Exercises for the Lecture
Topics in Mathematical Physics

Attendance Sheet

Exercise 1: (6=1+1+2+2 Points)

Consider a point particle in $\mathbb{R}^3$ under the action of a radial potential

$$K(x) = K(|x|), \quad x \in \mathbb{R}^3.$$

(i) Show that the angular momentum

$$L := x \wedge p$$

is preserved under the time evolution.

HINT: Show that each of the three components of L is conserved.

(ii) Deduce from (i) that the particle moves on a fixed plane orthogonal to L and passing through the origin.

(iii) Using (i), solve the equation of motion.

HINT: Parametrize the position of the particle using polar coordinates

$$x = (r\cos\phi, r\sin\phi)$$

and express the energy in terms of (r, ϕ). Define the effective potential

$$U_{\text{eff}}(r) := \frac{L^2}{2mr^2} + K(r)$$

to get

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\big(E - U(r)\big)}.$$

(iv) Let

$$K(x) = -\frac{GMm}{|x|},$$

where M is the mass of the sun and m the mass of a planet in the solar system, and G is the gravitational constant. Show that trajectories are ellipses if $E < 0$, parabolas if $E = 0$, and hyperbolas if $E > 0$. (Kepler's first law)

Set up the energy conservation. Try the substitution $u = \frac{1}{r}$ and transform time derivatives into angular derivatives by the chain rule. The resulting ODE for u in the angle θ should be in the form $\frac{d^2u}{d\theta^2} = au + b$ for some numbers a, b.

Exercise 2: (4=1+2+1 Points)

Let $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ denote the Lebesgue space, $1 \le p \le \infty$, and let, for $f \in \mathcal{L}^p$,

$$\|f\|_p := \begin{cases} \left(\int |f|^p \, d\mu\right)^{1/p}, & 1 \le p < \infty, \\ \inf\{c > 0 : |f(x)| \le c \text{ a.e.}\}, & p = \infty. \end{cases}$$

Show that $\|\cdot\|_p$ is such that

(i) $\|\lambda f\|_p = |\lambda| \|f\|_p, \quad \forall \lambda \in \mathbb{C}, \forall f \in \mathcal{L}^p$

(ii) $\|f + g\|_p \le \|f\|_p + \|g\|_p, \quad \forall f, g \in \mathcal{L}^p$ (triangle inequality)

(iii) If $f \in \mathcal{L}^p$ with $\|f\|_p = 0$, we don't necessarily get $f \equiv 0$.

Exercise 3: (4=2+2 Points)

(i) Show that $\sim$ is an equivalence relation on $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$:

$$f \sim g \iff f(x) = g(x) \quad \mu\text{-a.e.}$$

(ii) Deduce that $\|\cdot\|_p$ defines a norm on $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)/\sim$.