Exercise Sheet
"Topics in Mathematical Physics"

Sheet Number 3

Due: Monday 10.11.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 6 (Young's Inequality): (3 Points)

Consider $p, q, r \in [1, \infty]$ such that

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}.$$

Let $f \in L^q(\mathbb{R}^d)$, $g \in L^r(\mathbb{R}^d)$; prove that

$$\|f * g\|_p \leq \|f\|_q \, \|g\|_r.$$

Hint: Consider the functions

$$\begin{aligned} \alpha(x,y) &:= |f(y)|^q |g(x-y)|^r, \\ \beta(y) &:= |f(y)|^q, \\ \gamma(x,y) &:= |g(x-y)|^r. \end{aligned}$$

Notice that

$$|f * g(x)| \leq \int_{\mathbb{R}^d} \alpha(x,y)^{\frac{1}{p}} \, \beta(y)^{\frac{p-q}{pq}} \, \gamma(x,y)^{\frac{p-r}{pr}} \, dy$$

and that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = 1$$

to apply Hölder's inequality.

Exercise 7: (4 Points)

Let A, B be bounded operators on a Hilbert space $\mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.
Prove the following equalities:

$$\begin{aligned} \mathrm{id}^* &= \mathrm{id}, \\ (A^*)^* &= A, \\ (AB)^* &= B^* A^*, \\ (\alpha A + \beta B)^* &= \bar{\alpha} A^* + \bar{\beta} B^*. \end{aligned}$$

Moreover, prove that A^* is bounded and that $\|A^*\| = \|A\|$.

Exercise 8: (3 Points)

Let $\mathcal{H}$ be a Hilbert space and A, B bounded self-adjoint operators on $\mathcal{H}$. Prove that

$$\frac{1}{i\hbar}[A, B] = \frac{1}{i\hbar}(AB - BA)$$

is self-adjoint.