Exercise Sheet
"Topics in Mathematical Physics"

Sheet Number 2

Due: Monday 03.11.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 4: (4=2+2 Points)

Suppose A is a symmetric operator on $\mathcal{H}$. Prove:

(i) For all $\psi \in \operatorname{dom}(A)$,

$$\langle \psi, A\psi \rangle \in \mathbb{R}$$

(ii) Suppose λ is an eigenvalue for A, i.e. $A\psi = \lambda\psi$ for some $\psi \in \operatorname{dom}(A)$, then $\lambda \in \mathbb{R}$.

Exercise 5: (8=1+2+2+ (Bonus) 3 Points)

Recall the definition of the Fourier transformation of $f \in L^1(\mathbb{R}^d)$:

$$\hat{f} : \mathbb{R}^d \to \mathbb{C}, \quad \hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ik\cdot x} dx$$

Show that

(i) The Fourier transformation, also written as $\mathcal{F}$, is bounded as a map $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$.

(ii) Convolutions: For $f, g \in L^1(\mathbb{R}^d)$,

$$\widehat{f * g}(k) = (2\pi)^{d/2} \hat{f}(k)\hat{g}(k)$$

(iii) Derivatives: Let $f \in L^1(\mathbb{R}^d)$, $g : \mathbb{R}^d \to \mathbb{C}^d$ s.t. $g(x) = -ixf(x)$.
If $g \in L^1(\mathbb{R}^d)$, then $\hat{f} \in C^1(\mathbb{R}^d)$ and $\nabla\hat{f} = \hat{g}$.

(iv) Plancherel's identity: If $f \in L^1 \cap L^2(\mathbb{R}^d)$, then $\hat{f} \in L^2(\mathbb{R}^d)$ and $\|\hat{f}\|_2 = \|f\|_2$

Exercise 6: (4=1+3 Points)

Show that every self-adjoint operator A is symmetric and show that the converse does not hold in general.