06

- Discuss symmetric products and their stratification.
- Introduce the Hilbert-Chow morphism.
- Discuss some properties of the Hilbert scheme of points of a one-dimensional variety. For example, one can show that $(\mathbb{A}^1)^{[n]} = S^n\mathbb{A}^1$, $S^n\mathbb{P}^1 \simeq \mathbb{P}^n$, or that if $\dim X = 1$ and X is smooth, then $X^{[n]} \simeq S^n X$ is also smooth.
- State and prove Theorem 1.15 from [4].

References: Sections 1.1 and 1.3 of [4]. For the discussion of one-dimensional varieties Section 3 of [6] is also useful.

4 Representations of Heisenberg algebras

In the remaining talks we understand how one may view the homology groups of Hilbert schemes of points on surfaces as representations of the Heisenberg Lie superalgebra.

Talk 8: Heisenberg algebras

Date: 15.06

- Introduce Lie algebras and their representations.
- Describe the Heisenberg and Clifford algebras as in 8.1 of [4].
- Define Lie superalgebras and describe the Heisenberg Lie superalgebra.
- State the character formula for the Heisenberg superalgebra.

References: Section 8.1 of [4] for the Heisenberg and Clifford Lie algebras and the Heisenberg Lie superalgebra. For generalities on Lie algebras see Section 8.1 of [7] and on Lie superalgebras Appendix 3B of [8].

Talk 9: Correspondences for manifolds

Date: 22.06

- Discuss Borel-Moore homology groups and the corresponding intersection pairing.
- Define correspondences (and their adjoints) in terms of fundamental classes of submanifolds of a Cartesian product of manifolds. Describe compositions of correspondences.
- Discuss how correspondences may be interpreted as generalisations of maps between manifolds.

References: Section 8.2 of [4].

Talk 10: Correspondences for Hilbert schemes of points on surfaces

Date: 29.06

- Following [4] introduce the cycles $P[i]$, $P_\alpha[i]$, and $P_\beta[-i]$.
- State the main Theorem 8.13 and its Corollary 8.16.
- Present Example 8.18.

References: Section 8.3 of [4].

Talk 11: Proof of the main Theorem: Part I

Date: 06.07

- Give the proof of Theorem 8.13 following [4]. Present only Case a).

References: Section 8.4 of [4].

Talk 12: Proof of the main Theorem: Part II

Date: 13.07

- Finish the proof of Theorem 8.13 following [4].

References: Section 8.4 of [4]

Talk 13: Chern classes of tautological bundles

Date: 20.07

- Introduce the notion of tautological bundles over Hilbert schemes of points.
- Motivate why it is interesting to be able to compute its characteristic classes.
- Discuss Theorem 4.6 of [9] in which one obtains a formula for the Chern classes of tautological bundles in terms of the Nakajima operators introduced in the previous talks.

References: Section 4 of [9].

References

[1] K. E. Smith, L. Kahanpää, P. Kekäläinen, W. Traves. *An Invitation to Algebraic Geometry,* Springer, 2000

[2] J. Harris. *Algebraic Geometry - A First Course,* Springer, 1992.

[3] E. Miller, B. Sturmfels. *Combinatorial Commutative Algebra,* Springer, 2005.

[4] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces,* AMS, 1999.

[5] J. Kock, I. Vainsencher. *An Invitation to Quantum Cohomology,* Birkhäuser, 2007.

[6] F. Rota. `http://www.math.utah.edu/~filipazz/seminar_notes/fall_2016/Hilbert_scheme_points.pdf`

[7] W. Fulton, J. Harris. *Representation Theory - A First Course,* Springer, 2004.

[8] D. Huybrechts. *Complex Geometry - An Introduction,* Springer, 2005.

[9] M. Lehn. *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces,* Inventiones Mathematicae **136**, 157-207 (1999).